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Jeffrey D. Adler, Michael Cassel,
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Lifting representations of finite reductive groups: a character relation

Jeffrey D. Adler, Michael Cassel,
Joshua M. Lansky, Emma Morgan and Yifei Zhao

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Given a connected reductive group $\widetilde{G}$ over a finite field k, and a semisimple k-automorphism ε of $\widetilde{G}$ of finite order, let G denote the connected part of the group of ε-fixed points. Two of the authors have previously shown that there exists a natural lifting from series of representations of $G(k)$ to series for $\widetilde{G}(k)$. In the case of Deligne–Lusztig representations, we show that this lifting satisfies a character relation analogous to that of Shintani.

0. Introduction

Suppose k is a finite field, $\widetilde{G}$ is a connected reductive k-group, and ε is a semisimple k-automorphism of $\widetilde{G}$ of finite order ℓ. Let G be the connected part of the group $\widetilde{G}^{\varepsilon}$ of ε-fixed points of $\widetilde{G}$. We will see (Proposition 1) that G is also a connected reductive k-group. Two of the authors have constructed a natural lifting [Adler and Lansky 2014] from series of irreducible representations of $G(k)$ to analogous series for $\widetilde{G}(k)$. In certain cases, this lifting is known to coincide with the Shintani lifting (see [Gyoja 1979, Theorem 7.2], [Digne 1987, Corollary 3.6], or [Silberger and Zink 2005, Proposition B4.4], for example), so it is natural to ask whether there is a relation, analogous to that of Shintani, between the character of a representation π of $G(k)$ and the ε-twisted character of its lift $\tilde{\pi}$. The purpose of this note is to prove an affirmative answer in the case where π is a Deligne–Lusztig representation, irreducible or not.

Let $\widetilde{G}^*$ and G^* denote the duals of $\widetilde{G}$ and G. For each semisimple element $z \in G^*(k)$, one obtains a collection $\mathcal{E}_z(G(k))$ of irreducible representations of $G(k)$, and these collections, known as *rational Lusztig series*, partition the set $\mathcal{E}(G(k))$ of (equivalence classes of) irreducible representations of $G(k)$ [Lusztig 1984, §14.1]. For example, suppose that z is regular in G^*, and let $T^* \subseteq G^*$ be the unique maximal

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k-torus containing z. Then the pair (T^*, z) corresponds to a pair (T, θ), where $T \subseteq G$ is a maximal k-torus, and θ is a complex-valued character of $T(k)$. This latter pair is uniquely determined up to $G(k)$-conjugacy. The Lusztig series $\mathcal{E}_z(G(k))$ corresponding to z is the set of irreducible components of the Deligne–Lusztig virtual representation whose character is $\mathrm{R}_T^G\,\theta$. An earlier work [Adler and Lansky 2014, Corollary 11.3] presents a natural map from semisimple conjugacy classes in $G^*(k)$ to semisimple classes in $\widetilde{G}^*(k)$, thus lifting each Lusztig series for $G(k)$ to one for $\widetilde{G}(k)$. The series of representations coming from $\pm\mathrm{R}_T^G\theta$ lifts to that coming from $\pm\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\tilde{\theta}$, where we will see that $\widetilde{T} := C_{\widetilde{G}}(T)$ is a maximal k-torus in $\widetilde{G}$, and $\tilde{\theta} = \theta \circ \mathcal{N}$, where $\mathcal{N} : \widetilde{T} \to T$ is the norm map defined by

$$\mathcal{N}(t) = t\varepsilon(t) \cdots \varepsilon^{\ell-1}(t).$$

(Of course, one could define a similar map $\mathcal{N}$ on any ε-invariant torus in $\widetilde{G}$.) In order to understand better this lifting of representations, one would like to have a relation between the character $\mathrm{R}_T^G\,\theta$ (for θ an arbitrary character of $T(k)$, not necessarily associated to a regular element of $G^*(k)$) and the ε-twisted character $(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\tilde{\theta})_\varepsilon$ associated to the ε-invariant character $\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\tilde{\theta}$. We prove that such a relation holds at sufficiently regular points, provided that one gathers together a "packet" of characters of $G(k)$, including $\mathrm{R}_T^G\,\theta$, that each lift to $\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\tilde{\theta}$.

Suppose we have a *Borel-torus pair* (B, T) for G; that is, we have a maximal k-torus $T \subseteq G$, together with a Borel subgroup $B \subseteq G$ containing T, not necessarily defined over k. From Proposition 1, we will obtain an ε-invariant Borel-torus pair $(\widetilde{B}, \widetilde{T})$ in $\widetilde{G}$, where $\widetilde{T} = C_{\widetilde{G}}(T)$ as above. Let $\langle\varepsilon\rangle$ be the group generated by ε. Recall that an element of a reductive group is *regular* if the connected part of its centralizer is a torus.

Theorem. *Suppose $\tilde{s} \in \widetilde{G}(k)$ belongs to an ε-invariant, maximal k-torus $\tilde{S}$ and $\mathcal{N}(\tilde{s})$ is regular in $\widetilde{G}$. Then*

$$(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\tilde{\theta})_\varepsilon(\tilde{s}) = \Bigg(\sum_{\tilde{g} \in G(k)\backslash(\widetilde{G}(k)/\widetilde{T}(k))^\varepsilon} \mathrm{R}_{{}^{\tilde{g}}T}^{G}\,{}^{\tilde{g}}\theta\Bigg)(\mathcal{N}(\tilde{s})). \tag{0-1}$$

Remark. Let us comment on some of the terms in (0-1).

(a) Here is what we mean by $(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\tilde{\theta})_\varepsilon$. Since $\tilde{\theta}$ is an ε-invariant character of $\widetilde{T}(k)$, we have that ε acts on the Deligne–Lusztig variety corresponding to $(\widetilde{B}, \widetilde{T}, \tilde{\theta})$, and thus on the virtual representation whose character is $\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\tilde{\theta}$. That is, even if this representation is reducible, we can form its ε-twisted character. To do so, extend $\tilde{\theta}$ to a character of $\widetilde{T}(k) \rtimes \langle\varepsilon\rangle$ by setting $\tilde{\theta}(\varepsilon) = 1$. Define the ε-twisted Deligne–Lusztig character $(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\tilde{\theta})_\varepsilon$ induced from $\tilde{\theta}$ by $(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\tilde{\theta})_\varepsilon(g) = (\mathrm{R}_{\widetilde{T}\rtimes\langle\varepsilon\rangle}^{\widetilde{G}\rtimes\langle\varepsilon\rangle}\,\tilde{\theta})(g\varepsilon)$ for $g \in \widetilde{G}(k)$. (See [Digne and Michel 1994] for the definition of Deligne–Lusztig induction for nonconnected groups.)

Note that if ε is quasicentral, i.e., the Weyl groups $W(G, T)$ and $W(\widetilde{G}, \widetilde{T})$ satisfy $W(G, T) = W(\widetilde{G}, \widetilde{T})^{\varepsilon}$ (see [Digne and Michel 1994, Définition-Théorème 1.15] for equivalent formulations), then $(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\, \tilde{\theta})_{\varepsilon}$ doesn't depend on the choice of $\widetilde{B}$, from the remark after [Digne and Michel 1994, Théorème 4.5]. More generally, there could be several Borel subgroups $\widetilde{B} \subseteq \widetilde{G}$ such that $(\widetilde{B}^{\varepsilon})^{\circ} = B$, and we don't know if the twisted character is independent of the choice of $\widetilde{B}$. However, our theorem will remain valid for any such choice.

(b) On the right-hand side of (0-1), $\tilde{g}$ runs over a set of double coset representatives. For each such $\tilde{g}$, we have that ${}^{\tilde{g}}T$ is a maximal k-torus in G and ${}^{\tilde{g}}\theta$ is a complex-valued character of ${}^{\tilde{g}}T(k)$. The choice of representative $\tilde{g}$ affects the pair $({}^{\tilde{g}}T, {}^{\tilde{g}}\theta)$ only up to $G(k)$-conjugacy, so it does not affect the character $\mathrm{R}_{{}^{\tilde{g}}T}^{G}\, {}^{\tilde{g}}\theta$ appearing in the corresponding summand.

(c) We note that $G(k)\backslash(\widetilde{G}(k)/\widetilde{T}(k))^{\varepsilon}$ can be viewed as the set of $G(k)$-conjugacy classes of ε-invariant Borel-torus pairs that are $\tilde{G}(k)$-conjugate to $(\tilde{B}, \widetilde{T})$. As $\tilde{g}$ runs over a set of double coset representatives, the k-tori ${}^{\tilde{g}}T \subset G$ are related in a way that is analogous to stable conjugacy, where $\widetilde{G}(k)$ plays the role usually played by $G(\bar{k})$, where $\bar{k}$ is an algebraic closure of k. Moreover, the set of characters appearing in the right-hand side of (0-1) is analogous to an L-packet, and their sum is "stable", in the sense that it is constant on the intersections with $G(k)$ of appropriate conjugacy classes in $\widetilde{G}(k)$.

Now let's consider some special cases.

(a) One can show that when ε is quasicentral, the index set for the summation is a singleton, so the theorem asserts that $(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\, \tilde{\theta})_{\varepsilon}(\tilde{s}) = (\mathrm{R}_{T}^{G}\, \theta)(\mathcal{N}(\tilde{s}))$.

(b) In particular, suppose $\widetilde{G} = R_{E/k}G$ is obtained from G via restriction of scalars over a cyclic extension E/k and ε is the algebraic k-automorphism of $\widetilde{G}$ associated to the action of a generator of the Galois group $\mathrm{Gal}(E/k)$. Given a representation π of $G(k)$, one often has an associated representation $\tilde{\pi}$ of $\widetilde{G}(k)$, known as the *Shintani lift* of π. (See [Kawanaka 1987] for a discussion.) The character of π and the ε-twisted character of $\tilde{\pi}$ satisfy the Shintani relation $\Theta_{\tilde{\pi},\varepsilon} = \Theta_{\pi} \circ \mathcal{N}$, where $\mathcal{N}$ has been extended to a map on all (not necessarily semisimple) conjugacy classes. From work of Digne [1999, Corollaire 3.6], one already knows that if $\mathrm{R}_{T}^{G}\, \theta$ has a Shintani lift, then it must be $\mathrm{R}_{\widetilde{T}}^{\tilde{G}}\, \tilde{\theta}$. Thus, if one restricts attention to Deligne–Lusztig characters and to the kind of elements that we consider, our character relation is a generalization of Shintani's.

(c) Consider the automorphism of GL(2) given by $\varepsilon(g) = {}^{t}g^{-1}$. Then a relation analogous to that in our theorem holds for all irreducible characters, not just those of Deligne–Lusztig type. Moreover, the relation can be extended to unipotent elements, but it fails if $\tilde{s}$ is regular but $\mathcal{N}(\tilde{s})$ is singular.

Comparing our theorem with [Digne and Michel 1994, Proposition 2.12], we see the latter shows that the restriction to $G(k)$ of a twisted character of $\widetilde{G}(k)$ is a certain twisted character of $G(k)$, at least in the case where ε is quasicentral. On the other hand, our theorem concerns a lifting of characters from $G(k)$ to $\widetilde{G}(k)$ via the norm map, and this lifting is not an inverse of restriction. As expected, the two formulas agree in those situations where both are applicable, but such situations are rare. Moreover, while we make regularity assumptions on our element $\tilde{s}$, we do not assume that ε is quasicentral.

1. Preliminary results

Given any algebraic group H, we denote by H° the connected component of the identity in H. If S is an algebraic subgroup of H, then $C_H(S)$ denotes the centralizer of S in H. Similarly, for an element $h \in H$, we let $C_H(h)$ denote its centralizer.

As in Section 0, let k denote a finite field, $\tilde{G}$ a connected reductive k-group, and ε a semisimple k-automorphism of $\tilde{G}$ of finite order ℓ, and let $G = (\widetilde{G}^\varepsilon)^\circ$.

Most of the following result appears in the statement or proof of [Steinberg 1968, Theorem 8.2]. Other versions appear in [Kottwitz and Shelstad 1999, Theorem 1.1.A; Digne and Michel 1994, Théorème 1.8]. A version that includes the rationality of G is in [Adler and Lansky 2014, Proposition 3.5].

Proposition 1. *With notation as above, one has:*

- *G is a connected reductive k-group.*
- *For every ε-invariant Borel-torus pair $(\widetilde{B}, \widetilde{T})$ for $\widetilde{G}$, one has a Borel-torus pair $((\widetilde{B}^\varepsilon)^\circ, (\widetilde{T}^\varepsilon)^\circ)$ for G. Moreover, $(\widetilde{T}^\varepsilon)^\circ = \widetilde{T} \cap G$.*
- *For every Borel-torus pair (B, T) for G, one has an ε-invariant Borel-torus pair $(\widetilde{B}, \widetilde{T})$, where $\widetilde{T} = C_{\widetilde{G}}(T)$ and such that $(\widetilde{B}^\varepsilon)^\circ = B$.*

From now on, we fix a maximal k-torus $T \subseteq G$, and thus obtain an ε-invariant maximal k-torus $\widetilde{T} = C_{\widetilde{G}}(T)$ as in Proposition 1, and a norm map $\mathcal{N} : \widetilde{T} \to T$ as in Section 0.

The following result concerns conjugacy and twisted conjugacy in G and $\tilde{G}$.

Lemma 2. *Suppose $\widetilde{S}$ is an ε-invariant maximal k-torus in $\widetilde{G}$. Let $S = \widetilde{S} \cap G$. Suppose that $\tilde{s} \in \widetilde{S}(k)$ and that $s := \mathcal{N}(\tilde{s}) \in S(k)$ is regular in $\widetilde{G}$. Let $\tilde{g} \in \widetilde{G}(k)$.*

(i) *There is an ε-invariant Borel subgroup of $\widetilde{G}$ containing $\widetilde{S}$.*

(ii) *If $\tilde{g}^{-1} s \tilde{g} \in T(k)$, then $\tilde{g}^{-1} \widetilde{S} \tilde{g} = \widetilde{T}$.*

(iii) *If $\tilde{g}^{-1} \tilde{s} \tilde{g} \in \widetilde{T}(k)$ and $\tilde{g}^{-1} \varepsilon(\tilde{g}) \in \widetilde{T}(k)$, then $\tilde{g}^{-1} S \tilde{g} = T$.*

(iv) *If $\tilde{g}^{-1} \tilde{s} \varepsilon(\tilde{g}) \in \widetilde{T}(k)$, then $\tilde{g}^{-1} \varepsilon(\tilde{g}) \in \widetilde{T}(k)$.*

Proof. To prove (i), let S' be a maximal k-torus in G such that $s \in S'(k)$, and let $\widetilde{S}' = C_{\widetilde{G}}(S')$. From Proposition 1, we have that $\widetilde{S}'$ is a maximal torus in $\widetilde{G}$, and it is contained in an ε-invariant Borel subgroup of $\widetilde{G}$. But $\widetilde{S}' = C_{\widetilde{G}}(S') \subseteq C_{\widetilde{G}}(s)^\circ = \widetilde{S}$, and therefore, $\widetilde{S}' = \widetilde{S}$.

To prove (ii), note that $\tilde{g}^{-1}\tilde{S}\tilde{g} = \tilde{g}^{-1}C_{\tilde{G}}(s)^\circ\tilde{g} = C_{\tilde{G}}(\tilde{g}^{-1}s\tilde{g})^\circ = \widetilde{T}$.

For (iii), it follows immediately from the assumptions that $\tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g}) \in \widetilde{T}(k)$. Then $\tilde{g}^{-1}s\tilde{g} = \mathcal{N}(\tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g})) \in \mathcal{N}(\widetilde{T}(k)) \subseteq T(k)$. From (ii), $\tilde{g}^{-1}\tilde{S}\tilde{g} = \widetilde{T}$. From (i) and Proposition 1, it is thus enough to show that $(\tilde{g}^{-1}\tilde{S}\tilde{g})^\varepsilon = \tilde{g}^{-1}\tilde{S}^\varepsilon\tilde{g}$. For $x \in \widetilde{S}(\bar{k})$,

$$\begin{aligned}\varepsilon(\tilde{g}^{-1}x\tilde{g}) &= \varepsilon(\tilde{g})^{-1}\varepsilon(x)\varepsilon(\tilde{g})\\ &= [\tilde{g}^{-1}\varepsilon(\tilde{g})]^{-1}\cdot\tilde{g}^{-1}\varepsilon(x)\tilde{g}\cdot[\tilde{g}^{-1}\varepsilon(\tilde{g})]\\ &= \tilde{g}^{-1}\varepsilon(x)\tilde{g},\end{aligned}$$

so $\varepsilon(x) = x$ if and only if $\varepsilon(\tilde{g}^{-1}x\tilde{g}) = \tilde{g}^{-1}x\tilde{g}$.

To prove (iv), note that as above, $\tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g}) \in \widetilde{T}(k)$ implies $\tilde{g}^{-1}s\tilde{g} \in T(k)$. By (ii), we have $\tilde{g}^{-1}\tilde{s}\tilde{g} \in \tilde{T}(k)$. Therefore,

$$\tilde{g}^{-1}\varepsilon(\tilde{g}) = (\tilde{g}^{-1}\tilde{s}\tilde{g})\ (\tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g}))^{-1} \in \widetilde{T}(k). \qquad \square$$

Now we consider a property of certain double coset spaces, whose proof is straightforward.

Lemma 3. *Let $\tilde{R}$ denote a set of representatives for the double coset space*

$$G(k)\backslash(\widetilde{G}(k)/\widetilde{T}(k))^\varepsilon.$$

Define the map $\phi : G(k) \times \tilde{R} \to (\widetilde{G}(k)/\widetilde{T}(k))^\varepsilon$ by $(g, \tilde{r}) \mapsto g\tilde{r}\widetilde{T}(k)$. Then

(i) *the map ϕ is surjective*;

(ii) *we have $\phi(g, \tilde{r}) = \phi(g', \tilde{r}')$ if and only if $\tilde{r} = \tilde{r}'$ and $g^{-1}g' \in {}^{\tilde{r}}T(k)$.*

Now we recall some facts about Deligne–Lusztig virtual characters.

Lemma 4. *One has the following*:

(i) *If $x \in G(k)$ is a regular element and θ is a complex character of $T(k)$, then*

$$(\mathrm{R}_T^G\,\theta)(x) = \sum_{\substack{g\in G(k)/T(k)\\ g^{-1}xg\in T(k)}} \theta(g^{-1}xg).$$

(ii) *Suppose that $\tilde{x} \in \tilde{G}(k)$ and that $\tilde{x}\varepsilon$ is a regular element of $\tilde{G}(k) \rtimes \langle\varepsilon\rangle$. Let $\tilde{\theta}$ be an ε-invariant character of $\widetilde{T}(k)$, extended trivially to $\widetilde{T}(k) \rtimes \langle\varepsilon\rangle$. Then*

$$(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\tilde{\theta})_\varepsilon(\tilde{x}) = \frac{1}{|\widetilde{T}(k)\rtimes\langle\varepsilon\rangle|} \sum_{\substack{\tilde{h}\in\widetilde{G}(k)\rtimes\langle\varepsilon\rangle\\ \tilde{x}\varepsilon\in\tilde{h}(\widetilde{T}(k)\rtimes\langle\varepsilon\rangle)\tilde{h}^{-1}}} \tilde{\theta}(\tilde{h}^{-1}(\tilde{x}\varepsilon)\tilde{h}).$$

Proof. For each formula, see [Digne and Michel 1994, Proposition 2.6]. For the second formula, note that $C_{\widetilde{G}}(\tilde{x}\varepsilon)$ contains no nontrivial unipotent elements. Thus, if we let S denote its connected part, then S is a torus, and the Green function $Q^{S(k)}_{S(k)}$ that arises in this proposition takes the value $|S(k)|$ at $(1, 1)$. □

2. Proof of the main theorem

From Lemma 2(i), we know $\widetilde{S}$ is contained in an ε-invariant Borel subgroup of $\widetilde{G}$, so it follows from Proposition 1 that $S := \widetilde{S} \cap G$ is a maximal k-torus in G, and $\widetilde{S} = C_{\widetilde{G}}(S)$.

Since $(\tilde{s}\varepsilon)^\ell = \mathcal{N}(\tilde{s})$, which is assumed regular in $\widetilde{G}$ and thus in $\widetilde{G} \rtimes \langle\varepsilon\rangle$, we must have that $\tilde{s}\varepsilon$ is also regular in $\widetilde{G} \rtimes \langle\varepsilon\rangle$. Lemma 4(ii) then implies that the left-hand side of (0-1) (hereafter denoted LHS) is equal to

$$\frac{1}{\ell|\widetilde{T}(k)|}\sum \tilde{\theta}(\tilde{h}^{-1}(\tilde{s}\varepsilon)\tilde{h}) = \frac{1}{\ell|\widetilde{T}(k)|}\sum \tilde{\theta}(\tilde{h}^{-1}\cdot\tilde{s}\cdot\varepsilon(\tilde{h})\cdot\varepsilon),$$

where each sum is over the set

$$\bigl\{\tilde{h} \in \widetilde{G}(k) \rtimes \langle\varepsilon\rangle \bigm| \tilde{s}\varepsilon \in \tilde{h}(\widetilde{T}(k) \rtimes \langle\varepsilon\rangle)\tilde{h}^{-1}\bigr\}.$$

If $\tilde{g} \in \widetilde{G}(k)$, then $\tilde{g}\varepsilon^i$ belongs to the index set if and only if $\tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g}) \in \widetilde{T}(k)$. The set of such elements $\tilde{g}$ is a union of left cosets of $\widetilde{T}(k)$. Thus,

$$\begin{aligned}
\text{LHS} &= \frac{1}{\ell|\widetilde{T}(k)|}\sum_{i=0}^{\ell-1}\sum_{\substack{\tilde{g}\in\widetilde{G}(k)\\ \tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g})\in\widetilde{T}(k)}} \tilde{\theta}\bigl((\tilde{g}\varepsilon^i)^{-1}\cdot\tilde{s}\cdot\varepsilon(\tilde{g}\varepsilon^i)\cdot\varepsilon\bigr)\\
&= \frac{1}{|\widetilde{T}(k)|}\sum_{\substack{\tilde{g}\in\widetilde{G}(k)\\ \tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g})\in\widetilde{T}(k)}} \tilde{\theta}(\tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g}))\\
&= \sum_{\substack{\tilde{g}\in\widetilde{G}(k)/\widetilde{T}(k)\\ \tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g})\in\widetilde{T}(k)}} \tilde{\theta}(\tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g}))\\
&= \sum_{\substack{\tilde{g}\in(\widetilde{G}(k)/\widetilde{T}(k))^\varepsilon\\ \tilde{g}^{-1}\tilde{s}\tilde{g}\in\widetilde{T}(k)}} \tilde{\theta}(\tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g})),
\end{aligned}$$

where the last equality follows from Lemma 2(iv). Let $s = \mathcal{N}(\tilde{s})$ and note that $\mathcal{N}(\tilde{g}^{-1}\tilde{s}\varepsilon(\tilde{g})) = \tilde{g}^{-1}s\tilde{g}$ for $\tilde{g} \in \widetilde{G}(k)$. Thus LHS is equal to

$$\sum_{\substack{\tilde{g}\in(\widetilde{G}(k)/\widetilde{T}(k))^\varepsilon\\ \tilde{g}^{-1}\tilde{s}\tilde{g}\in\widetilde{T}(k)}} {}^{\tilde{g}}\theta(s). \tag{2-1}$$

On the other hand, it follows from Lemma 4(i) that the right-hand side of (0-1) is equal to

$$\begin{aligned}\sum_{\tilde{x}\in G(k)\backslash(\widetilde{G}(k)/\widetilde{T}(k))^{\varepsilon}} (\mathrm{R}^{G}_{\tilde{x}T}\, {}^{\tilde{x}}\theta)(s) &= \sum_{\tilde{x}\in G(k)\backslash(\widetilde{G}(k)/\widetilde{T}(k))^{\varepsilon}} \sum_{\substack{g\in G(k)/{}^{\tilde{x}}T(k)\\ g^{-1}sg\in{}^{\tilde{x}}T(k)}} {}^{\tilde{x}}\theta(g^{-1}sg)\\ &= \sum_{\tilde{x}\in G(k)\backslash(\widetilde{G}(k)/\widetilde{T}(k))^{\varepsilon}} \sum_{\substack{g\in G(k)/{}^{\tilde{x}}T(k)\\ (g\tilde{x})^{-1}s(g\tilde{x})\in T(k)}} {}^{g\tilde{x}}\theta(s)\\ &= \sum_{\substack{\tilde{g}\in(\widetilde{G}(k)/\widetilde{T}(k))^{\varepsilon}\\ \tilde{g}^{-1}s\tilde{g}\in T(k)}} {}^{\tilde{g}}\theta(s),\end{aligned}$$

where the final equality follows from Lemma 3. The last sum above is equal to (2-1) by Lemma 2(ii) and (iii), which together imply that for $\tilde{g}\in(\widetilde{G}(k)/\widetilde{T}(k))^{\varepsilon}$, we have $\tilde{g}^{-1}s\tilde{g}\in T(k)$ if and only if $\tilde{g}^{-1}\tilde{s}\tilde{g}\in\widetilde{T}(k)$.

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jadler@american.edu *Department of Mathematics and Statistics, American University, Washington, DC 20016, United States*

m.cassel@columbia.edu *Columbia University, New York, NY 10027, United States*

lansky@american.edu *Department of Mathematics and Statistics, American University, Washington, DC 20016, United States*

emma.morgan@tufts.edu *Office of Institutional Research and Evaluation, Tufts University, Medford, MA 02155, United States*

yifei@math.harvard.edu *Department of Mathematics, Harvard University, Cambridge, MA 02138, United States*

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