

involve

a journal of mathematics

Note on superpatterns

Daniel Gray and Hua Wang



Note on superpatterns

Daniel Gray and Hua Wang

(Communicated by Joshua Cooper)

Given a set P of permutations, a P -superpattern is a permutation that contains every permutation in P as a pattern. The study of the minimum length of a superpattern has been of interest. For P being the set of all permutations of a given length, bounds on the minimum length have been improved over the years, and the minimum length is conjectured to be asymptotic with k^2/e^2 . Similar questions have been considered for the set of layered permutations. We consider superpatterns with respect to packing colored permutations or multiple copies of permutations. Some simple but interesting observations will be presented. We also propose a few questions.

1. Introduction

Given a permutation π of length n , a pattern σ is said to be contained in π , or σ occurs in π , if a subsequence of π is order isomorphic to σ . For instance, the permutation $\pi = 51342$ contains two occurrences of the pattern $\sigma = 321$ as the subsequences 532 and 542. Much effort has been devoted to the study of occurrences of patterns in a permutation, most of which involves studying permutations which avoid a particular pattern, i.e., pattern avoidance.

As a symmetric problem to pattern avoidance, the concept of a superpattern concerns packing all patterns from a given set into a single permutation.

Definition. Let P be a set of permutations. A P -superpattern is a permutation that contains π for every $\pi \in P$.

Superpatterns were first introduced in [Arratia 1999]. The natural question immediately following this definition is to find the minimum length of a P -superpattern. When P is the set of all permutations of length k , this minimum length is denoted by $\text{sp}(k)$ and has been vigorously studied. The trivial upper bound of k^2 was improved to $\frac{2}{3}k^2$ in [Eriksson et al. 2002], and was conjectured to be asymptotic with $\frac{1}{2}k^2$. Later, it was shown in [Miller 2009] that $\text{sp}(k) \leq \frac{1}{2}k(k+1)$ through

MSC2010: 05A05.

Keywords: superpatterns, colored permutations.

This work was partially supported by grants from the Simons Foundation (#245307).

the construction of a zigzag k -superword. More recently, bounds on the minimum lengths of superpatterns containing all layered permutations were considered in [Gray 2015]. Constructions similar to that in [Miller 2009] were also used to study superpatterns containing simple patterns of length k in [Gray \geq 2016].

Definition. An m -colored permutation χ of length n is a permutation of length n in which each element is assigned one of m distinct colors.

For example, let $\chi = 3_1 2_1 5_1 1_1 4_2$ be a 2-colored permutation, where 3, 2, 5, and 1 have color 1, while 4 has color 2. Analogous to the case of noncolored patterns, the colored pattern $\phi = 2_1 1_1 3_2$ occurs in χ as the subsequences $3_1 1_1 4_2$, $2_1 1_1 4_2$, and $3_1 2_1 4_2$, but not $3_1 2_1 5_1$.

Colored permutations are of interest in the study of patterns and pattern avoidance [Savage and Wilf 2006]. Packing densities of colored permutations were also recently considered [Just and Wang 2016].

In this note we consider superpatterns of different sets of colored permutations. Some elementary, but interesting, observations will be presented. We also propose some questions from these studies.

2. Superpatterns containing all colored permutations

Let $S(k, m)$ denote the set of m -colored permutations of length k and

$$\text{sp}(k, m) = \min\{|p| : p \text{ is an } S(k, m)\text{-superpattern}\}.$$

The following presents a simple connection between $\text{sp}(k, m)$ and $\text{sp}(k)$.

Theorem 2.1. *For any positive integers k and m , we have*

$$\text{sp}(k, m) = m \cdot \text{sp}(k).$$

Proof. Let p' be an $S(k, m)$ -superpattern, and denote by p'_i the subsequence of p' of color i (for any $1 \leq i \leq m$). Then p'_i , without the color, is a superpattern containing all noncolored patterns of length k . Consequently $|p'_i| \geq \text{sp}(k)$ for any i and

$$|p'| = \sum_{i=1}^m |p'_i| \geq m \cdot \text{sp}(k).$$

On the other hand, let p be a permutation of length $\text{sp}(k)$ that contains all noncolored patterns of length k . Construct an m -colored permutation p'' from p by replacing each $1 \leq j \leq \text{sp}(k)$ in p by the sequence

$$s_j := [m(j-1) + 1]_1 [m(j-1) + 2]_2 \cdots [m(j-1) + m]_m.$$

Note that $|p''| = m \cdot |p| = m \cdot \text{sp}(k)$. For any pattern $\pi \in S(k, m)$, the noncolored version is contained in p and the corresponding colored pattern can be found in p'' by choosing corresponding digits in s_j with the required color. Thus,

$$\text{sp}(k, m) \leq |p''| = m \cdot \text{sp}(k). \quad \square$$

For example, $p = 132$ is a superpattern containing all patterns of length 2, and p is of length $\text{sp}(2) = 3$. An $S(2, 3)$ -superpattern p'' can be constructed as

$$1_1 2_2 3_3 7_1 8_2 9_3 4_1 5_2 6_3.$$

As an immediate consequence of Theorem 2.1, the established asymptotic bounds for $\text{sp}(k)$ apply directly to $\text{sp}(k, m)$. The trivial asymptotic lower bound k^2/e^2 for $\text{sp}(k)$ follows from

$$\binom{\text{sp}(k)}{k} \geq k!$$

and a standard application of Stirling’s approximation for factorials [Arratia 1999].

Corollary 2.2. *For any positive integers k and m ,*

$$mk^2/e^2 \leq \text{sp}(k, m) \leq \frac{1}{2}mk(k + 1).$$

Remark. The arguments in Theorem 2.1 establish the same relationship between the colored and noncolored versions of superpatterns containing any particular subset of the length- k permutations, such as the layered permutations [Gray 2015] and simple and alternating permutations [Gray \geq 2016], and consequently provide bounds on the minimum lengths of these colored superpatterns.

3. Monochromatic and nonmonochromatic patterns

Let $NMS(k, m)$ be the set of nonmonochromatic m -colored patterns of length k and $MS(k, m)$ be the set of all monochromatic m -colored patterns of length k . Then, $S(k, m)$ is the disjoint union of $NMS(k, m)$ and $MS(k, m)$. It is easy to see that

$$|MS(k, m)| = mk!,$$

and consequently,

$$\begin{aligned} |NMS(k, m)| &= |S(k, m)| - |MS(k, m)| = m^k k! - mk! = (m^k - m)k! \\ &= (m^{k-1} - 1) |MS(k, m)|. \end{aligned}$$

Given any $NMS(k, m)$ -superpattern of length n , we must have

$$\binom{n}{k} \geq (m^k - m)k!,$$

implying (by way of a standard application of Stirling’s approximation for factorials)

$$n \geq mk^2/e^2,$$

the same asymptotic lower bound for $\text{sp}(k, m)$ for general $S(k, m)$ -superpatterns.

Letting

$$\text{nmsp}(k, m) = \min\{|p| : p \text{ is an } NMS(k, m)\text{-superpattern}\},$$

we have the simple consequence that

$$mk^2/e^2 \leq \text{nmsp}(k, m) \leq \text{sp}(k, m) \leq \frac{1}{2}mk(k + 1). \tag{1}$$

On the other hand, exactly the same argument as that of Theorem 2.1 implies

$$\text{msp}(k, m) = m \cdot \text{sp}(k), \tag{2}$$

where

$$\text{msp}(k, m) = \min\{|p| : p \text{ is an } MS(k, m)\text{-superpattern}\}.$$

Remark. Equations (1) and (2) imply, in addition to the semitrivial bounds of $\text{msp}(k, m)$ and $\text{nmsp}(k, m)$, that

$$\text{msp}(k, m) = m \cdot \text{sp}(k) = \text{sp}(k, m) \geq \text{nmsp}(k, m), \tag{3}$$

a rather surprising fact given that $|NMS(k, m)| = (m^{k-1} - 1) |MS(k, m)|$.

While it may be a bit unexpected to see that $\text{msp}(k, m) = \text{sp}(k, m)$, a natural question follows.

Question 3.1. *Does strict inequality hold in (3)?*

In the special case for $k = 2$, the proposition below answers Question 3.1 in the affirmative.

Proposition 3.2. *For any positive integer m , we have*

$$3m = \text{msp}(2, m) > 3m - 2 \geq \text{nmsp}(2, m).$$

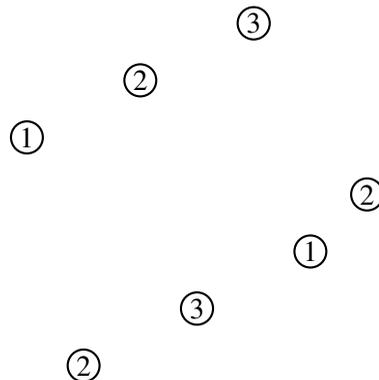
Proof. Clearly, $\text{sp}(2) = 3$, and hence $\text{msp}(2, m) = m \cdot \text{sp}(2) = 3m$. Let p be a permutation of length $3m - 2$ defined as

$$[2m - 1]_1 1_2 [2m]_2 2_3 \cdots [3m - 3]_{m-1} [m - 1]_m [3m - 2]_m m_1 [m + 1]_2 [m + 2]_3 \cdots [2m - 2]_{m-1}.$$

For instance, if $m = 3$, then

$$p = 5_1 1_2 6_2 2_3 7_3 3_1 4_2,$$

and the graph of p is depicted below:



In general, the graph of p will be the disjoint union of two increasing subsequences. The top row is of length m and the bottom row is of length $2m - 2$, and every entry of the top row is larger than every entry of the bottom row. The entries of p alternate from the top row to the bottom row until there are m entries in the top row. Then, $m - 2$ more entries are added to the bottom row. The i -th entry of the top row will have color i for $1 \leq i \leq m$, while the i -th entry of the bottom row will have color $i + 1 \pmod{m}$.

For $i, j \in [1, m]$ with $i \neq j$, the pattern $1_i 2_j$ is contained in the bottom row, the only exception being the pattern $1_1 2_j$, which is contained in the top row. The pattern $2_i 1_j$ can be found by selecting the unique entry colored i from the top row, and taking an entry in the bottom row colored j which lies to the right of the entry just selected. Then, p is an $NMS(2, m)$ -superpattern. \square

To answer Question 3.1 in general appears to be very difficult. In an effort to further understand the relationship between monochromatic and nonmonochromatic superpatterns, we also point out the following.

Proposition 3.3. *For any positive integers $k \geq 2$ and m , we have*

$$\text{msp}(k - 1, m) \leq \text{nmsp}(k, m) \leq \text{msp}(k, m).$$

Proof. The second inequality is implied by the remark on page 800. To see the first inequality, let q be an m -colored pattern of length k whose first $k - 1$ entries are colored by color i and whose k -th entry is colored by $j \neq i$. Then, the first $k - 1$ entries of q form a monochromatic pattern of length $k - 1$.

Since q is a nonmonochromatic m -colored pattern of length k , it must be contained in any $NMS(k, m)$ -superpattern. Noting that we could have colored the first $k - 1$ entries of q monochromatically using any of the m colors, any $NMS(k, m)$ -superpattern must contain all monochromatic m -colored patterns of length $k - 1$. Thus, $\text{nmsp}(k, m) \geq \text{msp}(k - 1, m)$. \square

4. Packing multiple copies of all patterns

The idea of superpatterns lies in the fact that they contain each permutation (from a given set of permutations) at least once. A natural generalization seems to be superpatterns that contain each permutation at least a given number of times.

Definition. For a given set P of permutations, a P_ℓ -superpattern is a permutation containing each pattern $\pi \in P$ at least ℓ times.

Define $\text{sp}_\ell(k)$, $\text{sp}_\ell(k, m)$, $\text{msp}_\ell(k, m)$, and $\text{nmsp}_\ell(k, m)$ accordingly. Some trivial facts follow immediately:

- $\text{sp}_\ell(k, m) = m \cdot \text{sp}_\ell(k)$. This can be seen by following exactly the same argument as that of Theorem 2.1.

- $\text{sp}_\ell(k) \leq \ell \cdot \text{sp}(k)$ and $\text{sp}_\ell(k, m) \leq \ell \cdot \text{sp}(k, m)$. For permutations p of length n and q of length m , the direct sum $p \oplus q$ is the permutation that has the first n entries from p and the next m entries from entries of q shifted by n . That is,

$$p \oplus q = p_1 p_2 \cdots p_n (q_1 + n)(q_2 + n) \cdots (q_m + n).$$

Given a permutation p of length $\text{sp}(k)$ that contains all patterns of length k , the permutation $\bigoplus_{i=1}^\ell p$ clearly contains each length- k pattern in each of the ℓ summands. Hence, $\text{sp}_\ell(k) \leq \ell \cdot \text{sp}(k)$. A similar argument holds for $\text{sp}_\ell(k, m) \leq \ell \cdot \text{sp}(k, m)$.

- $\text{sp}_m(k) \leq \text{msp}(k, m)$. This can be seen from removing the colors of an $MS(k, m)$ -superpattern.

The asymptotic lower bounds, for k large and ℓ constant, of $\text{sp}_\ell(k)$ or $\text{sp}_\ell(k, m)$ stay the same as $\text{sp}(k)$ or $\text{sp}(k, m)$. Given that the multiple copies of patterns need not be disjoint, it is natural to ask for improvement of the upper bounds above. The existing constructions (that provided upper bounds for the minimum lengths of various superpatterns) such as those in [Arratia 1999; Gray 2015; \geq 2016] do not directly generalize to the case of packing multiple copies of every permutation. We conclude this note by showing a nontrivial upper bound for the sp_ℓ function.

Definition. For $k, n \in \mathbb{N}$, let $q = q_1 q_2 q_3 \cdots q_k$ be a pattern of length k and let $w = w_1 w_2 w_3 \cdots w_n$ be a word of length n . We say that q is “contained exactly in w ” if there is a subsequence of length k , say

$$(w_{i_1}, w_{i_2}, \dots, w_{i_k}),$$

such that $w_{i_j} = q_j$ for all $1 \leq j \leq k$.

Theorem 4.1. For $k, \ell \in \mathbb{N}$, we have

$$\text{sp}_\ell(k) \leq \begin{cases} \frac{1}{2}(k+1)(k+\ell-1) & \text{if } k \text{ is odd,} \\ \frac{1}{2}(k+1)(k+\ell-1) & \text{if } k \text{ is even and } \ell \text{ is odd,} \\ \frac{1}{2}(k+1)(k+\ell-1) + 1 & \text{if } k \text{ is even and } \ell \text{ is even.} \end{cases}$$

Proof. We begin with Allison Miller’s construction [2009] of the zigzag k -superword. Let k_o (resp. k_e) be the smallest odd (resp. even) integer at least as large as k . We make the following definitions:

$$\bar{k}_o = 1357 \cdots k_o \quad \text{and} \quad \bar{k}_e = k_e \cdots 8642.$$

Define

$$w = \bar{k}_o \bar{k}_e \bar{k}_o \bar{k}_e \cdots \bar{k}_o \bar{k}_e$$

if k is even or

$$w = \bar{k}_o \bar{k}_e \bar{k}_o \bar{k}_e \cdots \bar{k}_o \bar{k}_e \bar{k}_o$$

if k is odd, where the combined number of copies of \bar{k}_o and \bar{k}_e is exactly k . The word w is what Miller calls the zigzag k -superword.

Let q be a pattern of length k , and let $q + 1$ be the permutation of the set $\{2, 3, 4, \dots, k + 1\}$ obtained by adding 1 to each entry of q . Number the runs of w from left to right in increasing order and let $m(q)$ be the number of runs needed (in $\bar{k}_o\bar{k}_e\bar{k}_o\bar{k}_e\cdots$) to contain q . Miller shows that

$$m(q) + m(q + 1) \leq 2k + 1, \tag{4}$$

but in fact, the same steps can be used to show equality in (4). Hence, either $m(q)$ or $m(q + 1)$ is at most k , which implies either q or $q + 1$ is contained exactly in w .

Now, consider the finite word

$$w(\ell) = \begin{cases} \bar{k}_o\bar{k}_e\bar{k}_o\bar{k}_e\cdots\bar{k}_o\bar{k}_e & \text{if } k \text{ is even,} \\ \bar{k}_o\bar{k}_e\bar{k}_o\bar{k}_e\cdots\bar{k}_o\bar{k}_e\bar{k}_o & \text{if } k \text{ is odd,} \end{cases}$$

where the combined number of copies of \bar{k}_o and \bar{k}_e is exactly $k + \ell - 1$. Suppose without loss of generality that $m(q) \leq k$, and recall, $m(q + 1) = 2k + 1 - m(q)$. Since q is contained in the first $m(q)$ copies of $w(\ell)$, and each run of $w(\ell)$ is repeated every two times, there is another copy of q contained between the third run and the $(m(q)+2)$ -th run, yet another copy of q contained between the fifth and $(m(q)+4)$ -th runs, and so on for as long as we do not exceed $k + \ell - 1$ runs. Thus, there are at least

$$1 + \lfloor \frac{1}{2}((k + \ell - 1) - m(q)) \rfloor = \lfloor \frac{1}{2}(\ell + (k - m(q)) + 1) \rfloor$$

copies of q contained exactly in $w(\ell)$. Hence, if $k - m(q) \geq \ell - 1$, we successfully have at least ℓ copies of q . Then, let us suppose that $k - m(q) < \ell - 1$. For the same reason as above, there are at least

$$1 + \lfloor \frac{1}{2}((k + \ell - 1) - (2k + 1 - m(q))) \rfloor = \lfloor \frac{1}{2}(\ell - (k - m(q))) \rfloor \geq 1$$

copies of $q + 1$ contained exactly in $w(\ell)$. Hence, the combined number of copies of $m(q)$ and $m(q + 1)$ is at least

$$\lfloor \frac{1}{2}(\ell + (k - m(q)) + 1) \rfloor + \lfloor \frac{1}{2}(\ell - (k - m(q))) \rfloor = \ell.$$

Finding a permutation p representing $w(\ell)$ is routine. Note that p will contain at least ℓ copies of q . Let us consider the length of $w(\ell)$. First suppose k is odd. Then, there are $\frac{1}{2}(k + 1)$ entries each in \bar{k}_o and \bar{k}_e . Miller shows that w is of length $\frac{1}{2}k(k + 1)$, to which we add $\ell - 1$ more runs. Hence, the length of $w(\ell)$ is

$$\frac{1}{2}k(k + 1) + (\ell - 1)\frac{1}{2}(k + 1) = \frac{1}{2}(k + 1)(k + \ell - 1).$$

Now suppose that k is even. Then, there are $\frac{1}{2}k$ entries in \bar{k}_e and $1 + \frac{1}{2}k = \frac{1}{2}(k + 2)$ entries in \bar{k}_o . If ℓ is odd, then we have added $\frac{1}{2}(\ell - 1)$ copies each of \bar{k}_e and \bar{k}_o .

Thus, the length of $w(\ell)$ is

$$\frac{1}{2}k(k+1) + \frac{1}{2}(\ell-1)\frac{1}{2}k + \frac{1}{2}(\ell-1)\frac{1}{2}(k+2) = \frac{1}{2}(k+1)(k+\ell-1).$$

If ℓ is even, then we have added $\frac{1}{2}(\ell-2)$ copies of \bar{k}_e and $\frac{1}{2}\ell$ copies of \bar{k}_o . Therefore, the length of $w(\ell)$ is

$$\frac{1}{2}k(k+1) + \frac{1}{2}(\ell-2)\frac{1}{2}k + \frac{1}{2}\ell\frac{1}{2}(k+2) = \frac{1}{2}(k+1)(k+\ell-1) + 1. \quad \square$$

Remark. The above argument can be easily modified, by using the construction in Theorem 2.1, to provide less trivial upper bounds for $\text{sp}_\ell(k, m)$.

Remark. It is also interesting to note that, if one takes a superpattern from $S(k, m)$ achieving $\text{sp}(k, m)$ and removes colors, the resulting noncolored permutation is a superpattern that contains each k -pattern m^k times (since there are m^k different ways to color a k -pattern with m colors).

Acknowledgement

We greatly appreciate many helpful suggestions and corrections from the anonymous referee.

References

- [Arratia 1999] R. Arratia, “On the Stanley–Wilf conjecture for the number of permutations avoiding a given pattern”, *Electron. J. Combin.* **6** (1999), Note N1. MR 1710623 Zbl 0922.05002
- [Eriksson et al. 2002] H. Eriksson, K. Eriksson, S. Linusson, and J. Wästlund, “Dense packing of patterns in a permutation”, in *FPSAC '02: 14th Conference on Formal Power Series and Algebraic Combinatorics* (Melbourne 2002), 2002.
- [Gray 2015] D. Gray, “Bounds on superpatterns containing all layered permutations”, *Graphs Combin.* **31**:4 (2015), 941–952. MR 3357666 Zbl 1316.05001
- [Gray \geq 2016] D. Gray, “Simple permutations, alternating permutations, and the superpatterns that contain them”, preprint. To appear in *Ars Combin.*
- [Just and Wang 2016] M. Just and H. Wang, “Note on packing patterns in colored permutations”, *Online J. Anal. Comb.* **11** (2016). Zbl 06574956
- [Miller 2009] A. Miller, “Asymptotic bounds for permutations containing many different patterns”, *J. Combin. Theory Ser. A* **116**:1 (2009), 92–108. MR 2469250 Zbl 1177.05012
- [Savage and Wilf 2006] C. D. Savage and H. S. Wilf, “Pattern avoidance in compositions and multiset permutations”, *Adv. in Appl. Math.* **36**:2 (2006), 194–201. MR 2199988 Zbl 1087.05002

Received: 2015-05-01 Revised: 2015-09-10 Accepted: 2015-09-17

dgray1@ufl.edu

*Department of Mathematics, University of Florida,
Gainesville, FL 32611, United States*

hwang@georgiasouthern.edu

*Department of Mathematical Sciences, Georgia Southern
University, Statesboro, GA 30460, United States*

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA	Suzanne Lenhart	University of Tennessee, USA
John V. Baxley	Wake Forest University, NC, USA	Chi-Kwong Li	College of William and Mary, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Robert B. Lund	Clemson University, USA
Martin Bohner	Missouri U of Science and Technology, USA	Gaven J. Martin	Massey University, New Zealand
Nigel Boston	University of Wisconsin, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA	Emil Minchev	Ruse, Bulgaria
Pietro Cerone	La Trobe University, Australia	Frank Morgan	Williams College, USA
Scott Chapman	Sam Houston State University, USA	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Joshua N. Cooper	University of South Carolina, USA	Zuhair Nashed	University of Central Florida, USA
Jem N. Corcoran	University of Colorado, USA	Ken Ono	Emory University, USA
Toka Diagana	Howard University, USA	Timothy E. O'Brien	Loyola University Chicago, USA
Michael Dorff	Brigham Young University, USA	Joseph O'Rourke	Smith College, USA
Sever S. Dragomir	Victoria University, Australia	Yuval Peres	Microsoft Research, USA
Behrouz Emamizadeh	The Petroleum Institute, UAE	Y.-F. S. Pétermann	Université de Genève, Switzerland
Joel Foisy	SUNY Potsdam, USA	Robert J. Plemmons	Wake Forest University, USA
Errin W. Fulp	Wake Forest University, USA	Carl B. Pomerance	Dartmouth College, USA
Joseph Gallian	University of Minnesota Duluth, USA	Vadim Ponomarenko	San Diego State University, USA
Stephan R. Garcia	Pomona College, USA	Bjorn Poonen	UC Berkeley, USA
Anant Godbole	East Tennessee State University, USA	James Propp	U Mass Lowell, USA
Ron Gould	Emory University, USA	József H. Przytycki	George Washington University, USA
Andrew Granville	Université Montréal, Canada	Richard Rebarber	University of Nebraska, USA
Jerrold Griggs	University of South Carolina, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Jim Haglund	University of Pennsylvania, USA	James A. Sellers	Penn State University, USA
Johnny Henderson	Baylor University, USA	Andrew J. Sterge	Honorary Editor
Jim Hoste	Pitzer College, USA	Ann Trenk	Wellesley College, USA
Natalia Hritonenko	Prairie View A&M University, USA	Ravi Vakil	Stanford University, USA
Glenn H. Hurlbert	Arizona State University, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
Charles R. Johnson	College of William and Mary, USA	Ram U. Verma	University of Toledo, USA
K. B. Kulasekera	Clemson University, USA	John C. Wierman	Johns Hopkins University, USA
Gerry Ladas	University of Rhode Island, USA	Michael E. Zieve	University of Michigan, USA

PRODUCTION

Silvio Levy, Scientific Editor

Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2016 is US \$160/year for the electronic version, and \$215/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2016 Mathematical Sciences Publishers

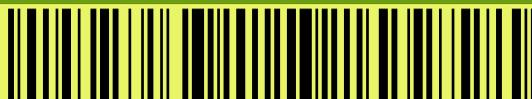
involve

2016

vol. 9

no. 5

An iterative strategy for Lights Out on Petersen graphs BRUCE TORRENCE AND ROBERT TORRENCE	721
A family of elliptic curves of rank ≥ 4 FARZALI IZADI AND KAMRAN NABARDI	733
Splitting techniques and Betti numbers of secant powers REZA AKHTAR, BRITTANY BURNS, HALEY DOHRMANN, HANNAH HOGANSON, OLA SOBIESKA AND ZEROTTI WOODS	737
Convergence of sequences of polygons ERIC HINTIKKA AND XINGPING SUN	751
On the Chermak–Delgado lattices of split metacyclic p -groups ERIN BRUSH, JILL DIETZ, KENDRA JOHNSON-TESCH AND BRIANNE POWER	765
The left greedy Lie algebra basis and star graphs BENJAMIN WALTER AND AMINREZA SHIRI	783
Note on superpatterns DANIEL GRAY AND HUA WANG	797
Lifting representations of finite reductive groups: a character relation JEFFREY D. ADLER, MICHAEL CASSEL, JOSHUA M. LANSKY, EMMA MORGAN AND YIFEI ZHAO	805
Spectrum of a composition operator with automorphic symbol ROBERT F. ALLEN, THONG M. LE AND MATTHEW A. PONS	813
On nonabelian representations of twist knots JAMES C. DEAN AND ANH T. TRAN	831
Envelope curves and equidistant sets MARK HUIBREGTSE AND ADAM WINCHELL	839
New examples of Brunnian theta graphs BYOUNGWOOK JANG, ANNA KRONAEUR, PRATAP LUITEL, DANIEL MEDICI, SCOTT A. TAYLOR AND ALEXANDER ZUPAN	857
Some nonsimple modules for centralizer algebras of the symmetric group CRAIG DODGE, HARALD ELLERS, YUKIHIDE NAKADA AND KELLY POHLAND	877
Acknowledgement	899



1944-4176(2016)9:5;1-0