involve
a journal of mathematics

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msp

2017 vol. 10, no. 1

INVOLVE 10:1 (2017)
dx.doi.org/10.2140/involve.2017.10.65

The lifting of graphs to 3-uniform hypergraphs and some applications to hypergraph Ramsey theory

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(Communicated by Joshua Cooper)

Given a simple graph Γ, we describe a "lifting" to a 3-uniform hypergraph $\varphi(\Gamma)$ that sends the complement of Γ to the complement of $\varphi(\Gamma)$. We consider the effects of this lifting on cycles, complete subhypergraphs, and complete subhypergraphs missing a single hyperedge. Our results lead to natural lower bounds for some hypergraph Ramsey numbers.

1. Introduction

The subject of extremal graph theory arose from the observation that as the cardinality of a set increases, one becomes able to predict the existence of specific complex structures within the set. In particular, Ramsey theory provides a plentiful garden, ripe with open problems for all extremal graph theorists. In Ramsey theory, mathematicians focus their attention on the determination of the Ramsey number $R(K_s, K_t)$, defined to the be least natural number n with the following property: if a graph G has order at least n, then G contains a K_s-subgraph (a subgraph isomorphic to the complete graph K_s on s vertices), or the complement $\overline{G}$ contains a K_t-subgraph. While only a handful of Ramsey numbers are known, many Ramsey numbers can be found to live within certain bounds (see Radziszowski's dynamic survey [1994] for a current list of known values and restrictions). Since determining exact values for Ramsey numbers is very difficult, we often shift our attention to finding a specific graph H that does not contain a K_s-subgraph and whose complement $\overline{H}$ does not contain a K_t-subgraph to improve upon known lower bounds for these elusive numbers.

The self-complementary graphs known as Paley graphs provide a natural lower bound for the diagonal Ramsey numbers, which take the form $R(K_s, K_s)$. To define the Paley graph G_q, let

$$q = p^f \equiv 1 \pmod 4$$

MSC2010: primary 05C65, 05C55; secondary 05C35.
Keywords: cliques, Ramsey number, Turán graphs.

be a power of the prime number p and let $\mathbb{F}_q$ denote the finite field containing exactly q elements. Then G_q has vertex set $V(G_q) := \mathbb{F}_q$ and edge set

$$E(G_q) := \{ab \mid b - a \in \mathbb{F}_q^{\times 2}\},$$

where $\mathbb{F}_q^{\times 2}$ denotes the subgroup of the multiplicative group $\mathbb{F}_q^{\times}$ consisting of squares:

$$\mathbb{F}_q^{\times 2} := \{y \in \mathbb{F}_q^{\times} \mid y = x^2 \text{ for some } x \in \mathbb{F}^{\times}\}.$$

Note that the assumed congruence $q \equiv 1 \pmod 4$ implies that $-1 \in \mathbb{F}_q^{\times 2}$ and hence, $a - b \in \mathbb{F}_q^{\times 2}$ if and only if $b - a \in \mathbb{F}_q^{\times 2}$.

Determining the aforementioned lower bound for a Ramsey number coincides with finding the clique number of a graph G, which we shall denote by $\omega(G)$ throughout this paper, along with the clique number of its complement $\overline{G}$. The clique number of a hypergraph is denoted analogously. Early discovery of lower bounds for Ramsey numbers hinged upon the results $\omega(G_5) = 2$ and $\omega(G_{17}) = 3$, which gave us $R(K_3, K_3) \geq 6$ and $R(K_4, K_4) \geq 18$. In fact, these bounds are optimal since $R(K_3, K_3) = 6$ and $R(K_4, K_4) = 18$. With the algebraic structure of Paley graphs providing a methodology for the determination of certain clique numbers, numerous generalizations of the concept of a Paley graph have been introduced (see [Budden et al. 2011; 2013] and [Su et al. 2002; Wu et al. 2010], where new lower bounds for several Ramsey numbers resulted).

One generalization of Ramsey theory worth considering is the corresponding theory in the context of 3-uniform hypergraphs. With Paley graphs playing such a vital role in the determination of the diagonal Ramsey numbers, we wish to determine the analogues of Paley graphs in this context. After some investigation, we noticed that the Paley graph G_q can be used to define an analogous hypergraph $G_q^{(3)}$ by setting $V(G_q^{(3)}) := \mathbb{F}_q$ and defining the hyperedge set

$$E(G_q^{(3)}) := \{abc \mid (b-a)(c-b)(a-c) \in \mathbb{F}_q^{\times 2}\}.$$

Then $G_q^{(3)}$ is self-complementary and maintains much of the algebraic structure inherent in Paley graphs. In fact, using a character sum similar to the one used to enumerate triangles in character difference graphs in [Budden et al. 2011; 2013], one can easily show that $G_q^{(3)}$ contains exactly

$$\tfrac{1}{192} q(q-1)(q-3)(q-5)$$

subhypergraphs isomorphic to $K_4^{(3)}$ (where $K_n^{(3)}$ denotes the complete 3-uniform hypergraph on n vertices). From this calculation, we see that the first 3-uniform Paley graph that contains a $K_4^{(3)}$-subhypergraph is $G_{13}^{(3)}$, and it is well-known that $R(K_4^{(3)}, K_4^{(3)}; 3) = 13$ (see [McKay and Radziszowski 1991]). Here, $R(K_s^{(r)}, K_t^{(r)}; r)$ is the Ramsey number for r-uniform hypergraphs.

The observation that $G_q^{(3)}$ seems to be the appropriate analogue for Paley graphs in the 3-uniform setting led us to consider how an arbitrary graph might naturally be lifted to form a 3-uniform hypergraph, while maintaining properties that are useful to Ramsey theory. In Section 2, we describe a natural way to lift a graph to a 3-uniform hypergraph, show that our lifting preserves complements, and consider the lifting of cycles. In Section 3, we consider which graphs map to complete subhypergraphs and complete subhypergraphs missing a single hyperedge, allowing us to relate the clique number of a graph to that of its 3-uniform lifting.

In Section 4, we focus on applications of our results to generalized Ramsey theory. One of the more well-known results in hypergraph Ramsey theory is the "stepping-up" lemma, usually credited to Erdős and Hajnal (see [Graham et al. 1990]). It states that if $s > r \geq 3$, then

$$R(K_s^{(r)}, K_s^{(r)}; r) > m \quad \Longrightarrow \quad R(K_{2s+r-4}^{(r+1)}, K_{2s+r-4}^{(r+1)}; r+1) > 2^m.$$

Despite the strength of this result, it begins with $r = 3$, for which there exist only a small number of known lower bounds. In fact, the only known 3-uniform Ramsey number for complete hypergraphs is $R(K_4^{(3)}, K_4^{(3)}; 3) = 13$ (see [Radziszowski 1994]), but many new bounds have recently been determined for Ramsey numbers of various hypergraphs that are not complete; see [Budden et al. 2015]. A weak version of Theorem 9 in Section 4 implies that when $s \geq 3$ and $t \geq 3$, we have

$$R(K_{2s-1}^{(3)}, K_{2t-1}^{(3)}; 3) \geq R(K_s, K_t).$$

This allows one to use known lower bounds for diagonal Ramsey numbers to deduce bounds for corresponding higher-uniform Ramsey numbers via the stepping-up lemma.

2. Lifting graphs to 3-uniform hypergraphs

Let $\mathcal{G}_2$ denote the set of all (simple) graphs of order at least three and let $\mathcal{G}_3$ denote the set of all 3-uniform (simple) hypergraphs of order at least three. Define the map $\varphi : \mathcal{G}_2 \to \mathcal{G}_3$ to send a graph Γ to a 3-uniform hypergraph $\varphi(\Gamma)$ satisfying $V(\varphi(\Gamma)) = V(\Gamma)$, and letting $E(\varphi(\Gamma))$ consist of all unordered 3-tuples abc of distinct vertices in $V(\varphi(\Gamma))$ such that exactly one or all of ab, bc, and ac are in $E(\Gamma)$. We easily confirm that $\varphi(G_q) = G_q^{(3)}$, as we defined in the previous section. One can also check that if two graphs in $\mathcal{G}_2$ are isomorphic, then their images under the lifting φ must also be isomorphic. It is easily demonstrated that the converse is not true.

Denoting the complement of a graph (or hypergraph) Γ by $\overline{\Gamma}$, we note that $abc \in E(\varphi(\Gamma))$ if and only if all three of ab, bc, and ac are edges in Γ (and hence, none of them form edges in $\overline{\Gamma}$) or if exactly one of ab, bc, and ac is an edge in Γ (in which case, exactly two of them form edges in $\overline{\Gamma}$). Observing that $\varphi(\overline{\Gamma})$ consists of hyperedges abc such that exactly zero or two of ab, bc, and ac are in Γ, it follows that

$$\overline{\varphi(\Gamma)} \cong \varphi(\overline{\Gamma}).$$

In particular, if Γ is self-complementary, then $\varphi(\Gamma)$ is self-complementary. The preservation of complements under the map φ further emphasizes this choice of lifting for its potential implications in Ramsey theory.

In order to gain an understanding of the map φ, we begin by considering its effects on cycles. For any (hyper)graph G and subset $S \subseteq V(G)$, we shall use $G[S]$ to denote the sub(hyper)graph of G induced by S. We employ the standard notation of writing

$$x_1 - x_2 - x_3 - \cdots - x_n - x_1$$

to indicate that the vertices $x_1, x_2, x_3, \ldots, x_n$ in a graph $\Gamma \in \mathcal{G}_2$ form a cycle of length n. There are two possible concepts of cycles in the 3-uniform case: loose and tight cycles. We say that $x_1, x_2, x_3, \ldots, x_n$ form a loose cycle in $\varphi(\Gamma)$ if

$$x_1x_2x_3,\ x_3x_4x_5,\ x_5x_6x_7,\ \ldots,\ x_{n-1}x_nx_1$$

are all hyperedges. We say that $x_1, x_2, x_3, \ldots, x_n$ form a tight cycle in $\varphi(\Gamma)$ if

$$x_1x_2x_3,\ x_2x_3x_4,\ x_3x_4x_5,\ \ldots,\ x_nx_1x_2$$

are all hyperedges. Note that for loose cycles, it is necessary that n be even and every even tight cycle is also a loose cycle (having fewer hyperedges). Given a cycle $x_1 - x_2 - x_3 - \cdots - x_n - x_1$ in $\Gamma \in \mathcal{G}_2$, we first focus on when its image

$$\varphi(\Gamma)[x_1, x_2, x_3, \ldots, x_n]$$

forms a loose or tight cycle in $\varphi(\Gamma)$. The liftings of cycles when $n = 3, 4$ are easy to work out and the following two theorems handle the remaining cases.

Theorem 1. *Let $x_1 - x_2 - x_3 - \cdots - x_n - x_1$ be a cycle in $\Gamma \in \mathcal{G}_2$ with $n > 5$. If n is even and*

$$x_1 - x_3 - x_5 - \cdots - x_{n-1} - x_1$$

is a cycle in Γ, then $x_1, x_2, x_3, \ldots, x_n$ form a loose cycle in $\varphi(\Gamma)$, and if it is a cycle in $\overline{\Gamma}$, then $x_1, x_2, x_3, \ldots, x_n$ form a loose cycle in $\varphi(\overline{\Gamma})$.

Proof. Assuming that $x_1 - x_2 - x_3 - \cdots - x_n - x_1$ is a cycle in Γ, it follows that for each potential hyperedge $x_{i-1}x_ix_{i+1}$, both $x_{i-1}x_i$ and x_ix_{i+1} form edges in Γ. Thus, $x_{i-1}x_ix_{i+1}$ is a hyperedge in $\varphi(\Gamma)$ if and only if $x_{i-1}x_{i+1}$ is an edge in Γ and it is a hyperedge in $\varphi(\overline{\Gamma})$ if and only if $x_{i-1}x_{i+1}$ is an edge in $\overline{\Gamma}$. □

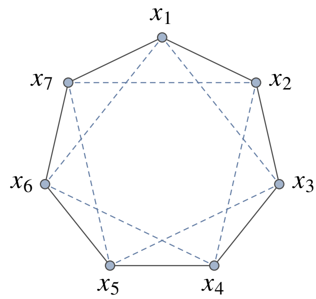

Figure 1. Parallel cycles when $n = 7$.

Theorem 2. *Let $x_1 - x_2 - x_3 - \cdots - x_n - x_1$ be a cycle in $\Gamma \in \mathcal{G}_2$ with $n \geq 5$. If n is odd and*

$$x_1 - x_3 - x_5 - \cdots - x_n - x_2 - x_4 - x_6 - \cdots - x_{n-1} - x_1$$

is a cycle in Γ, then $x_1, x_2, x_3, \ldots, x_n$ form a tight cycle in $\varphi(\Gamma)$, and if it is a cycle in $\overline{\Gamma}$, then $x_1, x_2, x_3, \ldots, x_n$ form a tight cycle in $\varphi(\overline{\Gamma})$. If n is even and both

$$x_1 - x_3 - x_5 - \cdots - x_{n-1} - x_1 \quad \textit{and} \quad x_2 - x_4 - x_6 - \cdots - x_n - x_2$$

form cycles in Γ, then $x_1, x_2, x_3, \ldots, x_n$ form a tight cycle in $\varphi(\Gamma)$, and if they are both cycles in $\overline{\Gamma}$, then $x_1, x_2, x_3, \ldots, x_n$ form a tight cycle in $\varphi(\overline{\Gamma})$.

Proof. This theorem follows from a similar argument to the one that was used in the proof of the previous theorem. The details are left to the reader. □

Figures 1 and 2 provide visual representations for the underlying cycles in Theorem 2 when $n = 7$ and $n = 16$, respectively. In each graph, the cycle

$$x_1 - x_2 - x_3 - \cdots - x_n - x_1$$

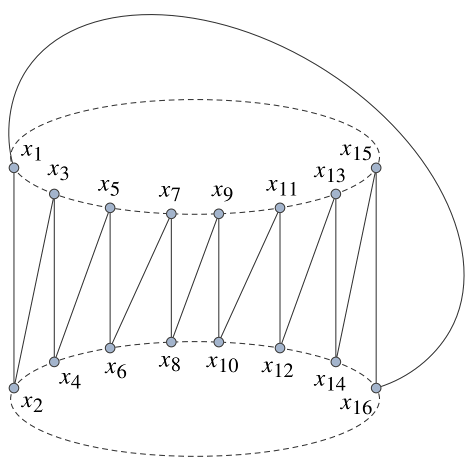

Figure 2. Parallel cycles when $n = 16$.

uses solid edges and all other cycles are represented with dashed edges. Whether or not dashed edges appear in Γ or $\overline{\Gamma}$ determines the location of the corresponding tight cycles in the lifting.

Although we are able to understand how cycles may lift, the subgraphs which map to loose and tight hypergraph cycles have a much less predictable structure, preventing such a nice characterization. So, we turn our attention to complete subhypergraphs and those that are missing a single hyperedge.

3. Complete hypergraphs and complete hypergraphs missing a single hyperedge

From the definition of φ, it is clear that if H is a complete subgraph of $\Gamma \in \mathcal{G}_2$ having order at least three, then its image $\varphi(H)$ is a complete subhypergraph of the same order in $\varphi(\Gamma)$. Now we turn our attention to understanding which subgraphs map to complete subhypergraphs under φ.

Lemma 3. *Suppose that $\Gamma \in \mathcal{G}_2$, $S \subseteq V(\Gamma)$ is a subset containing at least three elements, and $K := \Gamma[S]$. If $\varphi(K)$ is complete and C is a component of K, then C is complete.*

Proof. Suppose that C is a component of K that is not complete (which necessarily requires the order of C to be at least two). Then there exist vertices $b_1, b_2 \in V(C)$ that do not form an edge in C. If $x \in V(K) - V(C)$, then neither b_1x nor b_2x form edges in K, and b_1b_2x is not a hyperedge in $\varphi(K)$, contradicting the assumption that $\varphi(K)$ is complete. If no such x exists, then $V(K) = V(C)$, and for every vertex $y \in V(C) - \{b_1, b_2\}$, exactly one of b_1y and b_2y must be in $E(K)$. Since C is assumed to be connected, it must have order at least four. Let N_{b_1} and N_{b_2} denote the sets of neighbors of b_1 and b_2, respectively, in $V(C)$. Note that each N_{b_j} is nonempty or else b_j would be disconnected from the rest of C. Also, since C is connected, it follows that $N_{b_1} \cap N_{b_2} \neq \varnothing$. So, let $z \in N_{b_1} \cap N_{b_2}$. Then b_1b_2z is not a hyperedge in $\varphi(K)$, contradicting the assumption that $\varphi(K)$ is complete. Thus, we find that C must be complete. □

Lemma 3 greatly restricts the structure of the possible subgraphs of a graph Γ that can map to a complete subhypergraph of $\varphi(\Gamma)$. The following theorem completely classifies the relevant subgraphs.

Theorem 4. *Suppose that $\Gamma \in \mathcal{G}_2$, $S \subseteq V(\Gamma)$ is a subset containing at least three elements, and $K := \Gamma[S]$. Then $\varphi(K)$ is complete if and only if K is complete or K is the union of exactly two disjoint complete subgraphs.*

Proof. From Lemma 3, it suffices to prove that $\varphi(K)$ is complete if and only if K contains at most two components. To prove the forward implication, assume that $\varphi(K)$ is complete and K consists of at least three components. Suppose that

C_1, C_2, C_3 are three components of K and for each $1 \le i \le 3$, choose a vertex $a_i \in V(C_i)$. Since the components are disconnected from one another, it follows that $\varphi(K)$ lacks the hyperedge $a_1a_2a_3$, contradicting our assumption that $\varphi(K)$ is complete. Hence, K contains at most two components. Now we consider the converse. Clearly, if K is a complete subgraph of order at least three, then $\varphi(K)$ must also be complete. Otherwise, assume that K is the disjoint union of two complete subgraphs having vertex sets S_1 and S_2. For every three vertices $a, b, c \in V(K)$, either all three are in one of S_1 or S_2, and hence form a hyperedge in $\varphi(K)$, or they are divided up between S_1 and S_2. Without loss of generality, assume that $a \in S_1$ and $b, c \in S_2$. Exactly one of ab, bc, and ac are edges in K, making $abc \in E(\varphi(K))$. It follows that $\varphi(K)$ is complete. $\square$

The previous theorem gives us an immediate corollary pertaining to the lifting of a complete bipartite graph.

Corollary 5. *For the complete bipartite graph $K_{m,n}$ where either m or n is greater than* 2, *we have $\varphi(K_{m,n})$ is isomorphic to the empty* 3*-uniform hypergraph of order $m+n$.*

Proof. Recall that $\overline{K_{m,n}} \simeq K_m \cup K_n$. Since the lifting φ preserves complements, we just apply the previous theorem to obtain our desired result. $\square$

Since every complete subgraph with at least three vertices in $\Gamma \in \mathcal{G}_2$ maps to a complete subhypergraph of $\varphi(\Gamma)$, we have

$$\omega(\varphi(\Gamma)) = m \ge 3 \quad \Longrightarrow \quad \omega(\Gamma) \le m,$$

and the previous theorem implies that

$$\omega(\Gamma) = n \ge 3 \quad \Longrightarrow \quad \omega(\varphi(\Gamma)) \le 2n.$$

From these observations, we obtain the following corollary.

Corollary 6. *Every graph $\Gamma \in \mathcal{G}_2$ with $\omega(\Gamma) \ge 3$ satisfies*

$$\omega(\Gamma) \le \omega(\varphi(\Gamma)) \le 2\omega(\Gamma),$$
$$\tfrac{1}{2}\omega(\varphi(\Gamma)) \le \omega(\Gamma) \le \omega(\varphi(\Gamma)).$$

Now let H be a subgraph of $\Gamma \in \mathcal{G}_2$ of order at least three that is isomorphic to a complete graph with a single edge missing. Without loss of generality, assume that $V(H) = \{a_1, a_2, \ldots, a_k\}$ with $a_1a_2 \notin E(H)$. Then for every $x \in \{a_3, \ldots, a_k\}$, we have $a_1a_2x \notin E(\varphi(H))$. However, every unordered 3-tuple of distinct elements in $\{a_3, \ldots, a_k\}$ forms a hyperedge in $\varphi(H)$ as does any 3-tuple of vertices from $V(H)$ that contains exactly one of a_1 and a_2. So, $\varphi(H)$ is isomorphic to a complete 3-uniform hypergraph with exactly $k-2$ hyperedges missing (those containing

a_1 and a_2). Now we consider which graphs (if any) lift under φ to hypergraphs isomorphic to complete hypergraphs missing a single hyperedge.

Theorem 7. *Suppose $n \geq 4$ and $\Gamma \in \mathcal{G}_2$. The lifting $\varphi(\Gamma)$ cannot contain an induced subhypergraph isomorphic to $K_n^{(3)} - e$ (i.e., a complete 3-uniform hypergraph on n vertices that is missing a single hyperedge).*

Proof. Assume $\varphi(\Gamma)$ contains an induced subhypergraph isomorphic to $K_n^{(3)} - e$. Let $S = \{b_1, b_2, \ldots, b_n\}$ denote the vertices of the $(K_n^{(3)}-e)$-subhypergraph of $\varphi(\Gamma)$. Without loss of generality, let $b_1b_2b_3$ be the missing hyperedge. There exist two possibilities. Either none of b_1b_2, b_2b_3, and b_1b_3 are in $E(\Gamma)$ or exactly two of the aforementioned edges exist in Γ. In the former case, note that $b_1b_2b_4 \in E(\varphi(\Gamma))$, from which we see that exactly one of b_1b_4 and b_2b_4 must be in $E(\Gamma)$. Without loss of generality, assume that $b_1b_4 \in E(\Gamma)$. In a similar manner, it can be shown that $b_2b_3b_4 \in E(\varphi(\Gamma))$ implies that $b_3b_4 \in E(\Gamma)$. Then $b_1b_3b_4$ cannot be contained in $E(\varphi(\Gamma))$, contradicting the assumption that $b_1b_2b_3$ was the only missing hyperedge. In the latter case, exactly two of b_1b_2, b_2b_3, and b_1b_3 are in $E(\Gamma)$. Without loss of generality, assume that b_1b_2 and b_2b_3 are in $E(\Gamma)$. Then $b_1b_3b_4 \in E(\varphi(\Gamma))$ implies that exactly one of b_1b_4 and b_3b_4 is in $E(\Gamma)$. Without loss of generality, assume that $b_1b_4 \in E(\Gamma)$. Then $b_1b_2b_4 \in E(\varphi(\Gamma))$ implies that $b_2b_4 \in E(\Gamma)$. Similarly, $b_2b_3b_4 \in E(\varphi(\Gamma))$ implies that $b_3b_4 \in E(\Gamma)$, contradicting our assumption. Hence, we have shown in both cases that if $\varphi(\Gamma)$ contains a $(K_n^{(3)}-e)$-subhypergraph, then it must contain a $K_n^{(3)}$-subhypergraph. □

4. Applications to Ramsey theory

From the specific subhypergraphs that we have chosen to consider under the lifting φ, it should be obvious that our interests lie in applications to extremal graph theory. In particular, our focus on the behavior of complete sub(hyper)graphs indicates an underlying interest in Ramsey theory. The multicolor Ramsey number $R(G_1, G_2, \ldots, G_k)$ is defined to be the least natural number n such that for every arbitrary coloring of the edges of K_n with k colors, there exists a subgraph in color i isomorphic to G_i for some i. The multicolor 3-uniform hypergraph Ramsey number $R(G_1, G_2, \ldots, G_k; 3)$ is defined analogously.

When studying the behavior of cliques in graphs, a class of graphs known as Turán graphs possess certain optimal parameters. Suppose $n \geq 3$ and $q \geq 2$ are integers. By the division algorithm, there exist unique integers $m \geq 0$ and $0 \leq r < q$ such that $n = mq + r$. The Turán graph $T_q(n)$ is the complete q-partite graph whose vertices are partitioned into balanced sets (i.e., sets with cardinalities as equal as possible). Such graphs contain K_q-subgraphs but lack K_{q+1}-subgraphs. In fact, Turán [1941] proved that out of all graphs of order n, they possess the maximum number of edges possible without containing a K_{q+1}-subgraph. When considering the lifting of Turán graphs, we obtain the following theorem.

Theorem 8. *Let $n \geq 3$, $q \geq 2$, and $n = mq + r$, where $0 \leq r < q$. Then we have the following*:

(1) *If $n = qm$, then $R(K_{q+1}^{(3)} - e, K_{2m+1}^{(3)} - e; 3) > n$.*

(2) *If $n = qm + 1$, then $R(K_{q+1}^{(3)} - e, K_{2m+2}^{(3)} - e; 3) > n$.*

(3) *If $n = qm + r$ with $r \geq 2$, then $R(K_{q+1}^{(3)} - e, K_{2m+3}^{(3)} - e; 3) > n$.*

Proof. Regardless of the value of r, note that $T_q(n)$ contains a K_q-subgraph, but not a K_{q+1}-subgraph. Also, at most one vertex of a complete subgraph can come from any one connected set of vertices. So, $\varphi(T_q(n))$ contains a $K_q^{(3)}$-subhypergraph, but not a $K_{q+1}^{(3)}$-subhypergraph. Note that $\overline{T_q(n)}$ consists of disconnected complete subgraphs of orders m and $m+1$. By Theorem 4, we obtain the following cases. If $n = qm$, then all of the sets of vertices have cardinality m and $\varphi(\overline{T_q(n)})$ contains a $K_{2m}^{(3)}$-subhypergraph, but not a $K_{2m+1}^{(3)}$-subhypergraph. If $n = qm + 1$, then exactly one vertex set has cardinality $m+1$ and $\varphi(\overline{T_q(n)})$ contains a $K_{2m+1}^{(3)}$-subhypergraph, but not a $K_{2m+2}^{(3)}$-subhypergraph. For the remaining cases, in which $n = qm + r$ with $2 \leq r < q$, at least two vertex sets have cardinality $m+1$, and we find that $\varphi(\overline{T_q(n)})$ contains a $K_{2m+2}^{(3)}$-subhypergraph, but not a $K_{2m+3}^{(3)}$-subhypergraph. These results, along with the implication of Theorem 7, prove the theorem. □

Now we shift our attention to proving a relationship between standard Ramsey numbers and certain corresponding 3-uniform Ramsey numbers for complete hypergraphs missing a single hyperedge.

Theorem 9. *Let $s, t \in \mathbb{N}$ with $s \geq 3$ and $t \geq 3$. Then*

$$R(K_{2s-1}^{(3)} - e, K_{2t-1}^{(3)} - e; 3) \geq R(K_s, K_t).$$

Proof. Assume $m = R(K_s, K_t)$. Then there exists a graph G of order $m-1$ that does not contain a K_s-subgraph, and whose complement does not contain a K_t-subgraph. From Theorem 4, it follows that $\varphi(G)$ does not contain a $K_{2s-1}^{(3)}$-subhypergraph, and its complement does not contain a $K_{2t-1}^{(3)}$-subhypergraph. Theorem 7 then implies that $\varphi(G)$ does not contain a $(K_{2s-1}^{(3)} - e)$-subhypergraph, and its complement does not contain a $(K_{2t-1}^{(3)} - e)$-subhypergraph. Thus,

$$R(K_{2s-1}^{(3)} - e, K_{2t-1}^{(3)} - e; 3) > m - 1 = R(K_s, K_t) - 1,$$

completing the proof of the theorem. □

Note that Theorem 9 implies

$$R(K_{2s-1}^{(3)}, K_{2t-1}^{(3)}; 3) \geq R(K_s, K_t),$$

which can be used with the stepping-up lemma. Recently, Conlon, Fox, and Sudakov [Conlon et al. 2013] also proved an analogue of the stepping-up lemma, which lifts from graphs to 3-uniform hypergraphs. In the spirit of the original stepping-up lemma, it focused on the diagonal case. Namely, they proved that

$$R(K_s, K_s) > m \quad \Longrightarrow \quad R(K_{s+1}^{(3)}, K_{s+1}^{(3)}, K_{s+1}^{(3)}, K_{s+1}^{(3)}; 3) > 2^m.$$

Of course since $R(4, 4) = 18$, this result implies $R(5, 5, 5, 5; 3) > 131{,}072$. The following theorem handles some off-diagonal cases.

Theorem 10. *If* $q \geq 3$, *then*

$$R(K_5^{(3)}, K_{q+1}^{(3)} - e, K_{2s-1}^{(3)} - e, K_{2t-1}^{(3)} - e; 3) > q(R(K_s, K_t) - 1).$$

Proof. Suppose $m = R(K_s, K_t)$ and $q \geq 3$, and let $n = q(m-1)$. Denote the partitioned vertex sets in $T_q(n)$ by $V_1, V_2, \dots, V_k$. We have already noted that $\varphi(T_q(n))$ contains a $K_q^{(3)}$-subhypergraph, but not a $K_{q+1}^{(3)}$-subhypergraph. From Theorem 7, it follows that it does not contain a $(K_{q+1}^{(3)}-e)$-subhypergraph. Color the hyperedges in $\varphi(T_q(n))$ yellow. Note that $\overline{T_q(n)}$ consists of q disconnected K_{m-1}-subgraphs. Since $R(s, t) = m$, there exists a red/blue coloring of the edges of K_{m-1} that does not contain a red K_s-subgraph or a blue K_t-subgraph. When lifting just a single K_{m-1} colored in this way, the lifted hypergraph contains at most a red $K_{2s-2}^{(3)}$-subhypergraph or a blue $K_{2t-2}^{(3)}$-subhypergraph by Theorem 4. In fact by Theorem 9, the lifted hypergraph does not contain a red $(K_{2s-1}^{(3)}-e)$-subhypergraph or a blue $(K_{2t-1}^{(3)}-e)$-subhypergraph. We apply this coloring to the hyperedges in $\varphi(\overline{T_q(n)})$ that arise from the individual liftings of the disjoint vertex sets. The remaining hyperedges in $\varphi(\overline{T_q(n)})$ are precisely those that include one vertex from V_i and the other two vertices from V_j, where $i \neq j$. Color these hyperedges green. A complete subhypergraph formed using only these edges includes at most two vertices from any V_i and vertices from no more than two of the partitioned vertex sets. Hence, the green hyperedges may contain a $K_4^{(3)}$-subhypergraph, but not a $K_5^{(3)}$-subhypergraph. From this coloring, we find that

$$R(K_5^{(3)}, K_{q+1}^{(3)} - e, K_{2s-1}^{(3)} - e, K_{2t-1}^{(3)} - e; 3) > n = q(m-1),$$

from which the theorem follows. □

Although the result of [Conlon et al. 2013] is stronger than Theorem 10 for diagonal Ramsey numbers, our results improve on many known lower bounds for off-diagonal 4-color 3-uniform Ramsey numbers. For example, using the explicit known lower bounds in Table IIc of [Radziszowski 1994], we obtain the following bound on an off-diagonal Ramsey number:

$$R(22, 22) > 29{,}940 \quad \Longrightarrow \quad R(K_5^{(3)}, K_{43}^{(3)} - e, K_{43}^{(3)} - e, K_{43}^{(3)} - e; 3) > 1{,}257{,}480.$$

The main advantage to considering the lifting φ is that one is able to sufficiently restrict the structure of hypergraphs in the image by knowing the structure of graphs in the domain. Many open questions naturally arise from this construction. One obvious question is whether or not analogous liftings can be constructed from graphs to r-uniform hypergraphs. This question was recently considered in [Budden and Rapp 2015], but since the liftings did not preserve complements when $r > 3$, it did not lead to new implications in Ramsey theory. We conclude with a list of several other avenues of potential inquiry:

(1) Besides cycles, complete hypergraphs, and complete hypergraphs missing a single hyperedge, what other hypergraph images have predictable preimages?

(2) Can one classify the hypergraphs in $\mathcal{G}_3$ that are not in the range of φ?

(3) Is it possible to classify all graphs in the preimage of a particular hypergraph in the range of φ?

(4) The fact that the lifting φ preserves complements means that it can be thought of as mapping a 2-coloring of the edges of K_n to a 2-coloring of the hyperedges in $K_n^{(3)}$. Can φ be used to describe a mapping of a k-coloring of the edges in K_n to a k-coloring of the hyperedges in $K_n^{(3)}$? If so, one may be able to use known bounds for multicolor Ramsey numbers to obtain analogous results in the setting of 3-uniform hypergraphs.

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Received: 2015-08-16 Revised: 2015-12-02 Accepted: 2015-12-13

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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2017 vol. 10 no. 1