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The multiplicity of solutions for a system of second-order differential equations

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Making use of the Guo–Krasnosel'skiĭ fixed point theorem multiple times, we establish the existence of at least three positive solutions for the system of second-order differential equations $-u''(t) = g\big(t, u(t), u'(t), v(t), v'(t)\big)$ and $-v''(t) = \lambda f\big(t, u(t), u'(t), v(t), v'(t)\big)$ for $t \in (0, 1)$ with right focal boundary conditions $u(0) = v(0) = 0$, $u'(1) = a$, and $v'(1) = b$, where $f, g : [0, 1] \times [0, \infty)^4 \to [0, \infty)$ are continuous, $a, b, \lambda \geq 0$, and $a + b > 0$. Our technique involves transforming the system of differential equations to a new system with homogeneous boundary conditions prior to applying the aforementioned fixed point theorem.

1. Introduction

Showing the existence of multiple positive solutions for boundary value problems is an active field of study due to the applications that arise in modeling real world phenomena. A classic example based on beam analysis, presented by Agarwal [1989], gives an existence and uniqueness result of the fourth-order problem $x^{(4)} = f(t, x, x', x'', x^{(3)})$. Additionally, do Ó, Lorca, and Ubilla [do Ó et al. 2008] studied the fourth-order nonhomogeneous boundary value problem,

$$u^{(4)} = \lambda h(t, u, u''), \quad t \in (0, 1),$$
$$u(0) = u''(0) = 0,$$
$$u(1) = a, \quad u''(1) = b.$$

Utilizing a technique of rewriting the fourth-order problem as a system of second-order differential equations, the authors guaranteed existence of multiple positive solutions by ultimately applying the Guo–Krasnosel'skiĭ fixed point theorem [Krasnosel'skiĭ 1964]. Hopkins [2015] extended this process to establish multiple solutions to the differential equation $u^{(2n)} = \lambda h(t, u, u'', \dots, u^{2(n-1)})$ satisfying

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right focal boundary conditions. Henderson and Hopkins [2010] applied this same technique to a similar fourth-order difference equation. In this work, we consider the system of second-order differential equations

$$-u''(t) = g\big(t, u(t), u'(t), v(t), v'(t)\big), \tag{1}$$

$$-v''(t) = \lambda f\big(t, u(t), u'(t), v(t), v'(t)\big), \tag{2}$$

$$u(0) = v(0) = 0, \tag{3}$$

$$u'(1) = a, \quad v'(1) = b, \tag{4}$$

where $f, g : [0, 1] \times [0, \infty)^4 \to [0, \infty)$ are continuous, $\lambda, a, b \geq 0$ and $a + b > 0$. The novelty of our paper is that the functions f and g contain both even- and odd-order derivatives.

In Section 2 of this paper, we consider a transformation of (1)–(4) that satisfies homogeneous boundary conditions. We also introduce some preliminaries and the conditions under which we can eventually apply the Guo–Krasnosel'skiĭ fixed point theorem. In Section 3 we introduce and prove a sequence of lemmas giving bounds on a defined operator. This culminates in the main result, given in Section 4 where we apply the Guo–Krasnosel'skiĭ fixed point theorem multiple times, yielding at least three positive solutions.

2. Preliminaries

We will prove the existence of multiple solutions for the system of second-order differential equations (1)–(4) by applying the transformation $\bar{u}(t) = u(t) - at$ and $\bar{v}(t) = v(t) - bt$, which gives

$$-\bar{u}''(t) = g\big(t, \bar{u}(t) + ta, \bar{u}'(t) + a, \bar{v}(t) + tb, \bar{v}'(t) + b\big), \tag{5}$$

$$-\bar{v}''(t) = \lambda f\big(t, \bar{u}(t) + ta, \bar{u}'(t) + a, \bar{v}(t) + tb, \bar{v}'(t) + b\big), \tag{6}$$

$$\bar{u}(0) = \bar{v}(0) = 0, \tag{7}$$

$$\bar{u}'(1) = 0, \quad \bar{v}'(1) = 0, \tag{8}$$

where $a, b, \lambda \geq 0$ and $a + b > 0$. Notice that solutions to (5)–(8) are in one-to-one correspondence with (1)–(4). Furthermore, suppose the following hypotheses on f and g are satisfied.

(H0) The functions $f, g : [0, 1] \times [0, \infty)^4 \to [0, \infty)$ are continuous and are nondecreasing in the second and fourth variables and nonincreasing in the third and fifth variables.

(H1) There exist $\alpha, \beta \in (0, 1)$, $\alpha < \beta$, such that given $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 \neq 0$, there exists $k > 0$ such that for $t \in [\alpha, \beta]$,

$$f(t, x_1, x_2, x_3, x_4) > k.$$

(H2) For $t \in (0, 1)$,

$$\lim_{x_1+x_2+x_3+x_4 \to 0^+} \frac{f(t, x_1, x_2, x_3, x_4)}{x_1 + x_2 + x_3 + x_4} = 0$$

uniformly.

(H3) For $t \in (0, 1)$,

$$\lim_{x_1+x_2+x_3+x_4 \to \infty} \frac{f(t, x_1, x_2, x_3, x_4)}{x_1 + x_2 + x_3 + x_4} = 0$$

uniformly.

(H4) There exist $\gamma \in \left(0, \frac{2}{3}\right)$ and $q > 0$ such that for $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 < q$,

$$g(t, x_1, x_2, x_3, x_4) \le \gamma(x_1 + x_2 + x_3 + x_4) \quad \text{for } t \in [0, 1].$$

(H5) There exist $\eta \in \left(0, \frac{2}{3}\right)$ and $\hat{\rho} > 0$ such that for $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 > \hat{\rho}$,

$$g(t, x_1, x_2, x_3, x_4) \le \eta(x_1 + x_2 + x_3 + x_4) \quad \text{for } t \in [0, 1].$$

Solutions to (5)–(8), provided they exist, are of the form

$$\bar{u}(t) = \int_0^1 G(t, s) g\big(s, \bar{u}(s) + as, \bar{u}'(s) + a, \bar{v}(s) + bs, \bar{v}'(s) + b\big)\, ds, \tag{9}$$

$$\bar{v}(t) = \lambda \int_0^1 G(t, s) f\big(s, \bar{u}(s) + as, \bar{u}'(s) + a, \bar{v}(s) + bs, \bar{v}'(s) + b\big)\, ds, \tag{10}$$

where $G(t, s)$ is the Green's function

$$G(t, s) = \begin{cases} t & \text{if } 0 \le t \le s \le 1, \\ s & \text{if } 0 \le s \le t \le 1. \end{cases}$$

Since $G(t, s)$ is clearly nonnegative and f and g are nonnegative by assumption, it follows that solutions u and v are also nonnegative. Some other useful properties on $G(t, s)$ are that

$$\max_{t \in [0,1]} \int_0^1 G(t, s)\, ds = \frac{1}{2} \quad \text{and} \quad \max_{t \in [0,1]} \int_0^1 \left| \frac{\partial}{\partial t} G(t, s) \right| ds = 1.$$

In order to make use of the Guo–Krasnosel'skiĭ fixed point theorem, we will need a Banach space and a cone, as well as an operator T. Let $(X, \|\cdot\|)$ denote the Banach space $X = C^1([0, 1], \mathbb{R}) \times C^1([0, 1], \mathbb{R})$ endowed with the norm

$$\|(\bar{u}, \bar{v})\| = \|\bar{u}\|_\infty + \|\bar{u}'\|_\infty + \|\bar{v}\|_\infty + \|\bar{v}'\|_\infty,$$

where $\|\bar{u}\|_\infty = \sup_{t \in [0,1]} |\bar{u}(t)|$.

Recall that a cone, C, in X is a nonempty, closed, convex subset of X satisfying:

(1) If $x \in C$, and $\lambda > 0$, then $\lambda x \in C$.

(2) If $x \in C$ and $-x \in C$, then $x = 0$.

Define $C \subset X$ to be the cone

$$C = \big\{(\bar{u}, \bar{v}) \in X : (\bar{u}, \bar{v})(0) = (\bar{u}', \bar{v}')(1) = (0, 0) \text{ and } \bar{u}, \bar{v} \text{ are concave}\big\}.$$

The fact that C is a cone follows directly from the definition. Moreover, let Ω_p denote the open set $\Omega_p = \{(\bar{u}, \bar{v}) \in X : \|(\bar{u}, \bar{v})\| < p\}$. Finally, define $T : X \to X$ to be the operator $T(\bar{u}, \bar{v}) = (A_1(\bar{u}, \bar{v}),\ A_2(\bar{u}, \bar{v}))$, where

$$A_1 = \int_0^1 G(t, s) g\big(s, \bar{u}(s) + as, \bar{u}'(s) + a, \bar{v}(s) + bs, \bar{v}'(s) + b\big)\, ds$$

and

$$A_2 = \lambda \int_0^1 G(t, s) f\big(s, \bar{u}(s) + as, \bar{u}'(s) + a, \bar{v}(s) + bs, \bar{v}'(s) + b\big)\, ds.$$

Consider the following lemma, which provides a useful property of T.

Lemma 2.1. *The operator $T : C \to C$ is completely continuous.*

We note that one can use a standard Arzelà–Ascoli argument to show that T is completely continuous; see [Hopkins 2009].
In the next section, we will take advantage of the following lemma.

Lemma 2.2. *Let $\bar{u}(t)$ be a nonnegative concave function which is continuous on* $[0, 1]$. *Then for all $\alpha, \beta \in (0, 1)$, with $\alpha < \beta$, we have*

$$\inf_{t \in [\alpha, \beta]} \bar{u}(t) \geq \alpha(1 - \beta)\|\bar{u}\|_\infty.$$

For a proof of Lemma 2.2, see [Hopkins 2009].

Since we will be using the Guo–Krasnosel'skiĭ fixed point theorem multiple times to acquire our main result, we end the section with the statement of this theorem.

Theorem 2.3 (Guo–Krasnosel'skiĭ fixed point theorem). *Let $(X, \|\cdot\|)$ be a Banach space and $C \subset X$ be a cone. Suppose Ω_1, Ω_2 are open subsets of X satisfying $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $T : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ is a completely continuous operator such that either*

(1) *$\|Tu\| \leq \|u\|$ for $u \in C \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in C \cap \partial\Omega_2$, or*

(2) *$\|Tu\| \geq \|u\|$ for $u \in C \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in C \cap \partial\Omega_2$,*

then T has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Technical results

In this section we give a sequence of four lemmas that allow us to obtain the estimates needed to apply the Guo–Krasnosel'skiĭ fixed point theorem.

Lemma 3.1. *Suppose (H0) and (H1) hold and let $\rho^* > 0$. Then there is a $\Lambda > 0$ such that, for every $\lambda \ge \Lambda$ and $(a, b) \in [0, \infty)^2$,*

$$\|T(\bar{u}, \bar{v})\| \ge \|(\bar{u}, \bar{v})\|$$

for $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho^*}$.

Proof. Let $\rho^* > 0$ and let $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho^*}$. Let $r = \alpha(1-\beta)$, where α and β are as in (H1) and note $r \in (0, 1)$. Furthermore, choose $c \ge 1$ so that both $\bar{u}' + a \le c\|\bar{u}'\|_\infty$ and $\bar{v}' + b \le c\|\bar{v}'\|_\infty$ hold for $t \in [\alpha, \beta]$. Define

$$M = \inf\left\{ \frac{f(t, ra_1, ca_2, ra_3, ca_4)}{r(a_1 + a_3) + c(a_2 + a_4)} : t \in [\alpha, \beta],\ a_1, a_2, a_3 > 0,\ a_4 \ge 0, \text{ and } a_1 + a_2 + a_3 + a_4 = p^* \right\}.$$

The existence of a positive M follows from (H1). Set $\Lambda \ge \left[Mr\int_\alpha^\beta G(1, s)\,ds\right]^{-1}$.

As $(\bar{u}, \bar{v}) \in C$, by Lemma 2.2, we have $\bar{u}(t) + at \ge \bar{u}(t) \ge r\|\bar{u}\|_\infty$. Moreover, due to the nondecreasing property of f in the second and fourth variables and its nonincreasing property in the third and fifth variables, we see that

$$\begin{aligned}
\|T(\bar{u}, \bar{v})\| &\ge \|A_2(\bar{u}, \bar{v})\|_\infty \\
&\ge \lambda \int_0^1 G(1, s) f\big(s, \bar{u} + sa, \bar{u}' + a, \bar{v} + sb, \bar{v}' + b\big)\,ds \\
&\ge \lambda \int_\alpha^\beta G(1, s) f\big(s, r\|\bar{u}\|_\infty, c\|\bar{u}'\|_\infty, r\|\bar{v}\|_\infty, c\|\bar{v}'\|_\infty\big)\,ds \\
&\ge \lambda\big[r(\|\bar{u}\|_\infty + \|\bar{v}\|_\infty) + c(\|\bar{u}'\|_\infty + \|\bar{v}'\|_\infty)\big] \\
&\qquad \times \int_\alpha^\beta G(1, s) \frac{f\big(s, r\|\bar{u}\|_\infty, c\|\bar{u}'\|_\infty, r\|\bar{v}\|_\infty, c\|\bar{v}'\|_\infty\big)}{r(\|\bar{u}\|_\infty + \|\bar{v}\|_\infty) + c(\|\bar{u}'\|_\infty + \|\bar{v}'\|_\infty)}\,ds \\
&\ge \lambda M\big[r(\|\bar{u}\|_\infty + \|\bar{v}\|_\infty) + c(\|\bar{u}'\|_\infty + \|\bar{v}'\|_\infty)\big] \int_\alpha^\beta G(1, s)\,ds \\
&\ge \lambda Mr\big(\|\bar{u}\|_\infty + \|\bar{u}'\|_\infty + \|\bar{v}\|_\infty + \|\bar{v}'\|_\infty\big) \int_\alpha^\beta G(1, s)\,ds \\
&\ge \lambda Mr\|(\bar{u}, \bar{v})\| \int_\alpha^\beta G(1, s)\,ds \\
&\ge \Lambda Mr\|(\bar{u}, \bar{v})\| \int_\alpha^\beta G(1, s)\,ds \\
&\ge \|(\bar{u}, \bar{v})\|.
\end{aligned}$$

□

Lemma 3.2. *Fix $\Lambda > 0$. Suppose (H0) and (H1) hold. Then, for all $\lambda \geq \Lambda$ and for all $(a, b) \in [0, \infty)^2$, with $a + b > 0$, there exists a $\rho_1 = \rho_1(\Lambda, a, b)$ such that for every $\rho \in (0, \rho_1)$, we have*

$$\|T(\bar{u}, \bar{v})\| \geq \|(\bar{u}, \bar{v})\|$$

for all $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_\rho$.

Proof. Fix $\Lambda > 0$. By (H1) and the nonincreasing/nondecreasing properties of f, there exists $k > 0$ such that

$$f\big(t, \bar{u}+ta, \bar{u}'+a, \bar{v}+tb, \bar{v}'+b\big) \geq f\big(t, \alpha a, \|\bar{u}'\|_\infty+a, \alpha b, \|\bar{v}'\|_\infty+b\big) > k$$

for all $t \in (\alpha, \beta)$, where α and β are as in (H1). Take $\rho_1 = \Lambda k \int_\alpha^\beta G(1, s)\, ds$. Then, for $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_\rho$ where $\rho \leq \rho_1$,

$$\begin{aligned}
\|T(\bar{u}, \bar{v})\| \geq \|A_2(\bar{u}, \bar{v})\|_\infty &\geq \lambda \int_0^1 G(1, s) f\big(s, \bar{u}+sa, \bar{u}'+a, \bar{v}+sb, \bar{v}'+b\big)\, ds \\
&\geq \lambda \int_\alpha^\beta G(1, s) f\big(s, \alpha a\, \|\bar{u}'\|_\infty+a, \alpha b, \|\bar{v}'\|_\infty+b\big)\, ds \\
&> \lambda k \int_\alpha^\beta G(1, s)\, ds \\
&= \lambda k \|(\bar{u}, \bar{v})\| \int_\alpha^\beta \frac{G(1, s)}{\|(\bar{u}, \bar{v})\|}\, ds \\
&\geq \Lambda k \|(\bar{u}, \bar{v})\| \int_\alpha^\beta \frac{G(1, s)}{\|(\bar{u}, \bar{v})\|}\, ds \\
&= \frac{\rho_1}{\rho} \|(\bar{u}, \bar{v})\| \\
&\geq \|(\bar{u}, \bar{v})\|.
\end{aligned}$$

□

Lemma 3.3. *Suppose (H0), (H2) and (H4) hold and let $\rho^* > 0$ be fixed. Then given $\lambda > 0$, there is a $\rho_2 \in (0, \rho^*)$ and a $\delta > 0$ such that for every $(a, b) \in [0, \infty)^2$, with $0 < a + b < \delta$, we have*

$$\|T(\bar{u}, \bar{v})\| \leq \|(\bar{u}, \bar{v})\|$$

for $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho_2}$.

Proof. Let $\lambda > 0$. Pick $\epsilon > 0$ so that $\lambda\epsilon < \frac{1}{3}$. Then, by (H2), we can find a $\rho_2 \in (0, \rho^*)$ such that, for all $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 = \rho_2$ and $a + b \leq \rho_2$ with $\rho_2 < \frac{1}{2}q$, where $q > 0$ is as in (H4), we have

$$f(t, x_1 + a, x_2, x_3 + b, x_4) < \epsilon\big[(x_1 + a) + x_2 + (x_3 + b) + x_4\big]$$

for $t \in [0, 1]$.

Take $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho_2}$, and suppose $a+b \le \rho_2$. Notice that there exists $c \in (0, 1]$ such that $\bar{u}' + a \ge c\|\bar{u}'\|_\infty$ and $\bar{v}' + b \ge c\|\bar{v}'\|_\infty$. Then, for $t \in [0, 1]$, we have

$$\begin{aligned} A_2(\bar{u}, \bar{v})(t) &= \lambda \int_0^1 G(t, s) f\big(s, \bar{u}+sa, \bar{u}'+a, \bar{v}+sb, \bar{v}'+b\big)\, ds \\ &\le \lambda \int_0^1 G(t, s) f\big(s, \|\bar{u}\|_\infty+a, c\|\bar{u}'\|_\infty, \|\bar{v}\|_\infty+b, c\|\bar{v}'\|_\infty\big)\, ds \\ &< \lambda\epsilon\big[\|\bar{u}\|_\infty+c\|\bar{u}'\|_\infty+\|\bar{v}\|_\infty+c\|\bar{v}'\|_\infty+(a+b)\big] \int_0^1 G(t, s)\, ds \\ &\le \lambda\epsilon\big[\|(\bar{u}, \bar{v})\|+(a+b)\big] \int_0^1 G(t, s)\, ds \\ &\le 2\lambda\epsilon\|(\bar{u}, \bar{v})\| \int_0^1 G(t, s)\, ds \\ &\le \lambda\epsilon\|(\bar{u}, \bar{v})\|. \end{aligned}$$

Using a similar argument to the one above, we see that

$$\begin{aligned} A_2'(\bar{u}, \bar{v})(t) &= \lambda \int_0^1 \frac{\partial}{\partial t} G(t, s) f\big(s, \bar{u}+sa, \bar{u}'+a, \bar{v}+sb, \bar{v}'+b\big)\, ds \\ &\le 2\lambda\epsilon\|(\bar{u}, \bar{v})\| \int_0^1 \frac{\partial}{\partial t} G(t, s)\, ds \\ &\le 2\lambda\epsilon\|(\bar{u}, \bar{v})\|. \end{aligned}$$

In other words,

$$\|A_2(\bar{u}, \bar{v})\|_\infty + \|A_2'(\bar{u}, \bar{v})\|_\infty \le 3\lambda\epsilon\|(\bar{u}, \bar{v})\|.$$

By (H4), since $\big[(\|\bar{u}\|_\infty+a)+\|\bar{u}'\|_\infty+(\|\bar{v}\|_\infty+b)+\|\bar{v}'\|_\infty\big] \le 2\rho_2 < q$, we have

$$\begin{aligned} &g\big(t, \|\bar{u}\|_\infty+a, \|\bar{u}'\|_\infty, \|\bar{v}\|_\infty+b, \|\bar{v}'\|_\infty\big) \\ &\qquad\qquad \le \gamma\big(\|\bar{u}\|_\infty+a+\|\bar{u}'\|_\infty+\|\bar{v}\|_\infty+b+\|\bar{v}'\|_\infty\big). \end{aligned}$$

Let $\delta' < 1$ and set $\delta = \delta'\rho_2$. Then for $a+b < \delta$, $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho_2}$, and $t \in [0, 1]$, we have

$$\begin{aligned} A_1(\bar{u}, \bar{v})(t) &= \int_0^1 G(t, s) g\big(s, \bar{u}+sa, \bar{u}'+a, \bar{v}+sb, \bar{v}'+b\big)\, ds \\ &\le \int_0^1 G(t, s) g\big(s, \|\bar{u}\|_\infty+a, c\|\bar{u}'\|_\infty, \|\bar{v}\|_\infty+b, c\|\bar{v}'\|_\infty\big)\, ds \\ &\le \gamma\big[\|\bar{u}\|_\infty+c\|\bar{u}'\|_\infty+\|\bar{v}\|_\infty+c\|\bar{v}'\|_\infty+(a+b)\big] \int_0^1 G(t, s)\, ds \\ &\le \gamma\big[\|(\bar{u}, \bar{v})\|+(a+b)\big] \int_0^1 G(t, s)\, ds \end{aligned}$$

$$
\begin{aligned}
&< \gamma(1+\delta')\|(\bar{u},\bar{v})\| \int_0^1 G(t,s)\,ds \\
&\le \tfrac{1}{2}\gamma(1+\delta')\|(\bar{u},\bar{v})\|,
\end{aligned}
$$

where c is as above. And similarly,

$$
\begin{aligned}
A_1'(\bar{u},\bar{v})(t) &= \int_0^1 \frac{\partial}{\partial t}G(t,s)g\big(s,\bar{u}+sa,\bar{u}'+a,\bar{v}+sb,\bar{v}'+b\big)\,ds \\
&< \gamma(1+\delta')\|(\bar{u},\bar{v})\| \int_0^1 \frac{\partial}{\partial t}G(t,s)\,ds \\
&\le \gamma(1+\delta')\|(\bar{u},\bar{v})\|.
\end{aligned}
$$

Hence,

$$
\|A_1(\bar{u},\bar{v})\|_\infty + \|A_1'(\bar{u},\bar{v})\|_\infty < \tfrac{3}{2}\gamma(1+\delta')\|(\bar{u},\bar{v})\|.
$$

Thus, for $a+b<\delta$, we have

$$
\begin{aligned}
\|T(\bar{u},\bar{v})\| &= \|A_1(\bar{u},\bar{v})\|_\infty + \|A_1'(\bar{u},\bar{v})\|_\infty + \|A_2(\bar{u},\bar{v})\|_\infty + \|A_2'(\bar{u},\bar{v})\|_\infty \\
&< \big[\tfrac{3}{2}\gamma(1+\delta') + 3\lambda\epsilon\big]\|(\bar{u},\bar{v})\|.
\end{aligned}
$$

For small enough ϵ and δ', it follows that $\|T(\bar{u},\bar{v})\| \le \|(\bar{u},\bar{v})\|$. □

Lemma 3.4. *Let $\delta>0$. Suppose $0<a+b<\delta$ and (H0), (H3) and (H5) hold. Then, for every $\lambda>0$, there is a $\rho_3=\rho_3(\delta,\lambda)$ such that for all $\rho\ge\rho_3$,*

$$
\|T(\bar{u},\bar{v})\| \le \|(\bar{u},\bar{v})\|,
$$

where $(\bar{u},\bar{v})\in C\cap\partial\Omega_\rho$.

Proof. Let $\delta>0$, $0<a+b<\delta$ and let $(x_1,x_2,x_3,x_4)\in[0,\infty)^4$. By (H5) and the nondecreasing/nonincreasing properties of g as in (H0), given any $q_1\ge\hat{\rho}$, we have

$$
g(t,x_1+a,x_2,x_3+a,x_4) \le \eta(x_1+a+x_2+x_3+b+x_4)
$$

for $x_1+x_2+x_3+x_4\ge q_1$ and $t\in[0,1]$.

Let $\epsilon>0$ and pick $q_1\ge\hat{\rho}$ large enough so that $\epsilon>\eta\delta/q_1$. Let $x_1+x_2+x_3+x_4\ge q_1$. Then

$$
\begin{aligned}
g(t,x_1+a,x_2,x_3+a,x_4) &\le \eta(x_1+x_2+x_3+x_4)+\eta(a+b) \\
&< \eta(x_1+x_2+x_3+x_4)+\epsilon(x_1+x_2+x_3+x_4) \\
&= (\eta+\epsilon)(x_1+x_2+x_3+x_4).
\end{aligned}
$$

Let $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{q_1}$. Pick $c \in (0, 1]$ such that $\bar{u}' + a \geq c\|\bar{u}'\|_\infty$ and $\bar{v}' + b \geq c\|\bar{v}'\|_\infty$. Then for $t \in [0, 1]$,

$$\begin{aligned} A_1(\bar{u}, \bar{v})(t) &= \int_0^1 G(t, s) g\big(s,\, \bar{u} + sa,\, \bar{u}' + a,\, \bar{v} + sb,\, \bar{v}' + b\big)\, ds \\ &\leq \int_0^1 G(t, s) g\big(s,\, \|\bar{u}\|_\infty + a,\, c\|\bar{u}'\|_\infty,\, \|\bar{v}\|_\infty + b,\, c\|\bar{v}'\|_\infty\big)\, ds \\ &< (\eta + \epsilon)\|(\bar{u}, \bar{v})\| \int_0^1 G(t, s)\, ds. \end{aligned}$$

A similar argument shows that

$$\begin{aligned} A_1'(\bar{u}, \bar{v})(t) &= \int_0^1 \frac{\partial}{\partial t} G(t, s) g\big(s,\, \bar{u} + sa,\, \bar{u}' + a,\, \bar{v} + sb,\, \bar{v}' + b\big)\, ds \\ &< (\eta + \epsilon)\|(\bar{u}, \bar{v})\| \int_0^1 \frac{\partial}{\partial t} G(t, s)\, ds. \end{aligned}$$

Combining these inequalities, we see that

$$\|A_1(\bar{u}, \bar{v})\|_\infty + \|A_1'(\bar{u}, \bar{v})\|_\infty < \tfrac{3}{2}(\eta + \epsilon)\|(\bar{u}, \bar{v})\|.$$

Now consider $A_2(\bar{u}, \bar{v})(t)$. Let $\delta' > 0$. Then, by (H0) and (H3), there is a $q_2 > 0$ such that for all $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 \geq q_2$, we have

$$f(t,\, x_1 + a,\, x_2,\, x_3 + b,\, x_4) \leq \delta'(x_1 + a + x_2 + x_3 + b + x_4)$$

for every $t \in [0, 1]$. Let $q_3 = \max\{\delta, q_2\}$. Noting that $a + b < \delta$, for $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 \geq q_3$, we have

$$\begin{aligned} f(t,\, x_1 + a,\, x_2,\, x_3 + b,\, x_4) &\leq \delta'\big[(x_1 + x_2 + x_3 + x_4) + q_3\big] \\ &\leq 2\delta'(x_1 + x_2 + x_3 + x_4). \end{aligned}$$

Then for $t \in [0, 1]$ and any $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{q_3}$,

$$\begin{aligned} A_2(\bar{u}, \bar{v}) &= \lambda \int_0^1 G(t, s) f\big(s,\, \bar{u} + sa,\, \bar{u}' + a,\, \bar{v} + sb,\, \bar{v}' + b\big)\, ds \\ &\leq \lambda \int_0^1 G(t, s) f\big(s,\, \|\bar{u}\|_\infty + a,\, c\|\bar{u}'\|_\infty,\, \|\bar{v}\|_\infty + b,\, c\|\bar{v}'\|_\infty\big)\, ds \\ &< \lambda \cdot 2\delta'\|(\bar{u}, \bar{v})\| \int_0^1 G(t, s)\, ds, \end{aligned}$$

where c is as above. And similarly,

$$\begin{aligned} A_2(\bar{u},\bar{v}) &= \lambda \int_0^1 \frac{\partial}{\partial t} G(t,s) f\big(s,\,\bar{u}+sa,\,\bar{u}'+a,\,\bar{v}+sb,\,\bar{v}'+b\big)\,ds \\ &< \lambda\cdot 2\delta' \|(\bar{u},\bar{v})\| \int_0^1 \frac{\partial}{\partial t} G(t,s)\,ds. \end{aligned}$$

Combining these inequalities, we see that

$$\|A_2(\bar{u},\bar{v})\|_\infty + \|A_2'(\bar{u},\bar{v})\|_\infty < 3\lambda\delta'\|(\bar{u},\bar{v})\|.$$

Take $\rho_3 = \max\{q_1, q_3\}$ and let $\rho \ge \rho_3$. Then given $(\bar{u},\bar{v}) \in C \cap \partial\Omega_\rho$, we see that

$$\begin{aligned} \|T(\bar{u},\bar{v})\| &= \|A_1(\bar{u},\bar{v})\|_\infty + \|A_1'(\bar{u},\bar{v})\|_\infty + \|A_2(\bar{u},\bar{v})\|_\infty + \|A_2'(\bar{u},\bar{v})\|_\infty \\ &< \big[\tfrac{1}{2}(6\lambda\delta' + 3(\eta+\epsilon))\big]\,\|(\bar{u},\bar{v})\|. \end{aligned}$$

Recall by (H5) that $\eta \in \big(0, \tfrac{2}{3}\big)$. Pick ϵ and δ' small enough that $6\lambda\delta' + 3\epsilon \le 2 - 3\eta$. Thus, we have the desired result. □

4. The main result

Theorem 4.1. *Let continuous functions $f, g : [0,1] \times [0,\infty)^4 \to [0,\infty)$ satisfy hypotheses (H0)–(H5). Then there exists $\Lambda > 0$ such that given $\lambda \ge \Lambda$, there exists $\delta > 0$ such that for every $a, b \ge 0$ satisfying $0 < a + b < \delta$, the system (5)–(8) has at least three positive solutions.*

Proof. Suppose f, g satisfy hypotheses (H0)–(H5). Let $\rho^* > 0$ be fixed. By Lemma 3.1, there is $\Lambda > 0$ such that, for every $\lambda \ge \Lambda$ and $a, b \ge 0$,

$$\|T(\bar{u},\bar{v})\| \ge \|(\bar{u},\bar{v})\| \quad \text{for } (\bar{u},\bar{v}) \in C \cap \partial\Omega_{\rho^*}.$$

Now, fix $\lambda \ge \Lambda$. Lemmas 3.2 through 3.4 give that there is $\delta > 0$ and $\rho_1, \rho_2, \rho_3 > 0$ satisfying $\rho_1 < \rho_2 < \rho^* < \rho_3$ such that for $(a,b) \in [0,\infty)^2$ with $0 < a + b < \delta$,

$$\begin{aligned} \|T(\bar{u},\bar{v})\| &\ge \|(\bar{u},\bar{v})\| \quad \text{for } (\bar{u},\bar{v}) \in C \cap \partial\Omega_{\rho_1}, \\ \|T(\bar{u},\bar{v})\| &\le \|(\bar{u},\bar{v})\| \quad \text{for } (\bar{u},\bar{v}) \in C \cap \partial\Omega_{\rho_2}, \\ \|T(\bar{u},\bar{v})\| &\le \|(\bar{u},\bar{v})\| \quad \text{for } (\bar{u},\bar{v}) \in C \cap \partial\Omega_{\rho_3}. \end{aligned}$$

Applying the Guo–Krasnosel'skiĭ fixed point theorem three times, we get the existence of three positive solutions, $(\bar{u}_1,\bar{v}_1)$, $(\bar{u}_2,\bar{v}_2)$, $(\bar{u}_3,\bar{v}_3) \in C$ such that

$$\rho_1 < \|(\bar{u}_1,\bar{v}_1)\| < \rho_2 < \|(\bar{u}_2,\bar{v}_2)\| < \rho^* < \|(\bar{u}_3,\bar{v}_3)\| < \rho_3. \qquad \square$$

Recall that solutions to the system (5)–(8) are in one-to-one correspondence with those of the system (1)–(4). Thus we have our desired result.

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