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We study representations of the knot groups of twist knots into $SL_2(\mathbb{C})$. The set of nonabelian $SL_2(\mathbb{C})$ representations of a twist knot K is described as the zero set in $\mathbb{C} \times \mathbb{C}$ of a polynomial $P_K(x, y) = Q_K(y) + x^2 R_K(y) \in \mathbb{Z}[x, y]$, where x is the trace of a meridian. We prove some properties of $P_K(x, y)$. In particular, we prove that $P_K(2, y) \in \mathbb{Z}[y]$ is irreducible over \mathbb{Q} . As a consequence, we obtain an alternative proof of a result of Hoste and Shanahan that the degree of the trace field is precisely two less than the minimal crossing number of a twist knot.

1. Introduction

Let $J(k, l)$ be the two-bridge knot/link in [Figure 1](#), where $k, l \neq 0$ denote the numbers of half-twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left-handed) twists. Note that $J(k, l)$ is a knot if and only if kl is even. The knots $J(2, 2n)$, where $n \neq 0$, are known as twist knots. Moreover, $J(2, 2)$ is the trefoil knot and $J(2, -2)$ is the figure-eight knot. For more information about $J(k, l)$, see [\[Hoste and Shanahan 2004\]](#).

We study representations of the knot groups of twist knots into $SL_2(\mathbb{C})$, where $SL_2(\mathbb{C})$ denotes the set of all 2×2 matrices with determinant 1. From now on we fix a twist knot $J(2, 2n)$. By [\[Hoste and Shanahan 2001\]](#) the knot group of $J(2, 2n)$ has a presentation $\pi_1(J(2, 2n)) = \langle c, d \mid cu = ud \rangle$, where c, d are meridians and $u = (cd^{-1}c^{-1}d)^n$. This presentation is closely related to the standard presentation of the knot group of a two-bridge knot. Note that $J(2, 2n)$ is the twist knot K_{2n} in [\[Hoste and Shanahan 2001\]](#). In this note we will follow [\[Tran 2015b, Lemma 1.1\]](#) and use a different presentation,

$$\pi_1(J(2, 2n)) = \langle a, b \mid aw = wb \rangle,$$

where a, b are meridians and $w = (ab^{-1})^{-n}a(ab^{-1})^n$. This presentation has been shown to be useful for studying invariants of twist knots; see [\[Nagasato and Tran 2013; Tran 2013a; 2015a; 2015b\]](#).

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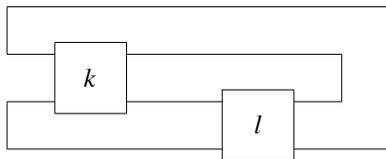


Figure 1. The two-bridge knot/link $J(k, l)$.

A representation $\rho : \pi_1(J(2, 2n)) \rightarrow \text{SL}_2(\mathbb{C})$ is called nonabelian if the image of ρ is a nonabelian subgroup of $\text{SL}_2(\mathbb{C})$. Suppose $\rho : \pi_1(J(2, 2n)) \rightarrow \text{SL}_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ 2 - y & s^{-1} \end{bmatrix},$$

where $s \neq 0$ and $y \neq 2$ satisfy a polynomial equation $P_n(s, y) = 0$. The polynomial P_n can be chosen so that $P_n(s, y) = P_n(s^{-1}, y)$, and hence it can be considered as a polynomial in the variables $x := s + s^{-1}$ and y . Note that $x = \text{tr } \rho(a) = \text{tr } \rho(b)$ and $y = \text{tr } \rho(ab^{-1})$. An explicit formula for $P_n(x, y)$ will be derived in Section 2 and it is given by

$$P_n(x, y) = 1 - (y + 2 - x^2)S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)),$$

where the $S_k(z)$ are the Chebychev polynomials of the second kind defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_k(z) = zS_{k-1}(z) - S_{k-2}(z)$ for all integers k . Note that $P_n(x, y)$ is different from the Riley polynomial [1984] of the two-bridge knot $J(2, 2n)$; see, e.g., [Nagasato and Tran 2013]. Moreover, $P_n(2, y)$ is also different from the polynomial $\Phi_{-n}(y)$ studied in [Hoste and Shanahan 2001].

In this note we prove the following two properties of $P_n(x, y)$.

Theorem 1. *Suppose $x_0^2 \in \mathbb{R}$ such that $4 - 1/|n| < x_0^2 \leq 4$. Then the polynomial $P_n(x_0, y)$ has no real roots y if $n < 0$, and has exactly one real root y if $n > 0$.*

Theorem 2. *The polynomial $P_n(2, y) \in \mathbb{Z}[y]$ is irreducible over \mathbb{Q} .*

A nonabelian representation $\rho : \pi_1(J(2, 2n)) \rightarrow \text{SL}_2(\mathbb{C})$ is called parabolic if the trace of a meridian is equal to 2. The zero set in \mathbb{C} of the polynomial $P_n(2, y)$ describes the set of all parabolic representations of the knot group of $J(2, 2n)$ into $\text{SL}_2(\mathbb{C})$. Theorem 1 is related to the problem of determining the existence of real parabolic representations in the study of the left-orderability of the fundamental groups of cyclic branched covers of two-bridge knots; see [Hu 2015; Tran 2015a].

As in the proof of [Hoste and Shanahan 2001, Theorem 1], Theorem 2 gives an alternative proof of a result of Hoste and Shanahan that the degree of the trace field is precisely two less than the minimal crossing number of a twist knot. Indeed, by definition the trace field of a hyperbolic knot K is the extension field

$\mathbb{Q}(\text{tr } \rho_0(g) : g \in \pi_1(K))$, where $\rho_0 : \pi_1(K) \rightarrow \text{SL}_2(\mathbb{C})$ is a discrete faithful representation. The representation ρ_0 is a parabolic representation. Since $P_n(2, y)$ is irreducible over \mathbb{Q} , the trace field of the twist knot $J(2, 2n)$ is $\mathbb{Q}(y_0)$, where y_0 is a certain complex root of $P_n(2, y)$ corresponding to the presentation ρ_0 . Consequently, the degree of $P_n(2, y)$ gives the degree of the trace field. The conclusion follows, since the minimal crossing number of $J(2, 2n)$ is $2n + 1$ if $n > 0$ and is $2 - 2n$ if $n < 0$.

The rest of this note is devoted to the proofs of Theorems 1 and 2.

2. Proofs of Theorems 1 and 2

In this section we first recall some properties of the Chebychev polynomials $S_k(z)$. We then compute the polynomial $P_n(x, y)$. Finally, we prove Theorems 1 and 2.

Chebychev polynomials. Recall that the $S_k(z)$ are the Chebychev polynomials defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_k(z) = zS_{k-1}(z) - S_{k-2}(z)$ for all integers k . Note that $S_k(2) = k + 1$ and $S_k(-2) = (-1)^k(k + 1)$. Moreover, if $z = t + t^{-1}$, where $t \neq \pm 1$, then

$$S_k(z) = \frac{t^{k+1} - t^{-(k+1)}}{t - t^{-1}}.$$

It is easy to see that $S_{-k}(z) = -S_{k-2}(z)$ for all integers k .

The following lemma is elementary; see, e.g., [Tran 2013b, Lemma 1.4].

Lemma 2.1. *One has*

$$S_k^2(z) - zS_k(z)S_{k-1}(z) + S_{k-1}^2(z) = 1$$

for all integers k .

Lemma 2.2. *For all $k \geq 1$ one has*

$$S_k(z) = \prod_{j=1}^k \left(z - 2 \cos \frac{j\pi}{k+1} \right),$$

$$S_k(z) - S_{k-1}(z) = \prod_{j=1}^k \left(z - 2 \cos \frac{(2j-1)\pi}{2k+1} \right).$$

Proof. We prove the second formula. The first one can be proved similarly.

Since $S_k(z) - S_{k-1}(z)$ is a polynomial of degree k , it suffices to show that its roots are

$$2 \cos \frac{(2j-1)\pi}{2k+1},$$

where $1 \leq j \leq k$. Let

$$\theta_j = \frac{(2j-1)\pi}{2k+1}.$$

Then $e^{i(2k+1)\theta_j} = -1$. Hence, if $z = 2 \cos \theta_j = e^{i\theta_j} + e^{-i\theta_j}$ then we have

$$S_k(z) = \frac{e^{i(k+1)\theta_j} - e^{-i(k+1)\theta_j}}{e^{i\theta_j} - e^{-i\theta_j}} = \frac{-e^{-ik\theta_j} + e^{ik\theta_j}}{e^{i\theta_j} - e^{-i\theta_j}} = S_{k-1}(z).$$

This means that $z = 2 \cos \theta_j$ is a root of $S_k(z) - S_{k-1}(z)$. □

Lemma 2.3. *Suppose $z \in \mathbb{R}$ such that $-2 \leq z \leq 2$. Then*

$$|S_{k-1}(z)| \leq |k|$$

for all integers k .

Proof. See [Tran 2015a, Lemma 2.6]. □

Lemma 2.4. *Suppose $M \in \text{SL}_2(\mathbb{C})$. Then*

$$M^k = S_{k-1}(z)M - S_{k-2}(z)I$$

for all integers k , where I is the 2×2 identity matrix and $z := \text{tr } M$.

Proof. Since $\det M = 1$, by the Cayley–Hamilton theorem we have $M^2 - zM + I = 0$. This implies that $M^k - zM^{k-1} + M^{k-2} = 0$ for all integers k . Then, by induction on k we have $M^k = S_{k-1}(z)M - S_{k-2}(z)I$ for all $k \geq 0$.

For $k < 0$, since $\text{tr } M^{-1} = \text{tr } M = z$ we have

$$\begin{aligned} M^k &= (M^{-1})^{-k} = S_{-k-1}(z)M^{-1} - S_{-k-2}(z)I \\ &= -S_{k-1}(z)(zI - M) + S_k(z)I. \end{aligned}$$

The lemma follows, since $zS_{k-1}(z) - S_k(z) = S_{k-2}(z)$. □

The polynomial P_n . Recall that the knot group of $J(2, 2n)$ has the presentation

$$\pi_1(J(2, 2n)) = \langle a, b \mid aw = wb \rangle,$$

where a, b are meridians and $w = (ab^{-1})^{-n}a(ab^{-1})^n$. See [Tran 2015b, Lemma 1.1].

Suppose $\rho : \pi_1(J(2, 2n)) \rightarrow \text{SL}_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ 2 - y & s^{-1} \end{bmatrix},$$

where $s \neq 0$ and $y \neq 2$ satisfy a polynomial equation $P_n(s, y) = 0$. We now compute the polynomial P_n from the matrix equation $\rho(aw) = \rho(wb)$.

Since

$$\rho(ab^{-1}) = \begin{bmatrix} y - 1 & s \\ s^{-1}(y - 2) & 1 \end{bmatrix},$$

by Lemma 2.4 we have

$$\begin{aligned} \rho((ab^{-1})^n) &= S_{n-1}(y)\rho(ab^{-1}) - S_{n-2}(y)I \\ &= \begin{bmatrix} (y-1)S_{n-1}(y) - S_{n-2}(y) & sS_{n-1}(y) \\ s^{-1}(y-2)S_{n-1}(y) & S_{n-1}(y) - S_{n-2}(y) \end{bmatrix}. \end{aligned}$$

Hence, by a direct (but lengthy) calculation we have

$$\begin{aligned} \rho(aw) - \rho(wb) &= \rho(a(ab^{-1})^{-n}a(ab^{-1})^n) - \rho((ab^{-1})^{-n}a(ab^{-1})^nb) \\ &= \begin{bmatrix} (y-2)P_n(s, y) & sP_n(s, y) \\ -s^{-1}(y-2)P_n(s, y) & 0 \end{bmatrix}, \end{aligned}$$

where

$$P_n(s, y) = (s^2 + s^{-2} + 1 - y)S_{n-1}^2(y) - (s^2 + s^{-2})S_{n-1}(y)S_{n-2}(y) + S_{n-2}^2(y).$$

By Lemma 2.1 we have $S_{n-1}^2(y) - yS_{n-1}(y)S_{n-2}(y) + S_{n-2}^2(y) = 1$. Hence

$$P_n(s, y) = 1 - (y - s^2 - s^{-2})S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)).$$

Since $P_n(s, y) = P_n(s^{-1}, y)$, from now on we consider P_n as a polynomial in the variables $x = s + s^{-1}$ and y . With these new variables we have

$$P_n(x, y) = 1 - (y + 2 - x^2)S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)).$$

Proof of Theorem 1. We first prove the following lemma.

Lemma 2.5. *Suppose $x_0^2 \in \mathbb{R}$ such that $4 - 1/|n| < x_0^2 \leq 4$. If $y \in \mathbb{R}$ satisfies $P_n(x_0, y) = 0$, then $y > 2$.*

Proof. Since $P_n(x_0, y) = 0$, we have $S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)) = (y + 2 - x_0^2)^{-1}$. Hence

$$\begin{aligned} ((y + 2 - x_0^2)S_{n-1}(y))^{-2} &= (S_{n-1}(y) - S_{n-2}(y))^2 \\ &= 1 + (y - 2)S_{n-1}(y)S_{n-2}(y) \\ &= 1 + (y - 2)(S_{n-1}^2(y) - (y + 2 - x_0^2)^{-1}), \end{aligned}$$

which implies that

$$1 = (y + 2 - x_0^2)(4 - x_0^2)S_{n-1}^2(y) + (y - 2)(y + 2 - x_0^2)^2S_{n-1}^4(y).$$

Assume $y \leq 2$. Then it follows from the above equation that

$$1 \leq (y + 2 - x_0^2)(4 - x_0^2)S_{n-1}^2(y). \tag{2-1}$$

In particular, $y > x_0^2 - 2 > -2$. Since $-2 < y \leq 2$, by Lemma 2.3 we have $S_{n-1}^2(y) \leq n^2$. Hence

$$(y + 2 - x_0^2)(4 - x_0^2)S_{n-1}^2(y) \leq (4 - x_0^2)^2n^2 < 1.$$

This contradicts (2-1). □

We now complete the proof of [Theorem 1](#). Suppose $x_0^2 \in \mathbb{R}$ and $4 - 1/|n| < x_0^2 \leq 4$. By [Lemma 2.5](#), it suffices to consider $P_n(x_0, y)$, where y is a real number greater than 2. The equation $P(x_0, y) = 0$ is equivalent to

$$x_0^2 - 4 = y - 2 - \frac{1}{S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y))}. \tag{2-2}$$

Denote by $f_n(y)$ the right-hand side of (2-2), where $y > 2$. We now use the factorizations of $S_{n-1}(y)$ and $S_{n-1}(y) - S_{n-2}(y)$ in [Lemma 2.2](#).

If $n = -1$ then

$$f_n(y) = y - 2 + \frac{1}{y-1} > 0 \geq x_0^2 - 4.$$

Hence $f_n(y) = x_0^2 - 4$ has no solutions on $(2, \infty)$.

If $n < -1$ then, by letting $m = -n > 1$, we have

$$\begin{aligned} f_n(y) &= y - 2 + \frac{1}{S_{m-1}(y)(S_m(y) - S_{m-1}(y))} \\ &= y - 2 + \frac{1}{\prod_{k=1}^{m-1} (y - 2 \cos \frac{k\pi}{m}) \prod_{l=1}^m (y - 2 \cos \frac{(2l-1)\pi}{2m+1})} > 0 \geq x_0^2 - 4. \end{aligned}$$

Hence $f_n(y) = x_0^2 - 4$ has no solutions on $(2, \infty)$.

If $n = 1$ then $f_n(y) = y - 3$. Since $x_0^2 > 3$, the equation $f_n(y) = x_0^2 - 4$ has a unique solution $y = x_0^2 - 1$ on $(2, \infty)$.

If $n > 1$ then we have

$$f_n(y) = y - 2 - \frac{1}{\prod_{k=1}^{n-1} (y - 2 \cos \frac{k\pi}{n}) \prod_{l=1}^{n-1} (y - 2 \cos \frac{(2l-1)\pi}{2n-1})}.$$

It is easy to see that $f_n(y)$ is increasing on $(2, \infty)$. Moreover, $\lim_{y \rightarrow \infty} f_n(y) = \infty$ and $\lim_{y \rightarrow 2} f_n(y) = -1/n < x_0^2 - 4$. Hence $f_n(y) = x_0^2 - 4$ has a unique solution on $(2, \infty)$.

The proof of [Theorem 1](#) is complete.

Proof of Theorem 2. We write $P_n(y)$ for $P_n(2, y)$. Let $y = t^2 + t^{-2}$. Then

$$\begin{aligned} P_n(y) &= (S_{n-1}(y) - S_{n-2}(y))^2 - (y - 2)S_{n-1}^2(y) \\ &= \frac{(t^{2n} + t^{2-2n})^2 - t^2(t^{2n} - t^{-2n})^2}{(t^2 + 1)^2} \\ &= \frac{(t^{2n} + t^{2-2n} + t^{2n+1} - t^{1-2n})(t^{2n} + t^{2-2n} - t^{2n+1} + t^{1-2n})}{(t^2 + 1)^2}. \end{aligned}$$

Up to a factor t^k , each of $t^{2n} + t^{2-2n} + t^{2n+1} - t^{1-2n}$ and $t^{2n} + t^{2-2n} - t^{2n+1} + t^{1-2n}$ is obtained from the other by replacing t by t^{-1} . To show that $P_n(y)$ is irreducible

over \mathbb{Q} , it suffices to show that

$$t^{4n} + t^{4n-1} + t - 1 = (t^2 + 1)Q_n(t), \quad (2-3)$$

where $Q_n(t) \in \mathbb{Z}[t]$ is irreducible over \mathbb{Q} .

As in the proof of [Baker and Petersen 2013, Lemma 6.8], we will use the following theorem of Ljunggren [1960]. Consider a polynomial of the form $R(t) = t^{k_1} + \varepsilon_1 t^{k_2} + \varepsilon_2 t^{k_3} + \varepsilon_3$, where $\varepsilon_j = \pm 1$ for $j = 1, 2, 3$. Then, if R has $r > 0$ roots of unity as roots then R can be decomposed into two factors, one of degree r which has these roots of unity as zeros and the other which is irreducible over \mathbb{Q} . Hence, to prove (2-3) it suffices to show that $\pm i$ are the only roots of unity which are roots of $t^{4n} + t^{4n-1} + t - 1$ and these occur with multiplicity 1.

Let t be a root of unity such that $t^{4n} + t^{4n-1} + t - 1 = 0$. Write $t = e^{i\theta}$, where $\theta \in \mathbb{R}$. Since $t^{2n-1} + t^{1-2n} + t^{2n} - t^{-2n} = 0$, we have

$$2 \cos(2n - 1)\theta + 2i \sin 2n\theta = 0,$$

which implies that both $\cos(2n - 1)\theta$ and $\sin 2n\theta$ are equal to zero. There exist integers k, l such that $(2n - 1)\theta = (k + \frac{1}{2})\pi$ and $2n\theta = l\pi$. This implies that $(2k + 1)/l = (2n - 1)/n$. Since $(2n - 1)/n$ is a reduced fraction, there exists an odd integer m such that $2k + 1 = m(2n - 1)$ and $l = mn$. Hence $\theta = \frac{1}{2}m\pi$, which implies that $t = e^{i\theta} = \pm i$. It is easy to verify that $\pm i$ are roots of $t^{4n} + t^{4n-1} + t - 1 = 0$ with multiplicity 1.

Ljunggren's theorem then completes the proof of Theorem 2.

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References

- [Baker and Petersen 2013] K. L. Baker and K. L. Petersen, “Character varieties of once-punctured torus bundles with tunnel number one”, *Internat. J. Math.* **24**:6 (2013), Article ID #1350048. MR 3078072 Zbl 1275.57016
- [Hoste and Shanahan 2001] J. Hoste and P. D. Shanahan, “Trace fields of twist knots”, *J. Knot Theory Ramifications* **10**:4 (2001), 625–639. MR 1831680 Zbl 1003.57014
- [Hoste and Shanahan 2004] J. Hoste and P. D. Shanahan, “A formula for the A-polynomial of twist knots”, *J. Knot Theory Ramifications* **13**:2 (2004), 193–209. MR 2047468 Zbl 1057.57010
- [Hu 2015] Y. Hu, “Left-orderability and cyclic branched coverings”, *Algebr. Geom. Topol.* **15**:1 (2015), 399–413. MR 3325741 Zbl 06425407

- [Ljunggren 1960] W. Ljunggren, “On the irreducibility of certain trinomials and quadrimomials”, *Math. Scand.* **8** (1960), 65–70. [MR 0124313](#) [Zbl 0095.01305](#)
- [Nagasato and Tran 2013] F. Nagasato and A. T. Tran, “Some families of minimal elements for a partial ordering on prime knots”, preprint, 2013. To appear in *Osaka J. Math.* [arXiv 1301.0138](#)
- [Riley 1984] R. Riley, “Nonabelian representations of 2-bridge knot groups”, *Quart. J. Math. Oxford Ser. (2)* **35**:138 (1984), 191–208. [MR 745421](#) [Zbl 0549.57005](#)
- [Tran 2013a] A. T. Tran, “On the twisted Alexander polynomial for representations into $SL_2(\mathbb{C})$ ”, *J. Knot Theory Ramifications* **22**:10 (2013), Article ID #1350059. [MR 3125898](#) [Zbl 1278.57023](#)
- [Tran 2013b] A. T. Tran, “The universal character ring of some families of one-relator groups”, *Algebr. Geom. Topol.* **13**:4 (2013), 2317–2333. [MR 3073918](#) [Zbl 1280.57014](#)
- [Tran 2015a] A. T. Tran, “On left-orderability and cyclic branched coverings”, *J. Math. Soc. Japan* **67**:3 (2015), 1169–1178. [MR 3376583](#) [Zbl 1328.57017](#)
- [Tran 2015b] A. T. Tran, “On left-orderable fundamental groups and Dehn surgeries on knots”, *J. Math. Soc. Japan* **67**:1 (2015), 319–338. [MR 3304024](#) [Zbl 06422613](#)

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