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Prime labelings of generalized Petersen graphs

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A graph G is called *prime* if the vertices of G can be assigned distinct labels $1, 2, \dots, |V(G)|$ such that the labels on any two adjacent vertices are relatively prime. By showing that for every even $n \leq 2.468 \times 10^9$ there exists $s \in [1, n-1]$ such that both $n+s$ and $2n+s$ are prime, we prove the generalized Peterson graph $P(n,1)$ is prime for all even $n \in [4, 2.468 \times 10^9]$. Moreover, for a fixed n we describe a method for labeling $P(n,k)$ that is a prime labeling for multiple values of k. Using this method, we prove $P(n,k)$ is prime for all even $n \leq 50$ and all odd $k \in [1, n/2)$.

1. Introduction

For a simple graph G with vertex set $V = \{v_1, v_2, \dots, v_m\}$ and edge set E, a *prime labeling* of G is a bijection $f : V \to \{1, 2, \dots, m\}$ such that $f(v_i)$ and $f(v_j)$ are relatively prime for all $\{v_i, v_j\} \in E$. A graph is called *prime* if it admits a prime labeling. This concept was proposed by Entringer, who conjectured that all trees are prime, and the first appearance of this problem in print was due to Tout, Dabboucy, and Howalla [Tout et al. 1982]. Since then, many families of graphs have been shown to be prime, including all trees on $m \leq 50$ vertices [Pikhurko 2002; 2007] and the grid graph $P_m \times P_n$ when $m \leq n$ and n is prime [Sundaram et al. 2006]. More recently, Kh. Md. Mominul Haque, Lin Xiaohui, Yang Yuansheng and Zhao Pingzhong have shown that the generalized Petersen graph $P(n,k)$ is prime for all even $n \leq 2500$ when $k = 1$ [Haque et al. 2010] and for all even $n \leq 100$ when $k = 3$ [Haque et al. 2011]. Both Diefenderfer et al. [2015] and Prajapati and Gajjar [2014] have shown that $P(n,1)$ is prime for an infinite family of values of n, the former for $n-1$ prime and the latter for $n+1$ prime. Their results are presented as labelings of the prism graph $C_n \times P_2$, which is isomorphic to $P(n,1)$. For a more thorough treatment of graph labeling, including prime labeling, see [Gallian 2015].

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Keywords: graph labeling, generalized Petersen graph, prime graph.

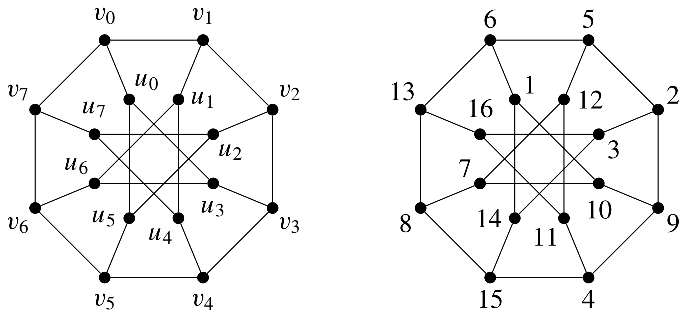

Figure 1. The graph $P(8, 3)$ and a prime labeling of $P(8, 3)$.

We follow the notation of [Haque et al. 2010; 2011]. Since all edges under consideration are undirected, we will write the edge $\{v, w\}$ as vw or wv. For integers $n \geq 3$ and $k \in [1, n/2)$, the *generalized Petersen graph* $P(n, k)$ is defined to be the graph with vertex set $V = \{v_i, u_i : i \in [0, n-1]\}$ and edge set $E = \{v_i v_{i+1},\ v_i u_i,\ u_i u_{i+k} : i \in [0, n-1]\}$, where all subscripts are reduced modulo n. We will refer to the vertices $v_0, v_1, \ldots, v_{n-1}$ as the *v-vertices* of $P(n, k)$ and the vertices $u_0, u_1, \ldots, u_{n-1}$ as the *u-vertices* of $P(n, k)$. An unlabeled $P(8, 3)$ and a prime labeling of $P(8, 3)$ are shown in Figure 1.

An *independent set* in a graph G is a subset of the vertices of G, no two of which are adjacent. Let $\alpha(G)$ denote the *independence number of* G, i.e., the size of a maximum independent set in G. Given a prime labeling of a graph G with m vertices, the vertices with even labels necessarily form an independent set; thus, $\alpha(G) \geq \lfloor m/2 \rfloor$. It was shown in [Fox et al. 2012] that $\alpha(P(n, k)) < n$ if n is odd or k is even, which leads immediately to the following result (an alternate proof is given in [Prajapati and Gajjar 2015]).

Theorem 1.1. *If n is odd or k is even, then $P(n, k)$ is not prime.*

In this paper, we build on the work appearing in [Diefenderfer et al. 2015; Haque et al. 2010; 2011; Prajapati and Gajjar 2014] by considering prime labelings of $P(n, 1)$ for $n > 2500$ and $P(n, k)$ for small n and $k > 3$. In Section 2, we conjecture that for every even n there exists $s \in [1, n-1]$ such that both $n + s$ and $2n + s$ are prime. In Section 3, we demonstrate a labeling scheme that relies on this conjecture and use computer-generated results to establish that $P(n, 1)$ is prime for all even $n \in [4, 2.468 \times 10^9]$, which improves considerably upon the upper bound given in [Haque et al. 2010]. In Section 4, we fix $n \leq 50$ and describe a method that produces a labeling of $P(n, k)$ that is prime for multiple values of k; with some minor ad hoc switching, this labeling method is used to show $P(n, k)$ is prime for all even $n \in [4, 50]$ and all odd $k \in [1, n/2)$. Together the results of Sections 3 and 4 provide evidence that $P(n, k)$ is prime precisely when n is even and k is odd (as conjectured in [Prajapati and Gajjar 2015]). Since this is a necessary and sufficient condition for $P(n, k)$ to be bipartite, we reformulate the conjecture as follows.

Conjecture 1.2. *$P(n, k)$ is prime if and only if it is bipartite.*

The property of being bipartite is, in general, neither a necessary nor a sufficient condition for a graph to be prime. The cycle C_n is prime but not bipartite for odd $n \geq 3$, and the complete bipartite graph $K_{n,n}$ is bipartite but not prime for all $n \geq 3$.

For positive integers a and b, we let $\gcd(a, b)$ denote the greatest common divisor of a and b. Then a and b are relatively prime if and only if $\gcd(a, b) = 1$. Note that if $d \mid a$ and $d \mid b$, then $d \mid (a + b)$ and $d \mid (a - b)$. From this we make the following observation.

Lemma 1.3. *Let a and b be positive integers. Then* $\gcd(a, b) = 1$ *if any of the following hold*:

(1) $a = 1$;

(2) $b = a + 1$;

(3) $a + b$ *is prime*;

(4) $a - b = p$ *is prime and a and b are not multiples of p*;

(5) $a = n + 1,\ b = 2n$ *for some even n.*

2. On the distribution of prime numbers

In a letter to Euler in 1742, Goldbach conjectured that every integer greater than 2 could be written as the sum of three primes (note that he was including 1 as a prime). Euler then reformulated this conjecture in the form in which it is now famous; see [Dickson 2005, pp. 421–424].

Goldbach conjecture. *Every even number greater than* 2 *can be written as the sum of two primes.*

A closely related conjecture, whose first appearance is due to Maillet [1905], states that every even number can be written as the *difference* of two primes; both this and the Goldbach conjecture remain unsolved. For more information on these conjectures, see [Dickson 2005].

We are going to strengthen Maillet's conjecture by requiring that the two primes be taken from specific intervals. Namely, we conjecture that every even integer n can be written as the difference of primes p_1 and p_2, where $n < p_1 < 2n$ and $2n < p_2 < 3n$. We reformulate this as follows.

Conjecture 2.1. *For every even integer $n \geq 2$, there exists $s \in [1, n - 1]$ such that $n + s$ and $2n + s$ are prime.*

n	s	S_n	n	s	S_n	n	s	S_n	n	s	S_n
2	1	1	52	9	3	102	7	10	152	27	6
4	3	1	54	5	5	104	3	5	154	3	7
6	1	2	56	15	4	106	21	5	156	1	16
8	3	1	58	15	3	108	23	5	158	15	7
10	3	2	60	7	8	110	3	5	160	33	8
12	5	2	62	27	2	112	15	6	162	29	11
14	3	2	64	3	5	114	13	11	164	3	6
16	15	1	66	5	9	116	51	3	166	15	6
18	1	3	68	3	5	118	21	6	168	11	15
20	3	1	70	9	5	120	11	15	170	9	9
22	9	2	72	7	6	122	27	3	172	9	8
24	5	4	74	9	4	124	3	6	174	5	12
26	15	2	76	21	3	126	5	12	176	15	8
28	3	2	78	1	7	128	21	4	178	3	8
30	1	6	80	3	5	130	9	6	180	13	13
32	9	2	82	15	4	132	5	11	182	9	9
34	3	2	84	5	10	134	3	8	184	15	5
36	1	7	86	21	3	136	21	4	186	7	14
38	3	3	88	15	5	138	1	11	188	3	7
40	3	4	90	11	11	140	27	9	190	3	12
42	5	5	92	9	6	142	9	4	192	5	15
44	9	3	94	3	5	144	5	10	194	33	8
46	15	2	96	1	9	146	21	6	196	27	7
48	5	6	98	3	5	148	15	6	198	1	16
50	3	3	100	27	7	150	7	16	200	33	5

Table 1. The minimum value of s such that $n+s$ and $2n+s$ are prime for even $n \in [2, 200]$. Additionally, the number of good s-values S_n is given for each even $n \in [2, 200]$.

For a fixed n, call s *good* if both $n+s$ and $2n+s$ are prime. Table 1 lists good s-values for even $n \in [2, 200]$, and we have obtained good s-values for all even $n \in [2, 2.468 \times 10^9]$ by computer.

Some interesting patterns also occur when we consider S_n, the *number* of good s-values for each n. The first 100 values of S_n are shown in Table 1. Note that $S_n = 1$ for half of the first 10 values of n, but then $S_n > 1$ for all n up to at least 300 million. Moreover, the graph of S_n appears to almost always have a peak at each multiple of 6 and higher peaks at multiples of 30, indicating a strong correlation between S_n and the number of distinct prime factors of n. The portion of this graph from $n = 190{,}100$ to $n = 190{,}200$, with straight line segments joining discrete data points, appears in Figure 2 (note that $S_{190{,}190}$ is a peak that does not occur at a multiple of 6).

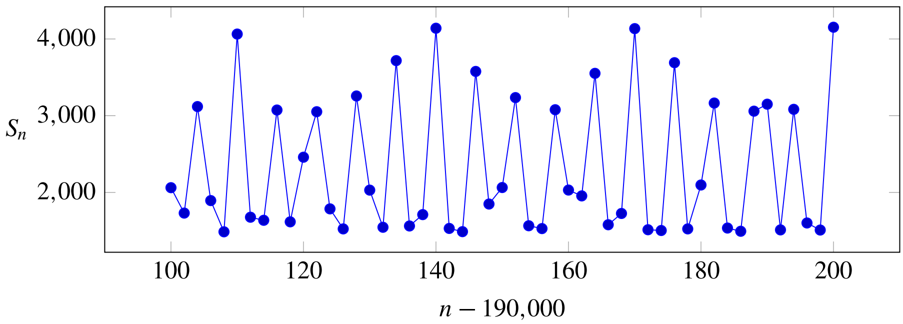

Figure 2. The graph of S_n from $n = 190{,}100$ to $n = 190{,}200$.

3. Prime labelings of $P(n, 1)$

The following labeling scheme for $P(n, 1)$ is an expansion and generalization of one employed in [Diefenderfer et al. 2015; Haque et al. 2010; Prajapati and Gajjar 2014].

For any value of k, we know v-vertices with consecutive indices are adjacent. Thus, if we set $f(v_i) = i + 1$ for all $i \in [0, n-1]$, adjacent v-vertices will either have consecutive integer labels or be labeled with 1 and n; these labels will be relatively prime by Lemma 1.3(2) or (1), respectively. Without loss of generality, we refer to this as a clockwise labeling of the v-vertices (see Figure 3).

If $k = 1$, then u-vertices with consecutive indices are also adjacent, so we label the u-vertices consecutively with $n+1, n+2, \ldots, 2n$. Now adjacent u-vertices

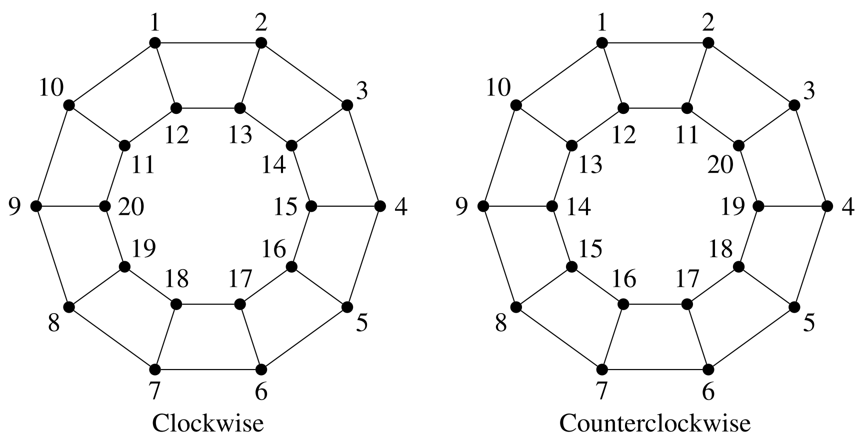

Figure 3. Prime labelings of $P(10, 1)$ using both a clockwise labeling and a counterclockwise labeling for the u-vertices. Note that the *difference* of inner and outer labels in the clockwise labeling is always 1 or 11, and the *sum* of inner and outer labels in the counterclockwise labeling is always 13 or 23.

will either have consecutive integer labels or be labeled with $n+1$ and $2n$; since n is even, these labels will be relatively prime by Lemma 1.3(2) or (5), respectively.

If we apply a clockwise labeling to the u-vertices as well, then labels on each adjacent vu-pair will have a constant difference d (modulo n). By varying our starting point (i.e., where we place the label $n+1$), we can obtain different values of d. The prime labeling of $P(10, 1)$ in Figure 3 employs a clockwise labeling on the u-vertices starting at u_9. For each i the labels on v_i and u_i differ by 1 or 11 ($d \equiv 1 \pmod{10}$), and since $d = 1$ implies consecutive integers and $d = 11$ is prime, these labels are relatively prime by Lemma 1.3(2) or (4), respectively.

If, instead, we apply a counterclockwise labeling to the u-vertices, then labels on each adjacent vu-pair will have a constant sum s (modulo n). Once again, varying the starting point will lead to different values of s. The prime labeling of $P(10, 1)$ in Figure 3 employs a counterclockwise labeling on the u-vertices starting at u_1. For each i the labels on v_i and u_i sum to 13 or 23 ($s \equiv 3 \pmod{10}$), and since both 13 and 23 are prime, these labels are relatively prime by Lemma 1.3(3).

The value of s in the preceding paragraph corresponds to a good s-value from Conjecture 2.1. Let $\mathcal{N}_s = \{n : n+s \text{ is prime}\}$ and $\mathcal{N}_s^* = \{n : 2n+s \text{ is prime}\}$.

Theorem 3.1 [Diefenderfer et al. 2015, Theorem 4.1]. *If $n \in \mathcal{N}_{-1}$, then $P(n, 1)$ is prime.*

Theorem 3.2 [Prajapati and Gajjar 2014, Theorem 2.10]. *If $n \in \mathcal{N}_1$, then $P(n, 1)$ is prime.*

Theorem 3.1 can be obtained by first applying a clockwise labeling to the v-vertices and a clockwise labeling to the u-vertices starting at u_1. Then, switching the labels 1 and $n-1$ on the vertices v_0 and v_{n-2} results in a prime labeling. Note that in [Diefenderfer et al. 2015] the labels n and $2n$ were also switched, but this is not necessary to obtain a prime labeling. Theorem 3.2 was obtained by applying a clockwise labeling to the v-vertices and a clockwise labeling to the u-vertices starting at u_{n-1}.

The following result utilizes a *counterclockwise* labeling on the u-vertices.

Theorem 3.3. *Let $n \geq 4$ be even, and suppose $n \in \mathcal{N}_1^*$ or $n \in \mathcal{N}_s \cap \mathcal{N}_s^*$ for some $s \in [3, n-1]$. Then $P(n, 1)$ is prime.*

Proof. Throughout this proof, we apply a clockwise labeling to the v-vertices starting at v_0 (given by $f(v_i) = i+1$ for all $i \in [0, n-1]$) and a counterclockwise labeling to the u-vertices starting at some u_j. We have already shown that adjacent v-vertices and adjacent u-vertices will have relatively prime labels, so it remains to consider the labels on v_i and u_i for each i.

If $n \in \mathcal{N}_1^*$, then apply a counterclockwise labeling to the u-vertices starting at u_{n-1}. Formally, let $f(u_i) = 2n - i$ for all $i \in [0, n-1]$. Then the sum of labels on the edge $v_i u_i$ is $i+1+2n-i=2n+1$ for all $i \in [0, n-1]$. Since $2n+1$ is prime, these labels are relatively prime by Lemma 1.3(3).

If $n \in \mathcal{N}_s \cap \mathcal{N}_s^*$ for some $s \in [3, n-1]$, then apply a counterclockwise labeling to the u-vertices starting at u_{s-2}. Formally, let

$$f(u_i) = \begin{cases} n+s-i-1 & \text{if } i \in [0, s-2], \\ 2n+s-i-1 & \text{if } i \in [s-1, n-1]. \end{cases}$$

Then the sum of labels on the edge $v_i u_i$ is either $i+1+n+s-i-1=n+s$ (if $i \in [0, s-2]$) or $i+1+2n+s-i-1=2n+s$ (if $i \in [s-1, n-1]$). Since both $n+s$ and $2n+s$ are prime, these labels are relatively prime by Lemma 1.3(3). □

For an example of the counterclockwise labeling used in Theorem 3.3, see Figure 3. Since $n=10 \in \mathcal{N}_3 \cap \mathcal{N}_3^*$, we apply a counterclockwise labeling to the u-vertices starting at u_1. The sum of labels on the edge $v_i u_i$ is $n+s=13$ for $i=0, 1$ and $2n+s=23$ for $i \in [2, 9]$.

We have verified Conjecture 2.1 for $n \in [4, 2.468 \times 10^9]$ by computer, and so we have the following result, which further supports the conjecture made in [Haque et al. 2010] that $P(n, 1)$ is prime for all even $n \geq 4$.

Corollary 3.4. $P(n, 1)$ *is prime for all even* $n \in [4, 2.468 \times 10^9]$.

In addition to this bound, note that $\mathcal{N}_{-1} \cup \mathcal{N}_1$ yields an infinite family of generalized Petersen graphs $P(n, 1)$ that are prime.

4. Prime labelings of $P(n, k)$ for even $n \leq 50$ and odd $k > 1$

Here we will give an example of how we produce a labeling of $P(n, k)$ that is a prime labeling for multiple values of $k > 1$. Specifically, we will describe a method of labeling $P(18, k)$ that is prime for $k \in \{3, 5\}$ and show that swapping just two labels also produces a prime labeling of $P(18, 7)$. The tables at the end of this section contain prime labelings of $P(n, k)$ for all even $n \leq 50$ and all odd $k \in [5, n/2)$ obtained using slight variations on the method for $P(18, k)$ (the case $k=3$ is handled in [Haque et al. 2011]).

If $k > 1$, then u-vertices with consecutive indices are no longer adjacent, so we abandon the clockwise and counterclockwise labeling schemes altogether. Instead, we seek to harness the structure of a bipartition of the vertices of $P(n, k)$. Let $A = \{u_0, v_1, u_2, v_3, \ldots, u_{n-2}, v_{n-1}\}$ and $B = \{v_0, u_1, v_2, u_3, \ldots, v_{n-2}, u_{n-1}\}$ denote the blocks of a bipartition of the vertices of $P(n, k)$; in Figure 4, the top row of vertices forms A while the bottom row forms B. We will place the odd labels on A and the even labels on B in such a way that the resulting labeling is prime for several different values of k.

We begin by placing the even multiples of 3 from left to right in ascending order on the vertices in B with the lowest index (whether v-vertices or u-vertices). We

u_0	v_1	u_2	v_3	u_4	v_5	u_6	v_7	u_8	v_9	u_{10}	v_{11}	u_{12}	v_{13}	u_{14}	v_{15}	u_{16}	v_{17}
							9		27		15	3	21		33		
6	12	18	24	30	36												
v_0	u_1	v_2	u_3	v_4	u_5	v_6	u_7	v_8	u_9	v_{10}	u_{11}	v_{12}	u_{13}	v_{14}	u_{15}	v_{16}	u_{17}

Figure 4. The graph $P(18, k)$ (edges suppressed) labeled with the multiples of 3.

will place the odd multiples of 3 only on v-vertices in A that are not adjacent to any vertex in B labeled with a multiple of 3. However, a simple counting argument shows there will always be one odd multiple of 3 that cannot be placed on a v-vertex; we assume this label is 3 and place the remaining odd multiples of 3 from left to right in ascending order based on the highest prime factor of the label (in the case of a tie, we simply place the smallest integer first — thus 15 is placed before 45, for example). To make this a prime labeling for small values of k, we place 3 on the u-vertex in A whose index is the furthest from the index of any u-vertex in B labeled with a multiple of 3. This first step for $P(18, k)$ is shown in Figure 4.

We continue in a likewise manner for the multiples of 5 and 7, placing the even multiples on the vertices in B with the lowest index and the odd multiples on the available v-vertices in A. If $n \geq 18$, then 35 — the only odd multiple of both 5 and 7 — is placed on v_{n-1}. Placing these labels may yield adjacent v-vertices that share 5 or 7 as divisors; to correct this, we rearrange the even multiples of 3. In the partial labeling of $P(18, k)$ shown in Figure 5, the labels 25 and 30 on the edge v_3v_4 have a conflict which is fixed in Figure 6 by swapping the labels on v_4 and u_5. It is also possible for larger values of n to have some multiples of 5 or 7 that do not fit on v-vertices; in that case, some additional ad hoc label switching is required.

The remaining even labels with an odd prime factor are placed on the unused vertices in B from left to right in ascending order based on smallest odd prime factor. If there is only one remaining label that is an even multiple of some prime p,

u_0	v_1	u_2	v_3	u_4	v_5	u_6	v_7	u_8	v_9	u_{10}	v_{11}	u_{12}	v_{13}	u_{14}	v_{15}	u_{16}	v_{17}
	5		25		7		9		27		15	3	21		33		35
6	12	18	24	30	36	10	20	14	28								
v_0	u_1	v_2	u_3	v_4	u_5	v_6	u_7	v_8	u_9	v_{10}	u_{11}	v_{12}	u_{13}	v_{14}	u_{15}	v_{16}	u_{17}

Figure 5. The graph $P(18, k)$ (edges suppressed) labeled with the multiples of 3, 5, and 7. Note that the labels 25 and 30 on the edge v_3v_4 have a conflict, which is addressed in Figure 6.

u_0	v_1	u_2	v_3	u_4	v_5	u_6	v_7	u_8	v_9	u_{10}	v_{11}	u_{12}	v_{13}	u_{14}	v_{15}	u_{16}	v_{17}
•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
	5		25		7		9		27		15	3	21		33		35
6	12	18	24	**36***	**30***	10	20	14	28	2	22	4	26	8	34	16	32
•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
v_0	u_1	v_2	u_3	v_4	u_5	v_6	u_7	v_8	u_9	v_{10}	u_{11}	v_{12}	u_{13}	v_{14}	u_{15}	v_{16}	u_{17}

Figure 6. The graph $P(18, k)$ (edges suppressed) labeled with all available even labels and the odd multiples of 3, 5, and 7. Note that the labels on v_4 and u_5 have been switched to fix the conflict on the edge v_3v_4.

u_0	v_1	u_2	v_3	u_4	v_5	u_6	v_7	u_8	v_9	u_{10}	v_{11}	u_{12}	v_{13}	u_{14}	v_{15}	u_{16}	v_{17}
•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
1	5	19	25	23	7	**29***	9	31	27	11	15	**3***	21	13	33	17	35
6	12	18	24	36	30	10	20	14	28	2	22	4	26	8	34	16	32
•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
v_0	u_1	v_2	u_3	v_4	u_5	v_6	u_7	v_8	u_9	v_{10}	u_{11}	v_{12}	u_{13}	v_{14}	u_{15}	v_{16}	u_{17}

Figure 7. A prime labeling of $P(18, k)$ (edges suppressed) for $k = 3$ or 5. For a prime labeling of $P(18, 7)$, swap the labels on u_6 and u_{12}.

then it is placed on the next available u-vertex. Finally, the multiples of 2 are placed on the unused B vertices, and any necessary label switching from earlier steps is completed. The partial labeling that results for $P(18, k)$ is shown in Figure 6.

Since $2n < 11^2$, the only unused odd multiple of a prime $p > 7$ is p itself; thus, the only odd labels remaining are 1 and the primes greater than 7 and less than $2n$. Each odd prime p less than n is placed on the u-vertex in A whose index is as close as possible to the indices of the u-vertices in B labeled with multiples of p. The remaining labels (1 and the primes greater than n) can be placed arbitrarily on the unlabeled vertices in A. For some n, additional ad hoc label switching may be required. However, for our example $n = 18$, the resulting labeling is prime for $k = 3$ and $k = 5$ (see Figure 7). To obtain a prime labeling for $k = 7$, simply swap the labels on u_6 and u_{12}. For the sake of comparison, the prime labeling of $P(18, 5)$ in Figure 7 is also given in Table 2.

The following result covers the aforementioned ad hoc switching and establishes that, for $n \in [4, 50]$, we have $P(n, k)$ prime precisely when n is even and k is odd (recall from Theorem 1.1 that $P(n, k)$ is *not* prime if n is odd or k is even).

Theorem 4.1. *$P(n, k)$ is prime for all even $n \in [4, 50]$ and all odd $k \in [1, n/2)$.*

Proof. The case $k = 1$ was covered in [Haque et al. 2010] and $k = 3$ in [Haque et al. 2011]. Tables 2 and 3 provide a prime labeling of $P(n, k)$ for every even

$P(12,5)$			$P(14,5)$			$P(16,5)$			$P(18,5)$			$P(20,5)$		
i	v_i	u_i	i	v_i	u_i	i	v_i	u_i	i	v_i	u_i	i	v_i	u_i
0	6	1	0	6	1	0	6	1	0	6	1	0	6	1
1	5	12	1	5	12	1	5	12	1	5	12	1	5	12
2	18	13	2	18	17	2	18	17	2	18	19	2	18	23
3	7	24	3	7	24	3	19	30	3	25	24	3	25	24
4	10	3	4	10	3	4	24	25	4	36	23	4	36	29
5	9	20	5	9	20	5	7	10	5	7	30	5	7	30
6	14	11	6	14	19	6	20	$\mathbf{23}^{*}$	6	10	$\mathbf{29}^{*}$	6	10	$\mathbf{31}^{*}$
7	15	2	7	27	28	7	9	14	7	9	20	7	9	20
8	22	17	8	22	23	8	28	11	8	14	31	8	40	37
9	21	4	9	15	2	9	27	22	9	27	28	9	27	14
10	8	19	10	26	11	10	2	$\mathbf{3}^{*}$	10	2	11	10	28	11
11	23	16	11	21	4	11	15	26	11	15	22	11	15	22
			12	8	13	12	4	13	12	4	$\mathbf{3}^{*}$	12	2	13
			13	25	16	13	21	8	13	21	26	13	21	26
						14	16	29	14	8	13	14	4	$\mathbf{3}^{*}$
						15	31	32	15	33	34	15	33	34
									16	16	17	16	8	17
									17	35	32	17	39	38
												18	16	19
												19	35	32

Table 2. A prime labeling of $P(n,5)$ for even $n \in [12, 22]$. A prime labeling of $P(n,k)$ for even $n \in [16, 22]$ and odd $k \in (n/3, n/2)$ can be obtained by swapping the starred labels within a column.

$n \in [12, 42]$ and every odd $k \in [5, n/3)$. If $k > n/3$ is odd (note that n is even, so $n/3$ is also even), then the u-vertex labeled with 3 is adjacent to another u-vertex labeled with a multiple of 3 in the given labeling. Swapping the starred labels in the prime labeling of $P(n,5)$ yields a prime labeling of $P(n,k)$ for even $n \in [16, 42]$ and every odd $k \in (n/3, n/2)$.

Prime labelings of $P(n,k)$ for even $n \in [44, 50]$ and all odd $k \in [5, n/2)$ are given in Tables 4–7. To check these tables, note that for any value of k the labels on all $v_i v_{i+1}$ edges are given by adjacent vertical pairs in the column labeled v_i and the labels on all $v_i u_i$ edges are given by adjacent horizontal pairs. The labels on all $u_i u_{i+k}$ edges are given by vertical pairs at a distance of k in the column labeled u_i. $\square$

Remark. If n is even one can expand the definition of generalized Petersen graphs to include $P(n, n/2)$. The result is a multigraph with two edges joining u_i and $u_{i+n/2}$ for all $i \in [0, n/2-1]$; for prime labeling purposes, these parallel edges can

$P(22, k)$, $k \in [5, 7]$			$P(24, k)$, $k \in [5, 7]$			$P(26, k)$, $k \in [5, 7]$			$P(28, k)$, $k \in [5, 9]$			$P(30, k)$, $k \in [5, 9]$			$P(32, k)$, $k \in [5, 9]$		
i	v_i	u_i	i	v_i	u_i	i	v_i	u_i	i	v_i	u_i	i	v_i	u_i	i	v_i	u_i
0	6	1	0	42	1	0	6	1	0	6	1	0	6	1	0	6	1
1	5	12	1	5	12	1	5	12	1	5	12	1	5	12	1	5	12
2	18	23	2	18	29	2	18	29	2	18	29	2	18	31	2	18	37
3	25	24	3	25	24	3	25	42	3	25	42	3	25	42	3	25	24
4	30	29	4	48	31	4	48	31	4	54	31	4	54	37	4	42	41
5	7	36	5	35	36	5	7	36	5	55	36	5	55	36	5	55	36
6	42	31	6	6	37	6	24	37	6	24	37	6	24	41	6	48	43
7	37	10	7	7	30	7	49	30	7	7	48	7	7	48	7	49	30
8	20	**41***	8	10	**41***	8	10	**41***	8	30	41	8	30	43	8	54	47
9	9	40	9	9	20	9	9	20	9	49	10	9	49	60	9	7	60
10	14	43	10	40	43	10	40	43	10	20	**43***	10	10	**47***	10	10	**53***
11	27	28	11	27	14	11	27	50	11	9	40	11	9	20	11	9	20
12	26	11	12	28	11	12	14	47	12	50	47	12	40	53	12	40	59
13	15	44	13	15	22	13	15	28	13	27	14	13	27	50	13	27	50
14	22	**3***	14	44	13	14	26	11	14	28	53	14	14	59	14	14	61
15	21	2	15	45	26	15	45	44	15	15	56	15	15	28	15	15	28
16	34	13	16	2	**3***	16	22	**3***	16	26	11	16	56	11	16	56	11
17	33	4	17	21	34	17	21	52	17	45	44	17	45	22	17	45	22
18	38	17	18	4	17	18	2	13	18	22	**3***	18	44	13	18	44	13
19	39	8	19	33	38	19	33	34	19	21	52	19	21	26	19	21	26
20	16	19	20	8	19	20	4	17	20	2	13	20	52	**3***	20	52	17
21	35	32	21	39	46	21	39	38	21	33	34	21	33	34	21	63	34
			22	16	23	22	8	19	22	4	17	22	2	17	22	2	**3***
			23	47	32	23	51	46	23	39	38	23	39	38	23	33	38
						24	16	23	24	8	19	24	4	19	24	4	19
						25	35	32	25	51	46	25	51	46	25	39	46
									26	16	23	26	8	23	26	8	23
									27	35	32	27	57	58	27	51	58
												28	16	29	28	16	29
												29	35	32	29	57	62
															30	32	31
															31	35	64

Table 3. A prime labeling of $P(n, k)$ for even $n \in [24, 42]$ and odd $k \in [5, n/3)$. A prime labeling of $P(n, k)$ for even $n \in [24, 42]$ and odd $k \in (n/3, n/2)$ can be obtained by swapping the starred labels within a column.

(Continued on next page.)

$P(34,k)$, $k\in[5,11]$			$P(36,k)$, $k\in[5,11]$			$P(38,k)$, $k\in[5,11]$			$P(40,k)$, $k\in[5,13]$			$P(42,k)$, $k\in[5,13]$		
i	v_i	u_i	i	v_i	u_i	i	v_i	u_i	i	v_i	u_i	i	v_i	u_i
0	6	1	0	6	1	0	6	1	0	6	41	0	6	1
1	5	12	1	5	12	1	5	12	1	5	12	1	5	12
2	18	37	2	18	37	2	18	41	2	18	43	2	18	43
3	25	24	3	25	24	3	25	24	3	25	42	3	25	24
4	66	41	4	36	41	4	36	43	4	78	47	4	42	47
5	65	36	5	55	48	5	55	48	5	55	36	5	55	36
6	42	43	6	42	43	6	42	47	6	24	53	6	84	53
7	55	48	7	65	54	7	65	54	7	65	48	7	65	48
8	54	47	8	66	47	8	66	53	8	54	59	8	66	59
9	7	60	9	7	72	9	7	72	9	7	60	9	7	54
10	30	53	10	30	53	10	30	59	10	66	61	10	78	61
11	49	10	11	49	60	11	49	60	11	49	72	11	49	72
12	20	$\mathbf{59}^*$	12	10	$\mathbf{59}^*$	12	10	$\mathbf{61}^*$	12	30	67	12	30	67
13	9	40	13	9	20	13	9	20	13	77	10	13	77	60
14	50	61	14	40	61	14	40	67	14	20	$\mathbf{71}^*$	14	10	$\mathbf{71}^*$
15	27	14	15	27	50	15	27	50	15	9	40	15	9	20
16	28	67	16	70	67	16	70	71	16	50	73	16	40	73
17	15	56	17	39	14	17	39	14	17	27	70	17	27	50
18	26	11	18	28	71	18	28	73	18	2	79	18	70	79
19	45	44	19	45	56	19	45	56	19	15	14	19	81	80
20	22	13	20	26	11	20	26	11	20	28	11	20	14	83
21	21	52	21	21	44	21	75	44	21	45	56	21	15	28
22	34	$\mathbf{3}^*$	22	22	13	22	22	13	22	22	13	22	56	11
23	63	68	23	63	52	23	21	52	23	75	44	23	45	22
24	2	17	24	34	$\mathbf{3}^*$	24	34	$\mathbf{3}^*$	24	26	17	24	44	13
25	33	38	25	33	68	25	63	68	25	21	52	25	75	26
26	4	19	26	38	17	26	38	17	26	34	$\mathbf{3}^*$	26	52	17
27	39	46	27	15	2	27	33	76	27	63	68	27	63	34
28	8	23	28	46	19	28	46	19	28	38	23	28	68	$\mathbf{3}^*$
29	51	58	29	51	4	29	15	2	29	33	76	29	21	38
30	16	29	30	58	23	30	58	23	30	80	19	30	20	19
31	57	62	31	57	8	31	51	4	31	39	46	31	33	46
32	32	31	32	62	29	32	62	29	32	4	29	32	76	23
33	35	64	33	69	16	33	57	8	33	17	58	33	39	58
			34	32	31	34	74	31	34	8	31	34	4	29
			35	35	64	35	69	16	35	57	62	35	51	62
						36	32	37	36	16	37	36	8	31
						37	35	64	37	69	74	37	57	74
									38	32	1	38	16	37
									39	35	64	39	69	82
												40	32	41
												41	35	64

Table 3. (Continued from previous page.)

i	v_i	u_i	departures	i	v_i	u_i	departures
0	6	1		22	56	11	
1	5	12		23	45	22	
2	18	47		24	44	13	
3	55	84		25	75	88	
4	78	53		26	26	17	
5	85	36		27	21	52	
6	42	59		28	34	3	$u_{28}=73$ for $k\in[15, 21]$
7	65	48		29	63	68	
8	54	61		30	38	9	$u_{30}=77$ for $k=15$, $u_{30}=43$ for $k\in[17, 21]$
9	25	72					
10	66	67		31	33	76	
11	7	60		32	46	19	
12	30	71		33	39	2	
13	49	24		34	4	29	
14	10	73	$u_{14}=3$ for $k\in[15, 21]$	35	51	58	
15	77	20	$v_{15}=9$ for $k=15$	36	8	31	
16	40	79		37	57	62	
17	27	50		38	16	37	
18	70	83		39	69	74	
19	81	80		40	32	41	
20	14	23		41	87	82	
21	15	28		42	64	43	$u_{42}=9$ for $k\in[17, 21]$
				43	35	86	

Table 4. A prime labeling of $P(44, k)$ for all odd $k\in[15, 21]$.

be suppressed to a single edge. When $n\in[14, 50]$ and $n/2$ is odd, the prime labeling of $P(n, n/2-2)$ given in Theorem 4.1 is also a prime labeling of $P(n, n/2)$. It is a simple exercise to show that $P(6, 3)$ and $P(10, 5)$ are prime, and additional results concerning prime labelings of $P(n, n/2)$ can be found in [Prajapati and Gajjar 2015].

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i	v_i	u_i	departures	i	v_i	u_i	departures
0	6	1		22	28	11	
1	65	12		23	15	56	
2	18	47		24	22	13	
3	55	24		25	45	44	
4	84	53		26	88	17	
5	85	36		27	75	26	
6	42	59		28	52	3	$v_{28}=4$, $u_{28}=91$ for $k\in[15, 21]$
7	5	48		29	21	34	
8	78	61		30	68	9	$u_{30}=43$ for $k\in[17, 21]$
9	25	54		31	63	38	
10	66	67		32	76	23	
11	7	72		33	33	46	
12	30	71		34	92	19	
13	49	60		35	39	58	
14	90	73		36	2	29	
15	77	10		37	51	62	
16	20	79		38	4	31	$v_{38}=52$ for $k\in[15, 21]$
17	91	40	$v_{17}=3$ for $k\in[15, 21]$	39	57	74	
18	50	83		40	8	37	
19	27	70		41	69	82	
20	80	89		42	16	41	
21	81	14		43	87	86	
				44	32	43	$u_{44}=9$ for $k\in[17, 21]$

Table 5. A prime labeling of $P(46, k)$ for all odd $k\in[15, 21]$.

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i	v_i	u_i	departures	i	v_i	u_i	departures
0	6	43		24	56	11	
1	55	12		25	45	22	
2	18	47		26	44	13	
3	65	24		27	75	88	
4	84	1		28	26	17	
5	85	36		29	21	52	
6	42	53		30	34	27	$u_{30} = 77$ for $k \in [15, 23]$
7	95	48		31	63	68	
8	54	59		32	38	3	$u_{32} = 73$ for $k \in [17, 23]$
9	5	96		33	33	76	
10	66	61		34	46	9	$u_{34} = 91$ for $k \in [15, 23]$
11	25	72		35	39	92	
12	78	67		36	2	19	
13	7	30		37	51	58	
14	90	71		38	4	23	
15	49	60		39	57	62	
16	10	73	$u_{16} = 3$ for $k \in [17, 23]$	40	8	29	
17	77	20	$v_{17} = 27$ for $k \in [15, 23]$	41	69	74	
18	40	79		42	16	31	
19	91	50	$v_{19} = 9$ for $k \in [15, 23]$	43	87	82	
20	80	83		44	32	37	
21	81	70		45	93	86	
22	14	89		46	64	41	
23	15	28		47	35	94	

Table 6. A prime labeling of $P(48, k)$ for all odd $k \in [15, 23]$.

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i	v_i	u_i	departures	i	v_i	u_i	departures
0	6	43		25	45	56	
1	5	12		26	98	11	
2	18	47		27	75	22	
3	25	24		28	44	13	
4	84	1		29	21	88	
5	55	36		30	26	9	$u_{30}=77$ for $k\in[15, 23]$
6	42	53		31	63	52	
7	65	48		32	34	3	$u_{32}=73$ for $k\in[17, 23]$
8	54	59		33	15	68	
9	85	96		34	38	27	$u_{34}=91$ for $k\in[17, 23]$
10	66	61		35	99	76	
11	95	72		36	46	17	
12	78	67		37	39	92	
13	7	30		38	2	19	
14	90	71		39	51	58	
15	49	60		40	4	23	
16	10	73	$u_{16}=3$ for $k\in[17, 23]$	41	57	62	
17	77	20	$v_{17}=27$ for $k\in[15, 23]$	42	8	29	
18	40	79		43	69	74	
19	91	50	$v_{19}=9$ for $k\in[15, 23]$	44	16	31	
20	80	83		45	87	82	
21	81	70		46	32	37	
22	100	89		47	93	86	
23	33	14		48	64	41	
24	28	97		49	35	94	

Table 7. A prime labeling of $P(50, k)$ for all odd $k\in[15, 23]$.

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