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A necessary and sufficient condition for coincidence with the weak topology

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For a topological space X, it is a natural undertaking to compare its topology with the weak topology generated by a family of real-valued continuous functions on X. We present a necessary and sufficient condition for the coincidence of these topologies for an arbitrary family $\mathcal{A} \subset C(X)$. As a corollary, we give a new proof of the fact that families of functions which separate points on a compact space induce topologies that coincide with the original topology.

1. Introduction

Given a topological space (X, τ), let $C(X)$ denote the collection of all continuous functions from X to $\mathbb{R}$, where $\mathbb{R}$ is equipped with its usual topology. The weak topology induced by a family $\mathcal{A} \subset C(X)$, which we denote by $\tau_{\mathcal{A}}$, is the topology on X such that the collection of sets of the form

$$V(f, y, \epsilon) = \{x \in X : |f(x) - f(y)| < \epsilon\},$$

where $y \in X$, $f \in \mathcal{A}$, and $\epsilon > 0$, is a subbase. It is also characterized as the coarsest topology making all the functions in $\mathcal{A}$ continuous, and thus $\tau_{\mathcal{A}} \subset \tau$. This naturally leads one to ask when equality holds.

Gillman and Jerison [1976, Theorem 3.7] demonstrated that if $\tau = \tau_{\mathcal{A}}$, then the space X is completely regular; however, the converse does not hold in general. For example, if we take (X, τ) to be the real line with the discrete topology and the family $\mathcal{A}$ to consist of only the identity function, then $\tau_{\mathcal{A}}$ is the usual topology on $\mathbb{R}$ and so $\tau_{\mathcal{A}} \neq \tau$.

Conditions for the coincidence of τ and $\tau_{\mathcal{A}}$ are also given. A family $\mathcal{A} \subset C(X)$ is said to be *completely regular* if given a closed set $F \subset X$ and a point $x_0 \in X \setminus F$, there exists an $f \in \mathcal{A}$ with $f(x_0) \notin \operatorname{cl} f[F]$. It is known (see [Gillman and Jerison 1976, Problem 3H]) that if $\mathcal{A}$ is completely regular, then $\tau = \tau_{\mathcal{A}}$. The converse also fails to hold, as we will demonstrate with Example 1.

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At present, a condition that is both necessary *and* sufficient appears to be absent from the literature. To remedy this lapse, we propose the following improvement to the definition of completely regular family:

Definition. A family $\mathcal{A} \subset C(X)$ is said to be finitely completely regular if given a closed set $F \subset X$ and a point $x_0 \in X \setminus F$, there exist $f_1, \dots, f_n \in \mathcal{A}$ such that $0 \notin \operatorname{cl} g[F]$, where the map $g : X \to \mathbb{R}$ is defined by

$$g(x) = \max_{1 \le k \le n} |f_k(x) - f_k(x_0)|.$$

We will show that the condition of finite complete regularity is both necessary and sufficient for $\tau_{\mathcal{A}}$ and τ to coincide, discuss the implications of our result for families $\mathcal{A}$ on compact spaces, and present examples.

2. Main theorem

Theorem. *Let (X, τ) be a topological space and let $\mathcal{A} \subset C(X)$ be a family of real-valued continuous functions on X. The weak topology generated by $\mathcal{A}$ coincides with τ if and only if $\mathcal{A}$ is a finitely completely regular family.*

Proof. Suppose $\tau = \tau_{\mathcal{A}}$, let F be closed, and let $x_0 \notin F$. As the collection $V(f, y, \epsilon)$ forms a subbase for $\tau_{\mathcal{A}}$, there exist $f_1, \dots, f_n \in \mathcal{A}$ and an $\epsilon > 0$ such that

$$x_0 \in \bigcap_{k=1}^{n} V(f_k, x_0, \epsilon) \subset X \setminus F,$$

and taking the complement yields

$$F \subseteq \bigcup_{k=1}^{n} X \setminus V(f_k, x_0, \epsilon).$$

Each set $X \setminus V(f_k, x_0, \epsilon)$ consists of all points $x \in X$ such that $|f_k(x) - f_k(x_0)| \ge \epsilon$, and so if $g : X \to \mathbb{R}$ is defined by $g(x) = \max\{|f_k(x) - f_k(x_0)| : 1 \le k \le n\}$, then $0 \notin \operatorname{cl} g(X \setminus V(f_k, x_0, \epsilon))$ for each k. Therefore, as

$$\operatorname{cl} g(F) \subseteq \bigcup_{k=1}^{n} \operatorname{cl} g(X \setminus V(f_k, x_0, \epsilon)),$$

we have $0 \notin \operatorname{cl} g(F)$ and thus the family $\mathcal{A}$ is finitely completely regular.

Now, let $\mathcal{A}$ be a finitely completely regular family. Given $U \in \tau$ and $x_0 \in U$, there exist $f_1, \dots, f_n \in \mathcal{A}$ such that $0 \notin \operatorname{cl} g(X \setminus U)$, where $g(x) = \max |f_k(x) - f_k(x_0)|$. Consequently, there exists an $\epsilon > 0$ such that $g(x) \ge \epsilon$ for all $x \in X \setminus U$, and we have

$$X \setminus U \subseteq \bigcup_{i=1}^{n} \{x \in X : |f_i(x) - f_i(x_0)| \ge \epsilon\},$$

which we complement to obtain

$$x_0 \in \bigcap_{i=1}^{n} \{x \in X : |f_i(x) - f_i(x_0)| < \epsilon\} \subseteq U.$$

Therefore $\tau \subset \tau_A$, and so $\tau = \tau_{\mathcal{A}}$. □

A family $\mathcal{A} \subset C(X)$ is said to *separate points* if for all distinct $x, y \in X$ there exists a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. It is well known that if a family separates points on a compact space, then $\tau_{\mathcal{A}} = \tau$ (see [Kaniuth 2009, Proposition 2.2.14], among others). The main theorem yields a new proof of this fact:

Corollary. *Let (X, τ) be a compact space. If $\mathcal{A} \subset C(X)$ is a family of functions that separates points then $\tau = \tau_{\mathcal{A}}$.*

Proof. We proceed by contraposition. Indeed, suppose $\tau \neq \tau_{\mathcal{A}}$. Then $\mathcal{A}$ fails to be finitely completely regular. Consequently, there exists a closed F and a point $x_0 \in X \setminus F$ such that $0 \in \operatorname{cl} g[F]$, where $g(x) = \max |f_k(x) - f_k(x_0)|$ for any finite collection $f_1, \ldots, f_n \in \mathcal{A}$. Since X is compact, g is a closed mapping and this implies that $\operatorname{cl} g[F] = g[F]$, which yields $0 \in g[F]$ and so there exists an $x \in F$ with $f_k(x) = f_k(x_0)$ for each $1 \leq k \leq n$.

Define the closed sets

$$F_f = \{x \in F : f(x) = f(x_0)\} \quad \text{and} \quad K = \bigcap_{f \in \mathcal{A}} F_f.$$

As any finite collection of functions $f_1, \ldots, f_n \in \mathcal{A}$ satisfies

$$\bigcap_{k=1}^{n} F_{f_k} \neq \varnothing,$$

the collection of closed sets $\{F_f : f \in \mathcal{A}\}$ has the finite intersection property and so there exists a $y \in K$. By construction, $f(y) = f(x_0)$ for all $f \in \mathcal{A}$ and since $y \in F$, it must be that $y \neq x_0$. Therefore, $\mathcal{A}$ does not separate points. □

3. Examples

We now give illustrative examples of families of continuous functions; one is finitely completely regular and the other fails to satisfy the definition.

Example 1. Consider the two functions $f, g \in C([0, 1])$ shown in Figure 1. The family $\mathcal{A} = \{f, g\}$ separates points, and thus the topology it induces on [0, 1] is the usual topology. This implies that $\mathcal{A}$ is finitely completely regular; however, it is worth noting that $\mathcal{A}$ fails to be completely regular. Indeed, let $F = \left[0, \frac{1}{9}\right] \cup \left[\frac{5}{9}, \frac{2}{3}\right]$ and $x_0 = \frac{1}{3}$; then $x_0 \notin F$ but $f(x_0) \in \operatorname{cl} f[F]$ and $g(x_0) \in \operatorname{cl} g[F]$.

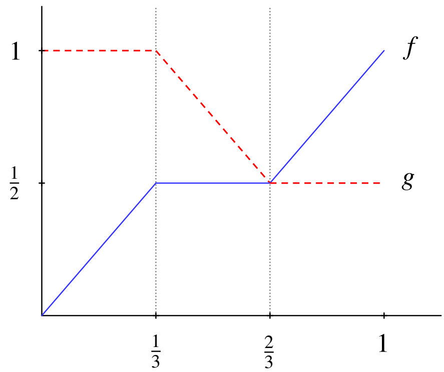

Figure 1. The family $\{f, g\}$ is finitely completely regular, but not completely regular.

It is interesting to note that the subfamily $\{f\}$ of the family in Figure 1 is not finitely completely regular because any interval of the form $\left(\frac{1}{3}+\epsilon, \frac{2}{3}-\epsilon\right)$ for $0<\epsilon<\frac{1}{6}$ is open in the usual topology of the unit interval, but not in the weak topology induced by $\{f\}$. The next example gives a family on $[0, \infty)$ that does not induce a topology that coincides with that of the original space.

Example 2. Let $\mathcal{A}=\{f(x)=\alpha x e^{-x} : \alpha \in \mathbb{R}^{+}\} \subset C([0, \infty))$, $F=[1, \infty)$, and $x_0=0$. For any finite collection $f_1, \ldots, f_n \in \mathcal{A}$, where $f_k(x)=\alpha_k x e^{-x}$, we have $0 \in \operatorname{cl} g(F)$, as $g(x)=\max |f_k(x)-f_k(x_0)|=\alpha_j x e^{-x}$ for some $1 \leq j \leq n$. Consequently, $\mathcal{A}$ fails to be finitely completely regular and so $\tau_{\mathcal{A}}$ is strictly coarser than the usual topology on $[0, \infty)$. See Figure 2 for an example.

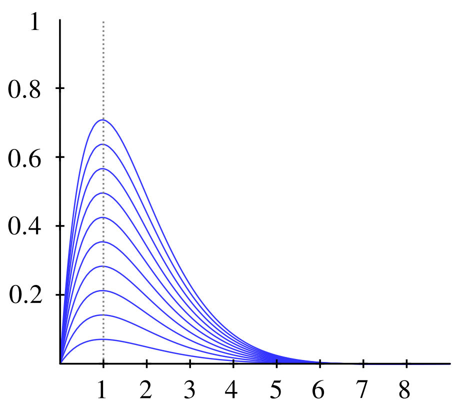

Figure 2. The finite collection $\left\{f_n(x)=\frac{1}{5} n x e^{-x} : n=1, \ldots, 10\right\} \subset \mathcal{A}$. Note that $f_n(x) \to 0$ as $x \to \infty$ for each $1 \leq n \leq 10$, and this forces $0 \in \operatorname{cl} g[[1, \infty)]$, where $g(x)=\max_{1 \leq k \leq 10} |f_k(x)-f_k(0)|$.

4. Concluding remarks

In this work we have given necessary and sufficient conditions for the coincidence of a topology and a weak topology induced by a family of continuous functions. In particular, this characterization yields a new, more direct proof of the fact that a family that separates points on a compact space will induce the original topology. The definition we introduce additionally reveals that coincidence of the two topologies is possible only when the functions in the family suitably interact with the topology, and our second example illustrates that this can fail even with uncountably many functions.

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