

involve

a journal of mathematics

The multiplicity of solutions for
a system of second-order differential equations

Olivia Bennett, Daniel Brumley, Britney Hopkins,
Kristi Karber and Thomas Milligan



The multiplicity of solutions for a system of second-order differential equations

Olivia Bennett, Daniel Brumley, Britney Hopkins,
Kristi Karber and Thomas Milligan

(Communicated by John Baxley)

Making use of the Guo–Krasnosel’skiĭ fixed point theorem multiple times, we establish the existence of at least three positive solutions for the system of second-order differential equations $-u''(t) = g(t, u(t), u'(t), v(t), v'(t))$ and $-v''(t) = \lambda f(t, u(t), u'(t), v(t), v'(t))$ for $t \in (0, 1)$ with right focal boundary conditions $u(0) = v(0) = 0$, $u'(1) = a$, and $v'(1) = b$, where $f, g : [0, 1] \times [0, \infty)^4 \rightarrow [0, \infty)$ are continuous, $a, b, \lambda \geq 0$, and $a + b > 0$. Our technique involves transforming the system of differential equations to a new system with homogeneous boundary conditions prior to applying the aforementioned fixed point theorem.

1. Introduction

Showing the existence of multiple positive solutions for boundary value problems is an active field of study due to the applications that arise in modeling real world phenomena. A classic example based on beam analysis, presented by Agarwal [1989], gives an existence and uniqueness result of the fourth-order problem $x^{(4)} = f(t, x, x', x'', x^{(3)})$. Additionally, do Ó, Lorca, and Ubilla [do Ó et al. 2008] studied the fourth-order nonhomogeneous boundary value problem,

$$\begin{aligned} u^{(4)} &= \lambda h(t, u, u''), & t \in (0, 1), \\ u(0) &= u''(0) = 0, \\ u(1) &= a, & u''(1) = b. \end{aligned}$$

Utilizing a technique of rewriting the fourth-order problem as a system of second-order differential equations, the authors guaranteed existence of multiple positive solutions by ultimately applying the Guo–Krasnosel’skiĭ fixed point theorem [Krasnosel’skiĭ 1964]. Hopkins [2015] extended this process to establish multiple solutions to the differential equation $u^{(2n)} = \lambda h(t, u, u'', \dots, u^{2(n-1)})$ satisfying

MSC2010: 34B18.

Keywords: differential equations, boundary value problem, multiple solutions, positive solutions.

right focal boundary conditions. Henderson and Hopkins [2010] applied this same technique to a similar fourth-order difference equation. In this work, we consider the system of second-order differential equations

$$-u''(t) = g(t, u(t), u'(t), v(t), v'(t)), \quad (1)$$

$$-v''(t) = \lambda f(t, u(t), u'(t), v(t), v'(t)), \quad (2)$$

$$u(0) = v(0) = 0, \quad (3)$$

$$u'(1) = a, \quad v'(1) = b, \quad (4)$$

where $f, g : [0, 1] \times [0, \infty)^4 \rightarrow [0, \infty)$ are continuous, $\lambda, a, b \geq 0$ and $a + b > 0$. The novelty of our paper is that the functions f and g contain both even- and odd-order derivatives.

In Section 2 of this paper, we consider a transformation of (1)–(4) that satisfies homogeneous boundary conditions. We also introduce some preliminaries and the conditions under which we can eventually apply the Guo–Krasnosel'skiĭ fixed point theorem. In Section 3 we introduce and prove a sequence of lemmas giving bounds on a defined operator. This culminates in the main result, given in Section 4 where we apply the Guo–Krasnosel'skiĭ fixed point theorem multiple times, yielding at least three positive solutions.

2. Preliminaries

We will prove the existence of multiple solutions for the system of second-order differential equations (1)–(4) by applying the transformation $\bar{u}(t) = u(t) - at$ and $\bar{v}(t) = v(t) - bt$, which gives

$$-\bar{u}''(t) = g(t, \bar{u}(t) + ta, \bar{u}'(t) + a, \bar{v}(t) + tb, \bar{v}'(t) + b), \quad (5)$$

$$-\bar{v}''(t) = \lambda f(t, \bar{u}(t) + ta, \bar{u}'(t) + a, \bar{v}(t) + tb, \bar{v}'(t) + b), \quad (6)$$

$$\bar{u}(0) = \bar{v}(0) = 0, \quad (7)$$

$$\bar{u}'(1) = 0, \quad \bar{v}'(1) = 0, \quad (8)$$

where $a, b, \lambda \geq 0$ and $a + b > 0$. Notice that solutions to (5)–(8) are in one-to-one correspondence with (1)–(4). Furthermore, suppose the following hypotheses on f and g are satisfied.

(H0) The functions $f, g : [0, 1] \times [0, \infty)^4 \rightarrow [0, \infty)$ are continuous and are nondecreasing in the second and fourth variables and nonincreasing in the third and fifth variables.

(H1) There exist $\alpha, \beta \in (0, 1)$, $\alpha < \beta$, such that given $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 \neq 0$, there exists $k > 0$ such that for $t \in [\alpha, \beta]$,

$$f(t, x_1, x_2, x_3, x_4) > k.$$

(H2) For $t \in (0, 1)$,

$$\lim_{x_1+x_2+x_3+x_4 \rightarrow 0^+} \frac{f(t, x_1, x_2, x_3, x_4)}{x_1 + x_2 + x_3 + x_4} = 0$$

uniformly.

(H3) For $t \in (0, 1)$,

$$\lim_{x_1+x_2+x_3+x_4 \rightarrow \infty} \frac{f(t, x_1, x_2, x_3, x_4)}{x_1 + x_2 + x_3 + x_4} = 0$$

uniformly.

(H4) There exist $\gamma \in (0, \frac{2}{3})$ and $q > 0$ such that for $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 < q$,

$$g(t, x_1, x_2, x_3, x_4) \leq \gamma(x_1 + x_2 + x_3 + x_4) \quad \text{for } t \in [0, 1].$$

(H5) There exist $\eta \in (0, \frac{2}{3})$ and $\hat{\rho} > 0$ such that for $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 > \hat{\rho}$,

$$g(t, x_1, x_2, x_3, x_4) \leq \eta(x_1 + x_2 + x_3 + x_4) \quad \text{for } t \in [0, 1].$$

Solutions to (5)–(8), provided they exist, are of the form

$$\bar{u}(t) = \int_0^1 G(t, s)g(s, \bar{u}(s) + as, \bar{u}'(s) + a, \bar{v}(s) + bs, \bar{v}'(s) + b) ds, \quad (9)$$

$$\bar{v}(t) = \lambda \int_0^1 G(t, s)f(s, \bar{u}(s) + as, \bar{u}'(s) + a, \bar{v}(s) + bs, \bar{v}'(s) + b) ds, \quad (10)$$

where $G(t, s)$ is the Green’s function

$$G(t, s) = \begin{cases} t & \text{if } 0 \leq t \leq s \leq 1, \\ s & \text{if } 0 \leq s \leq t \leq 1. \end{cases}$$

Since $G(t, s)$ is clearly nonnegative and f and g are nonnegative by assumption, it follows that solutions u and v are also nonnegative. Some other useful properties on $G(t, s)$ are that

$$\max_{t \in [0, 1]} \int_0^1 G(t, s) ds = \frac{1}{2} \quad \text{and} \quad \max_{t \in [0, 1]} \int_0^1 \left| \frac{\partial}{\partial t} G(t, s) \right| ds = 1.$$

In order to make use of the Guo–Krasnosel’skiĭ fixed point theorem, we will need a Banach space and a cone, as well as an operator T . Let $(X, \|\cdot\|)$ denote the Banach space $X = C^1([0, 1], \mathbb{R}) \times C^1([0, 1], \mathbb{R})$ endowed with the norm

$$\|(\bar{u}, \bar{v})\| = \|\bar{u}\|_\infty + \|\bar{u}'\|_\infty + \|\bar{v}\|_\infty + \|\bar{v}'\|_\infty,$$

where $\|\bar{u}\|_\infty = \sup_{t \in [0, 1]} |\bar{u}(t)|$.

Recall that a cone, C , in X is a nonempty, closed, convex subset of X satisfying:

- (1) If $x \in C$, and $\lambda > 0$, then $\lambda x \in C$.
- (2) If $x \in C$ and $-x \in C$, then $x = 0$.

Define $C \subset X$ to be the cone

$$C = \{(\bar{u}, \bar{v}) \in X : (\bar{u}, \bar{v})(0) = (\bar{u}', \bar{v}')(1) = (0, 0) \text{ and } \bar{u}, \bar{v} \text{ are concave}\}.$$

The fact that C is a cone follows directly from the definition. Moreover, let Ω_p denote the open set $\Omega_p = \{(\bar{u}, \bar{v}) \in X : \|(\bar{u}, \bar{v})\| < p\}$. Finally, define $T : X \rightarrow X$ to be the operator $T(\bar{u}, \bar{v}) = (A_1(\bar{u}, \bar{v}), A_2(\bar{u}, \bar{v}))$, where

$$A_1 = \int_0^1 G(t, s)g(s, \bar{u}(s) + as, \bar{u}'(s) + a, \bar{v}(s) + bs, \bar{v}'(s) + b) ds$$

and

$$A_2 = \lambda \int_0^1 G(t, s)f(s, \bar{u}(s) + as, \bar{u}'(s) + a, \bar{v}(s) + bs, \bar{v}'(s) + b) ds.$$

Consider the following lemma, which provides a useful property of T .

Lemma 2.1. *The operator $T : C \rightarrow C$ is completely continuous.*

We note that one can use a standard Arzelà–Ascoli argument to show that T is completely continuous; see [Hopkins 2009].

In the next section, we will take advantage of the following lemma.

Lemma 2.2. *Let $\bar{u}(t)$ be a nonnegative concave function which is continuous on $[0, 1]$. Then for all $\alpha, \beta \in (0, 1)$, with $\alpha < \beta$, we have*

$$\inf_{t \in [\alpha, \beta]} \bar{u}(t) \geq \alpha(1 - \beta)\|\bar{u}\|_\infty.$$

For a proof of Lemma 2.2, see [Hopkins 2009].

Since we will be using the Guo–Krasnosel’skiĭ fixed point theorem multiple times to acquire our main result, we end the section with the statement of this theorem.

Theorem 2.3 (Guo–Krasnosel’skiĭ fixed point theorem). *Let $(X, \|\cdot\|)$ be a Banach space and $C \subset X$ be a cone. Suppose Ω_1, Ω_2 are open subsets of X satisfying $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $T : C \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow C$ is a completely continuous operator such that either*

- (1) $\|Tu\| \leq \|u\|$ for $u \in C \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in C \cap \partial\Omega_2$, or
- (2) $\|Tu\| \geq \|u\|$ for $u \in C \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in C \cap \partial\Omega_2$,

then T has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Technical results

In this section we give a sequence of four lemmas that allow us to obtain the estimates needed to apply the Guo–Krasnosel’skiĭ fixed point theorem.

Lemma 3.1. *Suppose (H0) and (H1) hold and let $\rho^* > 0$. Then there is a $\Lambda > 0$ such that, for every $\lambda \geq \Lambda$ and $(a, b) \in [0, \infty)^2$,*

$$\|T(\bar{u}, \bar{v})\| \geq \|(\bar{u}, \bar{v})\|$$

for $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho^*}$.

Proof. Let $\rho^* > 0$ and let $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho^*}$. Let $r = \alpha(1 - \beta)$, where α and β are as in (H1) and note $r \in (0, 1)$. Furthermore, choose $c \geq 1$ so that both $\bar{u}' + a \leq c\|\bar{u}'\|_\infty$ and $\bar{v}' + b \leq c\|\bar{v}'\|_\infty$ hold for $t \in [\alpha, \beta]$. Define

$$M = \inf \left\{ \frac{f(t, ra_1, ca_2, ra_3, ca_4)}{r(a_1 + a_3) + c(a_2 + a_4)} : t \in [\alpha, \beta], a_1, a_2, a_3 > 0, a_4 \geq 0, \right. \\ \left. \text{and } a_1 + a_2 + a_3 + a_4 = p^* \right\}.$$

The existence of a positive M follows from (H1). Set $\Lambda \geq [Mr \int_\alpha^\beta G(1, s) ds]^{-1}$.

As $(\bar{u}, \bar{v}) \in C$, by Lemma 2.2, we have $\bar{u}(t) + at \geq \bar{u}(t) \geq r\|\bar{u}\|_\infty$. Moreover, due to the nondecreasing property of f in the second and fourth variables and its nonincreasing property in the third and fifth variables, we see that

$$\begin{aligned} \|T(\bar{u}, \bar{v})\| &\geq \|A_2(\bar{u}, \bar{v})\|_\infty \\ &\geq \lambda \int_0^1 G(1, s) f(s, \bar{u} + sa, \bar{u}' + a, \bar{v} + sb, \bar{v}' + b) ds \\ &\geq \lambda \int_\alpha^\beta G(1, s) f(s, r\|\bar{u}\|_\infty, c\|\bar{u}'\|_\infty, r\|\bar{v}\|_\infty, c\|\bar{v}'\|_\infty) ds \\ &\geq \lambda [r(\|\bar{u}\|_\infty + \|\bar{v}\|_\infty) + c(\|\bar{u}'\|_\infty + \|\bar{v}'\|_\infty)] \\ &\quad \times \int_\alpha^\beta G(1, s) \frac{f(s, r\|\bar{u}\|_\infty, c\|\bar{u}'\|_\infty, r\|\bar{v}\|_\infty, c\|\bar{v}'\|_\infty)}{r(\|\bar{u}\|_\infty + \|\bar{v}\|_\infty) + c(\|\bar{u}'\|_\infty + \|\bar{v}'\|_\infty)} ds \\ &\geq \lambda M [r(\|\bar{u}\|_\infty + \|\bar{v}\|_\infty) + c(\|\bar{u}'\|_\infty + \|\bar{v}'\|_\infty)] \int_\alpha^\beta G(1, s) ds \\ &\geq \lambda Mr (\|\bar{u}\|_\infty + \|\bar{u}'\|_\infty + \|\bar{v}\|_\infty + \|\bar{v}'\|_\infty) \int_\alpha^\beta G(1, s) ds \\ &\geq \lambda Mr \|(\bar{u}, \bar{v})\| \int_\alpha^\beta G(1, s) ds \\ &\geq \Lambda Mr \|(\bar{u}, \bar{v})\| \int_\alpha^\beta G(1, s) ds \\ &\geq \|(\bar{u}, \bar{v})\|. \end{aligned}$$

□

Lemma 3.2. Fix $\Lambda > 0$. Suppose (H0) and (H1) hold. Then, for all $\lambda \geq \Lambda$ and for all $(a, b) \in [0, \infty)^2$, with $a + b > 0$, there exists a $\rho_1 = \rho_1(\Lambda, a, b)$ such that for every $\rho \in (0, \rho_1)$, we have

$$\|T(\bar{u}, \bar{v})\| \geq \|(\bar{u}, \bar{v})\|$$

for all $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_\rho$.

Proof. Fix $\Lambda > 0$. By (H1) and the nonincreasing/nondecreasing properties of f , there exists $k > 0$ such that

$$f(t, \bar{u} + ta, \bar{u}' + a, \bar{v} + tb, \bar{v}' + b) \geq f(t, \alpha a, \|\bar{u}'\|_\infty + a, \alpha b, \|\bar{v}'\|_\infty + b) > k$$

for all $t \in (\alpha, \beta)$, where α and β are as in (H1). Take $\rho_1 = \Lambda k \int_\alpha^\beta G(1, s) ds$. Then, for $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_\rho$ where $\rho \leq \rho_1$,

$$\begin{aligned} \|T(\bar{u}, \bar{v})\| &\geq \|A_2(\bar{u}, \bar{v})\|_\infty \geq \lambda \int_0^1 G(1, s) f(s, \bar{u} + sa, \bar{u}' + a, \bar{v} + sb, \bar{v}' + b) ds \\ &\geq \lambda \int_\alpha^\beta G(1, s) f(s, \alpha a \|\bar{u}'\|_\infty + a, \alpha b, \|\bar{v}'\|_\infty + b) ds \\ &> \lambda k \int_\alpha^\beta G(1, s) ds \\ &= \lambda k \|(\bar{u}, \bar{v})\| \int_\alpha^\beta \frac{G(1, s)}{\|(\bar{u}, \bar{v})\|} ds \\ &\geq \Lambda k \|(\bar{u}, \bar{v})\| \int_\alpha^\beta \frac{G(1, s)}{\|(\bar{u}, \bar{v})\|} ds \\ &= \frac{\rho_1}{\rho} \|(\bar{u}, \bar{v})\| \\ &\geq \|(\bar{u}, \bar{v})\|. \end{aligned} \quad \square$$

Lemma 3.3. Suppose (H0), (H2) and (H4) hold and let $\rho^* > 0$ be fixed. Then given $\lambda > 0$, there is a $\rho_2 \in (0, \rho^*)$ and a $\delta > 0$ such that for every $(a, b) \in [0, \infty)^2$, with $0 < a + b < \delta$, we have

$$\|T(\bar{u}, \bar{v})\| \leq \|(\bar{u}, \bar{v})\|$$

for $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho_2}$.

Proof. Let $\lambda > 0$. Pick $\epsilon > 0$ so that $\lambda\epsilon < \frac{1}{3}$. Then, by (H2), we can find a $\rho_2 \in (0, \rho^*)$ such that, for all $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 = \rho_2$ and $a + b \leq \rho_2$ with $\rho_2 < \frac{1}{2}q$, where $q > 0$ is as in (H4), we have

$$f(t, x_1 + a, x_2, x_3 + b, x_4) < \epsilon [(x_1 + a) + x_2 + (x_3 + b) + x_4]$$

for $t \in [0, 1]$.

Take $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho_2}$, and suppose $a + b \leq \rho_2$. Notice that there exists $c \in (0, 1]$ such that $\bar{u}' + a \geq c\|\bar{u}'\|_\infty$ and $\bar{v}' + b \geq c\|\bar{v}'\|_\infty$. Then, for $t \in [0, 1]$, we have

$$\begin{aligned}
 A_2(\bar{u}, \bar{v})(t) &= \lambda \int_0^1 G(t, s) f(s, \bar{u} + sa, \bar{u}' + a, \bar{v} + sb, \bar{v}' + b) ds \\
 &\leq \lambda \int_0^1 G(t, s) f(s, \|\bar{u}\|_\infty + a, c\|\bar{u}'\|_\infty, \|\bar{v}\|_\infty + b, c\|\bar{v}'\|_\infty) ds \\
 &< \lambda \epsilon [\|\bar{u}\|_\infty + c\|\bar{u}'\|_\infty + \|\bar{v}\|_\infty + c\|\bar{v}'\|_\infty + (a + b)] \int_0^1 G(t, s) ds \\
 &\leq \lambda \epsilon [\|(\bar{u}, \bar{v})\| + (a + b)] \int_0^1 G(t, s) ds \\
 &\leq 2\lambda \epsilon \|(\bar{u}, \bar{v})\| \int_0^1 G(t, s) ds \\
 &\leq \lambda \epsilon \|(\bar{u}, \bar{v})\|.
 \end{aligned}$$

Using a similar argument to the one above, we see that

$$\begin{aligned}
 A'_2(\bar{u}, \bar{v})(t) &= \lambda \int_0^1 \frac{\partial}{\partial t} G(t, s) f(s, \bar{u} + sa, \bar{u}' + a, \bar{v} + sb, \bar{v}' + b) ds \\
 &\leq 2\lambda \epsilon \|(\bar{u}, \bar{v})\| \int_0^1 \frac{\partial}{\partial t} G(t, s) ds \\
 &\leq 2\lambda \epsilon \|(\bar{u}, \bar{v})\|.
 \end{aligned}$$

In other words,

$$\|A_2(\bar{u}, \bar{v})\|_\infty + \|A'_2(\bar{u}, \bar{v})\|_\infty \leq 3\lambda \epsilon \|(\bar{u}, \bar{v})\|.$$

By (H4), since $[(\|\bar{u}\|_\infty + a) + \|\bar{u}'\|_\infty + (\|\bar{v}\|_\infty + b) + \|\bar{v}'\|_\infty] \leq 2\rho_2 < q$, we have

$$\begin{aligned}
 g(t, \|\bar{u}\|_\infty + a, \|\bar{u}'\|_\infty, \|\bar{v}\|_\infty + b, \|\bar{v}'\|_\infty) \\
 \leq \gamma (\|\bar{u}\|_\infty + a + \|\bar{u}'\|_\infty + \|\bar{v}\|_\infty + b + \|\bar{v}'\|_\infty).
 \end{aligned}$$

Let $\delta' < 1$ and set $\delta = \delta' \rho_2$. Then for $a + b < \delta$, $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho_2}$, and $t \in [0, 1]$, we have

$$\begin{aligned}
 A_1(\bar{u}, \bar{v})(t) &= \int_0^1 G(t, s) g(s, \bar{u} + sa, \bar{u}' + a, \bar{v} + sb, \bar{v}' + b) ds \\
 &\leq \int_0^1 G(t, s) g(s, \|\bar{u}\|_\infty + a, c\|\bar{u}'\|_\infty, \|\bar{v}\|_\infty + b, c\|\bar{v}'\|_\infty) ds \\
 &\leq \gamma [\|\bar{u}\|_\infty + c\|\bar{u}'\|_\infty + \|\bar{v}\|_\infty + c\|\bar{v}'\|_\infty + (a + b)] \int_0^1 G(t, s) ds \\
 &\leq \gamma [\|(\bar{u}, \bar{v})\| + (a + b)] \int_0^1 G(t, s) ds
 \end{aligned}$$

$$\begin{aligned} &< \gamma(1+\delta')\|(\bar{u}, \bar{v})\| \int_0^1 G(t, s) ds \\ &\leq \frac{1}{2}\gamma(1+\delta')\|(\bar{u}, \bar{v})\|, \end{aligned}$$

where c is as above. And similarly,

$$\begin{aligned} A'_1(\bar{u}, \bar{v})(t) &= \int_0^1 \frac{\partial}{\partial t} G(t, s) g(s, \bar{u} + sa, \bar{u}' + a, \bar{v} + sb, \bar{v}' + b) ds \\ &< \gamma(1+\delta')\|(\bar{u}, \bar{v})\| \int_0^1 \frac{\partial}{\partial t} G(t, s) ds \\ &\leq \gamma(1+\delta')\|(\bar{u}, \bar{v})\|. \end{aligned}$$

Hence,

$$\|A_1(\bar{u}, \bar{v})\|_\infty + \|A'_1(\bar{u}, \bar{v})\|_\infty < \frac{3}{2}\gamma(1+\delta')\|(\bar{u}, \bar{v})\|.$$

Thus, for $a + b < \delta$, we have

$$\begin{aligned} \|T(\bar{u}, \bar{v})\| &= \|A_1(\bar{u}, \bar{v})\|_\infty + \|A'_1(\bar{u}, \bar{v})\|_\infty + \|A_2(\bar{u}, \bar{v})\|_\infty + \|A'_2(\bar{u}, \bar{v})\|_\infty \\ &< \left[\frac{3}{2}\gamma(1+\delta') + 3\lambda\epsilon \right] \|(\bar{u}, \bar{v})\|. \end{aligned}$$

For small enough ϵ and δ' , it follows that $\|T(\bar{u}, \bar{v})\| \leq \|(\bar{u}, \bar{v})\|$. \square

Lemma 3.4. *Let $\delta > 0$. Suppose $0 < a + b < \delta$ and (H0), (H3) and (H5) hold. Then, for every $\lambda > 0$, there is a $\rho_3 = \rho_3(\delta, \lambda)$ such that for all $\rho \geq \rho_3$,*

$$\|T(\bar{u}, \bar{v})\| \leq \|(\bar{u}, \bar{v})\|,$$

where $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_\rho$.

Proof. Let $\delta > 0$, $0 < a + b < \delta$ and let $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$. By (H5) and the nondecreasing/nonincreasing properties of g as in (H0), given any $q_1 \geq \hat{\rho}$, we have

$$g(t, x_1 + a, x_2, x_3 + a, x_4) \leq \eta(x_1 + a + x_2 + x_3 + b + x_4)$$

for $x_1 + x_2 + x_3 + x_4 \geq q_1$ and $t \in [0, 1]$.

Let $\epsilon > 0$ and pick $q_1 \geq \hat{\rho}$ large enough so that $\epsilon > \eta\delta/q_1$. Let $x_1 + x_2 + x_3 + x_4 \geq q_1$. Then

$$\begin{aligned} g(t, x_1 + a, x_2, x_3 + a, x_4) &\leq \eta(x_1 + x_2 + x_3 + x_4) + \eta(a + b) \\ &< \eta(x_1 + x_2 + x_3 + x_4) + \epsilon(x_1 + x_2 + x_3 + x_4) \\ &= (\eta + \epsilon)(x_1 + x_2 + x_3 + x_4). \end{aligned}$$

Let $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{q_1}$. Pick $c \in (0, 1]$ such that $\bar{u}' + a \geq c\|\bar{u}'\|_\infty$ and $\bar{v}' + b \geq c\|\bar{v}'\|_\infty$. Then for $t \in [0, 1]$,

$$\begin{aligned} A_1(\bar{u}, \bar{v})(t) &= \int_0^1 G(t, s)g(s, \bar{u} + sa, \bar{u}' + a, \bar{v} + sb, \bar{v}' + b) ds \\ &\leq \int_0^1 G(t, s)g(s, \|\bar{u}\|_\infty + a, c\|\bar{u}'\|_\infty, \|\bar{v}\|_\infty + b, c\|\bar{v}'\|_\infty) ds \\ &< (\eta + \epsilon)\|(\bar{u}, \bar{v})\| \int_0^1 G(t, s) ds. \end{aligned}$$

A similar argument shows that

$$\begin{aligned} A'_1(\bar{u}, \bar{v})(t) &= \int_0^1 \frac{\partial}{\partial t} G(t, s)g(s, \bar{u} + sa, \bar{u}' + a, \bar{v} + sb, \bar{v}' + b) ds \\ &< (\eta + \epsilon)\|(\bar{u}, \bar{v})\| \int_0^1 \frac{\partial}{\partial t} G(t, s) ds. \end{aligned}$$

Combining these inequalities, we see that

$$\|A_1(\bar{u}, \bar{v})\|_\infty + \|A'_1(\bar{u}, \bar{v})\|_\infty < \frac{3}{2}(\eta + \epsilon)\|(\bar{u}, \bar{v})\|.$$

Now consider $A_2(\bar{u}, \bar{v})(t)$. Let $\delta' > 0$. Then, by (H0) and (H3), there is a $q_2 > 0$ such that for all $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 \geq q_2$, we have

$$f(t, x_1 + a, x_2, x_3 + b, x_4) \leq \delta'(x_1 + a + x_2 + x_3 + b + x_4)$$

for every $t \in [0, 1]$. Let $q_3 = \max\{\delta, q_2\}$. Noting that $a + b < \delta$, for $(x_1, x_2, x_3, x_4) \in [0, \infty)^4$ with $x_1 + x_2 + x_3 + x_4 \geq q_3$, we have

$$\begin{aligned} f(t, x_1 + a, x_2, x_3 + b, x_4) &\leq \delta'[(x_1 + x_2 + x_3 + x_4) + q_3] \\ &\leq 2\delta'(x_1 + x_2 + x_3 + x_4). \end{aligned}$$

Then for $t \in [0, 1]$ and any $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_{q_3}$,

$$\begin{aligned} A_2(\bar{u}, \bar{v}) &= \lambda \int_0^1 G(t, s)f(s, \bar{u} + sa, \bar{u}' + a, \bar{v} + sb, \bar{v}' + b) ds \\ &\leq \lambda \int_0^1 G(t, s)f(s, \|\bar{u}\|_\infty + a, c\|\bar{u}'\|_\infty, \|\bar{v}\|_\infty + b, c\|\bar{v}'\|_\infty) ds \\ &< \lambda \cdot 2\delta'\|(\bar{u}, \bar{v})\| \int_0^1 G(t, s) ds, \end{aligned}$$

where c is as above. And similarly,

$$\begin{aligned} A_2(\bar{u}, \bar{v}) &= \lambda \int_0^1 \frac{\partial}{\partial t} G(t, s) f(s, \bar{u} + sa, \bar{u}' + a, \bar{v} + sb, \bar{v}' + b) ds \\ &< \lambda \cdot 2\delta' \|(\bar{u}, \bar{v})\| \int_0^1 \frac{\partial}{\partial t} G(t, s) ds. \end{aligned}$$

Combining these inequalities, we see that

$$\|A_2(\bar{u}, \bar{v})\|_\infty + \|A_2'(\bar{u}, \bar{v})\|_\infty < 3\lambda\delta' \|(\bar{u}, \bar{v})\|.$$

Take $\rho_3 = \max\{q_1, q_3\}$ and let $\rho \geq \rho_3$. Then given $(\bar{u}, \bar{v}) \in C \cap \partial\Omega_\rho$, we see that

$$\begin{aligned} \|T(\bar{u}, \bar{v})\| &= \|A_1(\bar{u}, \bar{v})\|_\infty + \|A_1'(\bar{u}, \bar{v})\|_\infty + \|A_2(\bar{u}, \bar{v})\|_\infty + \|A_2'(\bar{u}, \bar{v})\|_\infty \\ &< \left[\frac{1}{2}(6\lambda\delta' + 3(\eta + \epsilon)) \right] \|(\bar{u}, \bar{v})\|. \end{aligned}$$

Recall by (H5) that $\eta \in (0, \frac{2}{3})$. Pick ϵ and δ' small enough that $6\lambda\delta' + 3\epsilon \leq 2 - 3\eta$. Thus, we have the desired result. \square

4. The main result

Theorem 4.1. *Let continuous functions $f, g : [0, 1] \times [0, \infty)^4 \rightarrow [0, \infty)$ satisfy hypotheses (H0)–(H5). Then there exists $\Lambda > 0$ such that given $\lambda \geq \Lambda$, there exists $\delta > 0$ such that for every $a, b \geq 0$ satisfying $0 < a + b < \delta$, the system (5)–(8) has at least three positive solutions.*

Proof. Suppose f, g satisfy hypotheses (H0)–(H5). Let $\rho^* > 0$ be fixed. By Lemma 3.1, there is $\Lambda > 0$ such that, for every $\lambda \geq \Lambda$ and $a, b \geq 0$,

$$\|T(\bar{u}, \bar{v})\| \geq \|(\bar{u}, \bar{v})\| \quad \text{for } (\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho^*}.$$

Now, fix $\lambda \geq \Lambda$. Lemmas 3.2 through 3.4 give that there is $\delta > 0$ and $\rho_1, \rho_2, \rho_3 > 0$ satisfying $\rho_1 < \rho_2 < \rho^* < \rho_3$ such that for $(a, b) \in [0, \infty)^2$ with $0 < a + b < \delta$,

$$\|T(\bar{u}, \bar{v})\| \geq \|(\bar{u}, \bar{v})\| \quad \text{for } (\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho_1},$$

$$\|T(\bar{u}, \bar{v})\| \leq \|(\bar{u}, \bar{v})\| \quad \text{for } (\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho_2},$$

$$\|T(\bar{u}, \bar{v})\| \leq \|(\bar{u}, \bar{v})\| \quad \text{for } (\bar{u}, \bar{v}) \in C \cap \partial\Omega_{\rho_3}.$$

Applying the Guo–Krasnosel'skiĭ fixed point theorem three times, we get the existence of three positive solutions, $(\bar{u}_1, \bar{v}_1), (\bar{u}_2, \bar{v}_2), (\bar{u}_3, \bar{v}_3) \in C$ such that

$$\rho_1 < \|(\bar{u}_1, \bar{v}_1)\| < \rho_2 < \|(\bar{u}_2, \bar{v}_2)\| < \rho^* < \|(\bar{u}_3, \bar{v}_3)\| < \rho_3. \quad \square$$

Recall that solutions to the system (5)–(8) are in one-to-one correspondence with those of the system (1)–(4). Thus we have our desired result.

References

- [Agarwal 1989] R. P. Agarwal, “On fourth order boundary value problems arising in beam analysis”, *Differential Integral Equations* **2**:1 (1989), 91–110. [MR](#) [Zbl](#)
- [Henderson and Hopkins 2010] J. Henderson and B. Hopkins, “Multiple positive solutions for a discrete fourth order nonhomogeneous boundary value problem”, *Int. Electron. J. Pure Appl. Math.* **2**:2 (2010), 81–92.
- [Hopkins 2009] B. Hopkins, *Multiplicity of positive solutions of even-order nonhomogeneous boundary value problems*, thesis, Baylor University, Waco, TX, 2009, <http://search.proquest.com/docview/304849906>. [MR](#)
- [Hopkins 2015] B. Hopkins, “Multiplicity of positive solutions for an even-order right focal boundary value problem”, *Adv. Dyn. Syst. Appl.* **10**:2 (2015), 189–200. [MR](#)
- [Krasnosel’skiĭ 1964] M. A. Krasnosel’skiĭ, *Positive solutions of operator equations*, P. Noordhoff Ltd., Groningen, 1964. [MR](#) [Zbl](#)
- [do Ó et al. 2008] J. M. do Ó, S. Lorca, and P. Ubilla, “Multiplicity of solutions for a class of non-homogeneous fourth-order boundary value problems”, *Appl. Math. Lett.* **21**:3 (2008), 279–286. [MR](#) [Zbl](#)

Received: 2015-08-31 Accepted: 2016-01-14

oliviabennett15@gmail.com *Department of Mathematics and Statistics,
University of Central Oklahoma, 100 N. University Drive,
Edmond, OK 73034, United States*

dbrumley1@uco.edu *Department of Mathematics and Statistics,
University of Central Oklahoma, 100 N. University Drive,
Edmond, OK 73034, United States*

bhopkins3@uco.edu *Department of Mathematics and Statistics,
University of Central Oklahoma, 100 N. University Drive,
Edmond, OK 73034, United States*

kkarber1@uco.edu *Department of Mathematics and Statistics,
University of Central Oklahoma, 100 N. University Drive,
Edmond, OK 73034, United States*

tmilligan1@uco.edu *Department of Mathematics and Statistics,
University of Central Oklahoma, 100 N. University Drive,
Edmond, OK 73034, United States*

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA	Suzanne Lenhart	University of Tennessee, USA
John V. Baxley	Wake Forest University, NC, USA	Chi-Kwong Li	College of William and Mary, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Robert B. Lund	Clemson University, USA
Martin Bohner	Missouri U of Science and Technology, USA	Gaven J. Martin	Massey University, New Zealand
Nigel Boston	University of Wisconsin, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA	Emil Minchev	Ruse, Bulgaria
Pietro Cerone	La Trobe University, Australia	Frank Morgan	Williams College, USA
Scott Chapman	Sam Houston State University, USA	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Joshua N. Cooper	University of South Carolina, USA	Zuhair Nashed	University of Central Florida, USA
Jem N. Corcoran	University of Colorado, USA	Ken Ono	Emory University, USA
Toka Diagana	Howard University, USA	Timothy E. O'Brien	Loyola University Chicago, USA
Michael Dorff	Brigham Young University, USA	Joseph O'Rourke	Smith College, USA
Sever S. Dragomir	Victoria University, Australia	Yuval Peres	Microsoft Research, USA
Behrouz Emamizadeh	The Petroleum Institute, UAE	Y.-F. S. Pétermann	Université de Genève, Switzerland
Joel Foisy	SUNY Potsdam, USA	Robert J. Plemmons	Wake Forest University, USA
Erin W. Fulp	Wake Forest University, USA	Carl B. Pomerance	Dartmouth College, USA
Joseph Gallian	University of Minnesota Duluth, USA	Vadim Ponomarenko	San Diego State University, USA
Stephan R. Garcia	Pomona College, USA	Bjorn Poonen	UC Berkeley, USA
Anant Godbole	East Tennessee State University, USA	James Propp	U Mass Lowell, USA
Ron Gould	Emory University, USA	József H. Przytycki	George Washington University, USA
Andrew Granville	Université Montréal, Canada	Richard Rebarber	University of Nebraska, USA
Jerrold Griggs	University of South Carolina, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Jim Haglund	University of Pennsylvania, USA	James A. Sellers	Penn State University, USA
Johnny Henderson	Baylor University, USA	Andrew J. Sterge	Honorary Editor
Jim Hoste	Pitzer College, USA	Ann Trenk	Wellesley College, USA
Natalia Hritonenko	Prairie View A&M University, USA	Ravi Vakil	Stanford University, USA
Glenn H. Hurlbert	Arizona State University, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
Charles R. Johnson	College of William and Mary, USA	Ram U. Verma	University of Toledo, USA
K. B. Kulasekera	Clemson University, USA	John C. Wierman	Johns Hopkins University, USA
Gerry Ladas	University of Rhode Island, USA	Michael E. Zieve	University of Michigan, USA

PRODUCTION

Silvio Levy, Scientific Editor


Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2017 is US \$175/year for the electronic version, and \$235/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2017 Mathematical Sciences Publishers

involve

2017

vol. 10

no. 1

Intrinsically triple-linked graphs in $\mathbb{R}P^3$	1
JARED FEDERMAN, JOEL FOISY, KRISTIN MCNAMARA AND EMILY STARK	
A modified wavelet method for identifying transient features in time signals with applications to bean beetle maturation	21
DAVID MCMORRIS, PAUL PEARSON AND BRIAN YURK	
A generalization of the matrix transpose map and its relationship to the twist of the polynomial ring by an automorphism	43
ANDREW MCGINNIS AND MICHAELA VANCLIFF	
Mixing times for the rook's walk via path coupling	51
CAM MCLEMAN, PETER T. OTTO, JOHN RAHMANI AND MATTHEW SUTTER	
The lifting of graphs to 3-uniform hypergraphs and some applications to hypergraph Ramsey theory	65
MARK BUDDEN, JOSH HILLER, JOSHUA LAMBERT AND CHRIS SANFORD	
The multiplicity of solutions for a system of second-order differential equations	77
OLIVIA BENNETT, DANIEL BRUMLEY, BRITNEY HOPKINS, KRISTI KARBER AND THOMAS MILLIGAN	
Factorization of Temperley–Lieb diagrams	89
DANA C. ERNST, MICHAEL G. HASTINGS AND SARAH K. SALMON	
Prime labelings of generalized Petersen graphs	109
STEVEN A. SCHLUCHTER, JUSTIN Z. SCHROEDER, KATHRYN COKUS, RYAN ELLINGSON, HAYLEY HARRIS, ETHAN RARITY AND THOMAS WILSON	
A generalization of Zeckendorf's theorem via circumscribed m -gons	125
ROBERT DORWARD, PARI L. FORD, EVA FOURAKIS, PAMELA E. HARRIS, STEVEN J. MILLER, EYVINDUR PALSSON AND HANNAH PAUGH	
Loewner deformations driven by the Weierstrass function	151
JOAN LIND AND JESSICA ROBINS	
Rank disequilibrium in multiple-criteria evaluation schemes	165
JONATHAN K. HODGE, FAYE SPRAGUE-WILLIAMS AND JAMIE WOELK	