On nonabelian representations of twist knots

James C. Dean and Anh T. Tran
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We study representations of the knot groups of twist knots into $\text{SL}_2(\mathbb{C})$. The set of nonabelian $\text{SL}_2(\mathbb{C})$ representations of a twist knot $K$ is described as the zero set in $\mathbb{C} \times \mathbb{C}$ of a polynomial $P_K(x, y) = Q_K(y) + x^2 R_K(y) \in \mathbb{Z}[x, y]$, where $x$ is the trace of a meridian. We prove some properties of $P_K(x, y)$. In particular, we prove that $P_K(2, y) \in \mathbb{Z}[y]$ is irreducible over $\mathbb{Q}$. As a consequence, we obtain an alternative proof of a result of Hoste and Shanahan that the degree of the trace field is precisely two less than the minimal crossing number of a twist knot.

1. Introduction

Let $J(k, l)$ be the two-bridge knot/link in Figure 1, where $k, l \neq 0$ denote the numbers of half-twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left-handed) twists. Note that $J(k, l)$ is a knot if and only if $kl$ is even. The knots $J(2, 2n)$, where $n \neq 0$, are known as twist knots. Moreover, $J(2, 2)$ is the trefoil knot and $J(2, -2)$ is the figure-eight knot. For more information about $J(k, l)$, see [Hoste and Shanahan 2004].

We study representations of the knot groups of twist knots into $\text{SL}_2(\mathbb{C})$, where $\text{SL}_2(\mathbb{C})$ denotes the set of all $2 \times 2$ matrices with determinant 1. From now on we fix a twist knot $J(2, 2n)$. By [Hoste and Shanahan 2001] the knot group of $J(2, 2n)$ has a presentation $\pi_1(J(2, 2n)) = \langle c, d | cu = ud \rangle$, where $c, d$ are meridians and $u = (cd^{-1}c^{-1}d)^n$. This presentation is closely related to the standard presentation of the knot group of a two-bridge knot. Note that $J(2, 2n)$ is the twist knot $K_{2n}$ in [Hoste and Shanahan 2001]. In this note we will follow [Tran 2015b, Lemma 1.1] and use a different presentation,

$$\pi_1(J(2, 2n)) = \langle a, b | aw = wb \rangle,$$

where $a, b$ are meridians and $w = (ab^{-1})^{-n}a(ab^{-1})^n$. This presentation has been shown to be useful for studying invariants of twist knots; see [Nagasato and Tran 2013; Tran 2013a; 2015a; 2015b].

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Figure 1. The two-bridge knot/link $J(k, l)$.

A representation $\rho : \pi_1(J(2, 2n)) \to \text{SL}_2(\mathbb{C})$ is called nonabelian if the image of $\rho$ is a nonabelian subgroup of $\text{SL}_2(\mathbb{C})$. Suppose $\rho : \pi_1(J(2, 2n)) \to \text{SL}_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ 2 - y & s^{-1} \end{bmatrix},$$

where $s \neq 0$ and $y \neq 2$ satisfy a polynomial equation $P_n(s, y) = 0$. The polynomial $P_n$ can be chosen so that $P_n(s, y) = P_n(s^{-1}, y)$, and hence it can be considered as a polynomial in the variables $x := s + s^{-1}$ and $y$. Note that $x = \text{tr} \, \rho(a) = \text{tr} \, \rho(b)$ and $y = \text{tr} \, \rho(ab^{-1})$. An explicit formula for $P_n(x, y)$ will be derived in Section 2 and it is given by

$$P_n(x, y) = 1 - (y + 2 - x^2)S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)),$$

where the $S_k(z)$ are the Chebychev polynomials of the second kind defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_k(z) = zS_{k-1}(z) - S_{k-2}(z)$ for all integers $k$. Note that $P_n(x, y)$ is different from the Riley polynomial [1984] of the two-bridge knot $J(2, 2n)$; see, e.g., [Nagasato and Tran 2013]. Moreover, $P_n(2, y)$ is also different from the polynomial $\Phi_{-n}(y)$ studied in [Hoste and Shanahan 2001].

In this note we prove the following two properties of $P_n(x, y)$.

**Theorem 1.** Suppose $x_0^2 \in \mathbb{R}$ such that $4 - 1/|n| < x_0^2 \leq 4$. Then the polynomial $P_n(x_0, y)$ has no real roots $y$ if $n < 0$, and has exactly one real root $y$ if $n > 0$.

**Theorem 2.** The polynomial $P_n(2, y) \in \mathbb{Z}[y]$ is irreducible over $\mathbb{Q}$.

A nonabelian representation $\rho : \pi_1(J(2, 2n)) \to \text{SL}_2(\mathbb{C})$ is called parabolic if the trace of a meridian is equal to 2. The zero set in $\mathbb{C}$ of the polynomial $P_n(2, y)$ describes the set of all parabolic representations of the knot group of $J(2, 2n)$ into $\text{SL}_2(\mathbb{C})$. **Theorem 1** is related to the problem of determining the existence of real parabolic representations in the study of the left-orderability of the fundamental groups of cyclic branched covers of two-bridge knots; see [Hu 2015; Tran 2015a].

As in the proof of [Hoste and Shanahan 2001, Theorem 1], **Theorem 2** gives an alternative proof of a result of Hoste and Shanahan that the degree of the trace field is precisely two less than the minimal crossing number of a twist knot. Indeed, by definition the trace field of a hyperbolic knot $K$ is the extension field
\( \mathbb{Q}(\text{tr} \rho_0(g) : g \in \pi_1(K)) \), where \( \rho_0 : \pi_1(K) \to \text{SL}_2(\mathbb{C}) \) is a discrete faithful representation. The representation \( \rho_0 \) is a parabolic representation. Since \( P_n(2, y) \) is irreducible over \( \mathbb{Q} \), the trace field of the twist knot \( J(2, 2n) \) is \( \mathbb{Q}(y_0) \), where \( y_0 \) is a certain complex root of \( P_n(2, y) \) corresponding to the presentation \( \rho_0 \). Consequently, the degree of \( P_n(2, y) \) gives the degree of the trace field. The conclusion follows, since the minimal crossing number of \( J(2, 2n) \) is \( 2n+1 \) if \( n > 0 \) and is \( 2-2n \) if \( n < 0 \).

The rest of this note is devoted to the proofs of Theorems 1 and 2.

2. Proofs of Theorems 1 and 2

In this section we first recall some properties of the Chebychev polynomials \( S_k(z) \). We then compute the polynomial \( P_n(x, y) \). Finally, we prove Theorems 1 and 2.

**Chebychev polynomials.** Recall that the \( S_k(z) \) are the Chebychev polynomials defined by \( S_0(z) = 1 \), \( S_1(z) = z \) and \( S_k(z) = zS_{k-1}(z) - S_{k-2}(z) \) for all integers \( k \). Note that \( S_k(2) = k+1 \) and \( S_k(-2) = (-1)^k(k+1) \). Moreover, if \( z = t + t^{-1} \), where \( t \neq \pm 1 \), then

\[
S_k(z) = \frac{t^{k+1} - t^{-(k+1)}}{t - t^{-1}}.
\]

It is easy to see that \( S_{-k}(z) = -S_{k-2}(z) \) for all integers \( k \).

The following lemma is elementary; see, e.g., [Tran 2013b, Lemma 1.4].

**Lemma 2.1.** One has

\[
S_k^2(z) - zS_k(z)S_{k-1}(z) + S_{k-1}^2(z) = 1
\]

for all integers \( k \).

**Lemma 2.2.** For all \( k \geq 1 \) one has

\[
S_k(z) = \prod_{j=1}^{k} \left( z - 2 \cos \frac{j\pi}{k+1} \right),
\]

\[
S_k(z) - S_{k-1}(z) = \prod_{j=1}^{k} \left( z - 2 \cos \frac{(2j-1)\pi}{2k+1} \right).
\]

**Proof.** We prove the second formula. The first one can be proved similarly.

Since \( S_k(z) - S_{k-1}(z) \) is a polynomial of degree \( k \), it suffices to show that its roots are

\[
2 \cos \frac{(2j-1)\pi}{2k+1},
\]

where \( 1 \leq j \leq k \). Let

\[
\theta_j = \frac{(2j-1)\pi}{2k+1}.
\]
Then $e^{i(2k+1)\theta_j} = -1$. Hence, if $z = 2\cos \theta_j = e^{i\theta_j} + e^{-i\theta_j}$ then we have

$$S_k(z) = \frac{e^{i(k+1)\theta_j} - e^{-i(k+1)\theta_j}}{e^{i\theta_j} - e^{-i\theta_j}} = \frac{-e^{-ik\theta_j} + e^{ik\theta_j}}{e^{i\theta_j} - e^{-i\theta_j}} = S_{k-1}(z).$$

This means that $z = 2\cos \theta_j$ is a root of $S_k(z) - S_{k-1}(z)$. □

**Lemma 2.3.** Suppose $z \in \mathbb{R}$ such that $-2 \leq z \leq 2$. Then

$$|S_{k-1}(z)| \leq |k|$$

for all integers $k$.

**Proof.** See [Tran 2015a, Lemma 2.6]. □

**Lemma 2.4.** Suppose $M \in \text{SL}_2(\mathbb{C})$. Then

$$M^k = S_{k-1}(z)M - S_{k-2}(z)I$$

for all integers $k$, where $I$ is the $2 \times 2$ identity matrix and $z := \text{tr} M$.

**Proof.** Since $\det M = 1$, by the Cayley–Hamilton theorem we have $M^2 - zM + I = 0$. This implies that $M^k - zM^{k-1} + M^{k-2} = 0$ for all integers $k$. Then, by induction on $k$ we have $M^k = S_{k-1}(z)M - S_{k-2}(z)I$ for all $k \geq 0$.

For $k < 0$, since $\text{tr} M^{-1} = \text{tr} M = z$ we have

$$M^k = (M^{-1})^{-k} = S_{-k-1}(z)M^{-1} - S_{-k-2}(z)I$$

$$= -S_{k-1}(z)(zI - M) + S_k(z)I.$$

The lemma follows, since $zS_{k-1}(z) - S_k(z) = S_{k-2}(z)$. □

**The polynomial $P_n$.** Recall that the knot group of $J(2, 2n)$ has the presentation

$$\pi_1(J(2, 2n)) = \langle a, b \mid aw = wb \rangle,$$

where $a$, $b$ are meridians and $w = (ab^{-1})^{-n}a(ab^{-1})^n$. See [Tran 2015b, Lemma 1.1].

Suppose $\rho : \pi_1(J(2, 2n)) \to \text{SL}_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ 2 - y & s^{-1} \end{bmatrix},$$

where $s \neq 0$ and $y \neq 2$ satisfy a polynomial equation $P_n(s, y) = 0$. We now compute the polynomial $P_n$ from the matrix equation $\rho(aw) = \rho(wb)$.

Since

$$\rho(ab^{-1}) = \begin{bmatrix} y - 1 \\ s^{-1}(y - 2) & 1 \end{bmatrix},$$

we have

$$S_k(z) = \frac{e^{i(k+1)\theta_j} - e^{-i(k+1)\theta_j}}{e^{i\theta_j} - e^{-i\theta_j}} = \frac{-e^{-ik\theta_j} + e^{ik\theta_j}}{e^{i\theta_j} - e^{-i\theta_j}} = S_{k-1}(z).$$

This means that $z = 2\cos \theta_j$ is a root of $S_k(z) - S_{k-1}(z)$. □
by Lemma 2.4 we have
\[
\rho((ab^{-1})^n) = S_{n-1}(y)\rho(ab^{-1}) - S_{n-2}(y)I
\]
\[
= \begin{bmatrix}
(y - 1)S_{n-1}(y) - S_{n-2}(y) & sS_{n-1}(y) \\
-s^{-1}(y - 2)S_{n-1}(y) & S_{n-1}(y) - S_{n-2}(y)
\end{bmatrix}.
\]
Hence, by a direct (but lengthy) calculation we have
\[
\rho(aw) - \rho(wb) = \rho(a(ab^{-1})^{-n}a(ab^{-1})^n) - \rho((ab^{-1})^{-n}a(ab^{-1})^n b)
\]
\[
= \begin{bmatrix}
(y - 2)P_n(s, y) & sP_n(s, y) \\
-s^{-1}(y - 2)P_n(s, y) & 0
\end{bmatrix},
\]
where
\[
P_n(s, y) = (s^2 + s^{-2} + 1 - y)S_{n-1}(y) - (s^2 + s^{-2})S_{n-1}(y)S_{n-2}(y) + S_{n-2}^2(y).
\]
By Lemma 2.1 we have
\[
S_{n-1}(y) - yS_{n-1}(y)S_{n-2}(y) + S_{n-2}^2(y) = 1.
\]
Hence
\[
P_n(s, y) = 1 - (y - s^2 - s^{-2})S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)).
\]
Since \(P_n(s, y) = P_n(s^{-1}, y)\), from now on we consider \(P_n\) as a polynomial in the variables \(x = s + s^{-1}\) and \(y\). With these new variables we have
\[
P_n(x, y) = 1 - (y + 2 - x^2)S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)).
\]

**Proof of Theorem 1.** We first prove the following lemma.

**Lemma 2.5.** Suppose \(x_0^2 \in \mathbb{R}\) such that \(4 - 1/|n| < x_0^2 \leq 4\). If \(y \in \mathbb{R}\) satisfies \(P_n(x_0, y) = 0\), then \(y > 2\).

**Proof.** Since \(P_n(x_0, y) = 0\), we have \(S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)) = (y + 2 - x_0^2)^{-1}\).
Hence
\[
((y + 2 - x_0^2)S_{n-1}(y))^{-2} = (S_{n-1}(y) - S_{n-2}(y))^2
\]
\[
= 1 + (y - 2)S_{n-1}(y)S_{n-2}(y)
\]
\[
= 1 + (y - 2)(S_{n-1}^2(y) - (y + 2 - x_0^2)^{-1}),
\]
which implies that
\[
1 = (y + 2 - x_0^2)(4 - x_0^2)S_{n-1}^2(y) + (y - 2)(y + 2 - x_0^2)^2S_{n-1}^4(y).
\]
Assume \(y \leq 2\). Then it follows from the above equation that
\[
1 \leq (y + 2 - x_0^2)(4 - x_0^2)S_{n-1}^2(y).
\]
(2-1)
In particular, \(y > x_0^2 - 2 > -2\). Since \(-2 < y \leq 2\), by Lemma 2.3 we have \(S_{n-1}^2(y) \leq n^2\). Hence
\[
(y + 2 - x_0^2)(4 - x_0^2)S_{n-1}^2(y) \leq (4 - x_0^2)^2n^2 < 1.
\]
This contradicts (2-1). □
We now complete the proof of Theorem 1. Suppose \( x_0^2 \in \mathbb{R} \) and \( 4 - 1/|n| < x_0^2 \leq 4 \). By Lemma 2.5, it suffices to consider \( P_n(x_0, y) \), where \( y \) is a real number greater than 2. The equation \( P(x_0, y) = 0 \) is equivalent to

\[
x_0^2 - 4 = y - 2 - \frac{1}{S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y))}.
\tag{2-2}
\]

Denote by \( f_n(y) \) the right-hand side of (2-2), where \( y > 2 \). We now use the factorizations of \( S_{n-1}(y) \) and \( S_{n-1}(y) - S_{n-2}(y) \) in Lemma 2.2.

If \( n = -1 \) then

\[
f_n(y) = y - 2 + \frac{1}{y - 1} > 0 \geq x_0^2 - 4.
\]

Hence \( f_n(y) = x_0^2 - 4 \) has no solutions on \((2, \infty)\).

If \( n < -1 \) then, by letting \( m = -n > 1 \), we have

\[
f_n(y) = y - 2 + \frac{1}{S_{m-1}(y)(S_{m}(y) - S_{m-1}(y))}
= y - 2 + \frac{1}{\prod_{k=1}^{m-1} (y - 2 \cos \frac{k\pi}{m}) \prod_{l=1}^{m} (y - 2 \cos \frac{(2l-1)\pi}{2m+1})} > 0 \geq x_0^2 - 4.
\]

Hence \( f_n(y) = x_0^2 - 4 \) has no solutions on \((2, \infty)\).

If \( n = 1 \) then \( f_n(y) = y - 3 \). Since \( x_0^2 > 3 \), the equation \( f_n(y) = x_0^2 - 4 \) has a unique solution \( y = x_0^2 - 1 \) on \((2, \infty)\).

If \( n > 1 \) then we have

\[
f_n(y) = y - 2 - \frac{1}{\prod_{k=1}^{n-1} (y - 2 \cos \frac{k\pi}{n}) \prod_{l=1}^{n-1} (y - 2 \cos \frac{(2l-1)\pi}{2n-1})}.
\]

It is easy to see that \( f_n(y) \) is increasing on \((2, \infty)\). Moreover, \( \lim_{y \to \infty} f_n(y) = \infty \) and \( \lim_{y \to 2} f_n(y) = -1/n < x_0^2 - 4 \). Hence \( f_n(y) = x_0^2 - 4 \) has a unique solution on \((2, \infty)\).

The proof of Theorem 1 is complete.

**Proof of Theorem 2.** We write \( P_n(y) \) for \( P_n(2, y) \). Let \( y = t^2 + t^{-2} \). Then

\[
P_n(y) = (S_{n-1}(y) - S_{n-2}(y))^2 - (y - 2)S_{n-1}(y)
= \frac{(t^{2n} + t^{2-2n})^2 - t^2(t^{2n} - t^{-2n})^2}{(t^2 + 1)^2}
= \frac{(t^{2n} + t^{2-2n} + t^{2n+1} - t^{1-2n})(t^{2n} + t^{2-2n} - t^{2n+1} + t^{1-2n})}{(t^2 + 1)^2}.
\]

Up to a factor \( t^k \), each of \( t^{2n} + t^{2-2n} + t^{2n+1} - t^{1-2n} \) and \( t^{2n} + t^{2-2n} - t^{2n+1} + t^{1-2n} \) is obtained from the other by replacing \( t \) by \( t^{-1} \). To show that \( P_n(y) \) is irreducible
over $\mathbb{Q}$, it suffices to show that
\begin{equation}
  t^{4n} + t^{4n-1} + t - 1 = (t^2 + 1)Q_n(t),
\end{equation}
where $Q_n(t) \in \mathbb{Z}[t]$ is irreducible over $\mathbb{Q}$.

As in the proof of [Baker and Petersen 2013, Lemma 6.8], we will use the following theorem of Ljunggren [1960]. Consider a polynomial of the form $R(t) = t^{k_1} + \varepsilon_1 t^{k_2} + \varepsilon_2 t^{k_3} + \varepsilon_3$, where $\varepsilon_j = \pm 1$ for $j = 1, 2, 3$. Then, if $R$ has $r > 0$ roots of unity as roots then $R$ can be decomposed into two factors, one of degree $r$ which has these roots of unity as zeros and the other which is irreducible over $\mathbb{Q}$. Hence, to prove (2-3) it suffices to show that $\pm i$ are the only roots of unity which are roots of $t^{4n} + t^{4n-1} + t - 1$ and these occur with multiplicity 1.

Let $t$ be a root of unity such that $t^{4n} + t^{4n-1} + t - 1 = 0$. Write $t = e^{i\theta}$, where $\theta \in \mathbb{R}$. Since $t^{2n-1} + t^{1-2n} + t^{2n} - t^{-2n} = 0$, we have
\[
2 \cos(2n - 1)\theta + 2i \sin 2n\theta = 0,
\]
which implies that both $\cos(2n - 1)\theta$ and $\sin 2n\theta$ are equal to zero. There exist integers $k, l$ such that $(2n - 1)\theta = (k + \frac{1}{2})\pi$ and $2n\theta = l\pi$. This implies that $(2k + 1)/l = (2n - 1)/n$. Since $(2n - 1)/n$ is a reduced fraction, there exists an odd integer $m$ such that $2k + 1 = m(2n - 1)$ and $l = mn$. Hence $\theta = \frac{1}{2}m\pi$, which implies that $t = e^{i\theta} = \pm i$. It is easy to verify that $\pm i$ are roots of $t^{4n} + t^{4n-1} + t - 1 = 0$ with multiplicity 1.

Ljunggren’s theorem then completes the proof of Theorem 2.

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References


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ERIC HINTIKKA AND XINGPING SUN

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BENJAMIN WALTER AND AMINREZA SHIRI

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DANIEL GRAY AND HUA WANG

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JEFFREY D. ADLER, MICHAEL CASSEL, JOSHUA M. LANSKY, EMMA MORGAN AND YIFEI ZHAO

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ROBERT F. ALLEN, THONG M. LE AND MATTHEW A. PONS

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BYOUNGWOOK JANG, ANNA KRONAUR, PRATAP LUITEL, DANIEL MEDICI, SCOTT A. TAYLOR AND ALEXANDER ZUPAN

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