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Combinatorial curve neighborhoods for the affine flag manifold of type A_1^1

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Let X be the affine flag manifold of Lie type A_1^1. Its moment graph encodes the torus fixed points (which are elements of the infinite dihedral group D_∞) and the torus stable curves in X. Given a fixed point $u \in D_\infty$ and a degree $\boldsymbol{d} = (d_0, d_1) \in \mathbb{Z}^2_{\geq 0}$, the combinatorial curve neighborhood is the set of maximal elements in the moment graph of X which can be reached from u using a chain of curves of total degree $\leq \boldsymbol{d}$. In this paper we give a formula for these elements, using combinatorics of the affine root system of type A_1^1.

1. Introduction

Let X be an arbitrary algebraic variety and $\Omega \subset X$ be a subvariety. Fix a degree $\boldsymbol{d}$, i.e., an effective homology class in $H_2(X)$. The (geometric) *curve neighborhood* $\Gamma_{\boldsymbol{d}}(\Omega)$ is the locus of points $x \in X$ which can be reached from Ω by a rational curve of some effective degree $\leq \boldsymbol{d}$. For example, if $X = \mathbb{P}^2$ is the projective plane, and $\Omega = \text{pt}$, then any other point in X can be reached from the given point, using a projective line. This implies $\Gamma_1(\text{pt}) = \mathbb{P}^2$.

Curve neighborhoods have been recently defined by A. Buch and the first author in [Buch and Mihalcea 2015] in relation to the study of quantum cohomology and quantum K theory rings of generalized flag manifolds $X = G/B$, where G is a complex semisimple Lie group and B is a Borel subgroup. The curve neighborhoods which are relevant in that context are those when Ω is a Schubert variety in G/B. It turns out that in this situation the calculation of the curve neighborhoods is encoded in the *moment graph* of X. This is a graph encoding the T-fixed points and the T-stable curves in X, where T is a maximal torus of G. Similar considerations, but in the case when X is an *affine* flag manifold, led L. Mare and the first author to a definition of an affine version of the quantum cohomology ring; see [Mare

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and Mihalcea 2014]. The curve neighborhoods which were relevant for quantum cohomology calculations were those for certain "small" degrees. Those for "large" degrees, which seem to encode more refined information about the geometry and the combinatorics of affine flag manifolds, are still unknown.

In the current paper we give an explicit combinatorial formula for the curve neighborhoods of the simplest affine flag manifold, that of affine Lie type A_1^1. See, e.g., [Kumar 2002] for details on affine flag manifolds. Instead of introducing the geometry related to this flag manifold, we consider the more elementary — but equivalent — problem of calculating the *combinatorial* curve neighborhoods. These are encoded in the combinatorics of the moment graph of the affine flag manifold.

To state our main result, we briefly introduce some notation and recall a few definitions. Full details are given in Section 2 below. Let D_∞ be the infinite dihedral group, generated by reflections s_0 and s_1. (This is the affine Weyl group of Lie type A_1^1.) Each element of D_∞ has a unique reduced expression which involves a s_0's and b s_1's, where $|a-b| \leq 1$. There is a natural length function $\ell : D_\infty \to \mathbb{Z}_{\geq 0}$, and a (Bruhat) partial order on D_∞, denoted $<$. A *degree* $\boldsymbol{d}$ is a pair of nonnegative integers (d_0, d_1). The moment graph has vertices given by the elements of D_∞; there is an edge between $u, v \in D_\infty$ whenever there exists an (affine root) reflection $s_{(a,b)}$ such that $v = us_{(a,b)}$. This edge has degree $\boldsymbol{d} = (a, b)$ such that $|a - b| = 1$; see Section 2B below. A *chain* in the moment graph is a succession of adjacent edges, and its degree is equal to the sum of the degrees of each of its edges.

Finally, fix a degree $\boldsymbol{d} = (d_0, d_1)$ and $u \in D_\infty$. The (combinatorial) *curve neighborhood* $\Gamma_{\boldsymbol{d}}(u)$ is the set of elements in D_∞ such that (1) they can be joined to u (in the moment graph) by a chain of degree $\leq \boldsymbol{d}$, and (2) they are maximal among all elements satisfying (1). To each $u \in D_\infty$, one associates the degree $d(u) := (a, b)$, where u has a reduced expression with a s_0's and b s_1's.

Consider the set

$$\mathcal{A}_{\boldsymbol{d}}(u) := \{v \in D_\infty : \ell(uv) = \ell(u) + \ell(v),\ d(v) \leq \boldsymbol{d}\},$$

and denote by $\max \mathcal{A}_{\boldsymbol{d}}(u)$ the subset of its maximal elements. Our main result is:

Theorem 1.1. *Let $u \in D_\infty$ and $\boldsymbol{d} = (d_1, d_2)$ be a degree. Then the following hold*:

(a) *The curve neighborhood $\Gamma_{\boldsymbol{d}}(u)$ is given by*

$$\Gamma_{\boldsymbol{d}}(u) = \{uw : w \in \max \mathcal{A}_{\boldsymbol{d}}(u)\}.$$

(b) *Formulas* (3) *and* (4) *below give explicit combinatorial formulas for the elements in* $\max \mathcal{A}_{\boldsymbol{d}}(u)$. *In particular*, *the curve neighborhood $\Gamma_{\boldsymbol{d}}(u)$ has exactly two elements if $u = 1$ and $\boldsymbol{d} = (a, a)$*, *and one element otherwise.*

It is interesting to remark that the curve neighborhoods distinguish the degrees corresponding to "imaginary roots" (a, a) in this case. (See [Kac 1985] for more about this affine root system.) We plan to study further this phenomenon elsewhere. The theorem implies the "geometric" curve neighborhood for the Schubert variety indexed by u is either a single Schubert variety, or the union of two Schubert varieties, indexed by the elements in $\Gamma_d(u)$. We refer to [Mare and Mihalcea 2014] for a discussion of geometric curve neighborhoods.

This paper is the outcome of an undergraduate research project of Norton conducted under the direction of Mihalcea.

2. Preliminaries

2A. ***The infinite dihedral group.*** The infinite dihedral group D_∞ is the group with generators s_0, s_1 and relations $s_0^2 = s_1^2 = 1$. Each element $w \in D_\infty$ can be written *uniquely* as a product of s_0's and s_1's in such a way that no s_0's and no s_1's are consecutive. We call such an expression *reduced*. We define the *length* $\ell(w)$ of w to be the total number of s_0's and s_1's in the expression of w. For example, $\ell(s_0) = 1$ and $\ell(s_1 s_0 s_1 s_0) = 4$.

A (positive) *root* corresponding to D_∞ is a pair of nonnegative integers $\alpha = (a, b) \in \mathbb{Z}^2_{\geq 0}$ such that $|a - b| = 1$. For example, $\alpha_0 := (1, 0)$ and $\alpha_1 := (0, 1)$ are roots, and so is $\alpha = 2\alpha_0 + 3\alpha_1 = (2, 3)$. Fix a root $\alpha = (a_0, a_1)$. A *root reflection* s_α is the unique element of D_∞ which can be written as a product of a_0 s_0's and a_1 s_1's, and which has length $a + b$. For example,

$$s_{(2,3)} = s_1 s_0 s_1 s_0 s_1, \quad s_{(1,0)} = s_0.$$

The terminology follows from the fact that these are the positive roots of the affine Lie algebra of type A_1^1; see, e.g., [Kac 1985].

We record for later use the following properties:

Lemma 2.1. *Let $u, v \in D_\infty$. Then*:

(a) $\ell(u) = \ell(u^{-1})$.

(b) *u is a root reflection if and only if $\ell(u)$ is odd.*

(c) $\ell(uv) \leq \ell(u) + \ell(v)$.

(d) *If $\ell(u) \leq \ell(v)$, then*

$$\ell(uv) = \ell(u) + \ell(v) \quad \textit{or} \quad \ell(uv) = \ell(v) - \ell(u).$$

In particular, $\ell(uv) \equiv \ell(u) + \ell(v) \mod 2$.

Proof. This is an easy verification. □

2B. ***The moment graph and curve neighborhoods.*** The *moment graph G* associated to D_∞ is the graph given by the following data:

- The set V of *vertices* is the group D_∞.

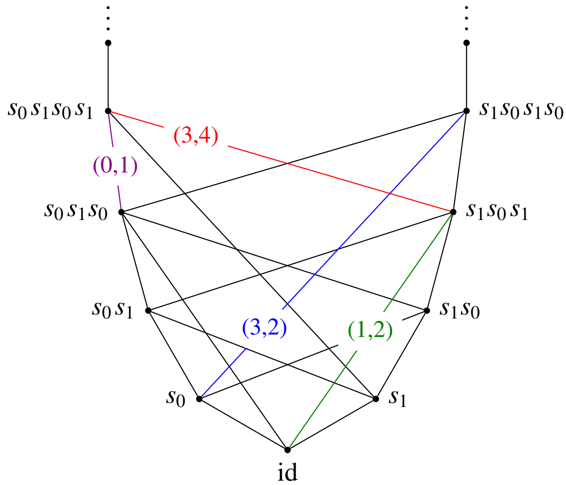

Figure 1. The moment graph G associated to D_∞.

- Let $u, v \in V$ be vertices. Then there is an edge from u to v if and only if there exists a root $\alpha = (a_0, a_1)$ such that $v = us_\alpha$. We denote this situation by

$$u \xrightarrow{\alpha} v,$$

and we say that the *degree* of this edge is α.

In Figure 1 we show the moment graph up to elements of length 4. We labeled a few of the edges by their corresponding degrees.

Remark 2.2. As mentioned in the Introduction, the vertices of this graph correspond to the T-fixed points, and its edges to the T-stable curves in the affine flag manifold of type A_1^1, where T is a maximal torus in an affine Kac–Moody group of type A_1^1. See, e.g., [Kumar 2002, Chapter 12], especially §12.2.E, for details.

A *chain* between u and v in the moment graph is a succession of adjacent edges starting with u and ending with v:

$$\pi : u = u_0 \xrightarrow{\beta_0} u_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{n-2}} u_{n-1} \xrightarrow{\beta_{n-1}} u_n = v.$$

The chain is called *increasing* if at each step the lengths increase, i.e., $\ell(u_i) > \ell(u_{i-1})$ for $1 \le i \le n$. The *degree* of the chain π is $\deg(\pi) = \beta_0 + \cdots + \beta_{n-1}$. Define a partial ordering on the elements of D_∞ by $u < v$ if and only if there exists an increasing chain starting with u and ending with v.

The next result gives an equivalent way to describe the partial ordering on D_∞:

Lemma 2.3. *Let $u, v \in D_\infty$. Then $u < v$ if and only if $\ell(u) < \ell(v)$.*

Proof. Clearly if $u < v$ then $\ell(u) < \ell(v)$ from the definition of an increasing chain. To prove the converse, we first notice that if $\ell(v) - \ell(u) = 1$, then $u^{-1}v$ is a root reflection s_α (possibly of length > 1); thus there exists an edge $u \xrightarrow{\alpha} v$. The general statement follows by induction on $\ell(v) - \ell(u) \geq 1$. □

A *degree* is a pair of nonnegative integers $\boldsymbol{d} = (d_0, d_1)$. There is a natural partial order on degrees. If $\boldsymbol{d} = (d_0, d_1)$ and $\boldsymbol{d}' = (d_0', d_1')$ then $\boldsymbol{d} \geq \boldsymbol{d}'$ if and only if $d_i \geq d_i'$ for $i \in \{0, 1\}$.

Definition 2.4. Fix a degree $\boldsymbol{d}$ and $u \in D_\infty$. The *(combinatorial) curve neighborhood* is the set $\Gamma_{\boldsymbol{d}}(u)$ consisting of elements $v \in D_\infty$ such that

(1) there exists a chain of some degree $\boldsymbol{d}' \leq \boldsymbol{d}$ from u to v in the moment graph G;

(2) the elements v are maximal among all of those satisfying the condition in (1).

For example,

$$\Gamma_{(1,0)}(\mathrm{id}) = \Gamma_{(2,0)}(\mathrm{id}) = \{s_0\}, \quad \Gamma_{(1,1)}(\mathrm{id}) = \{s_1 s_0,\ s_0 s_1\}.$$

Our main goal is to find a formula to determine $\Gamma_{\boldsymbol{d}}(u)$.

For $w \in D_\infty$, define the degree associated to w to be $d(w) = (d_0, d_1)$, where $d_i :=$ number of reflections s_i in the reduced word of w. The following holds.

Lemma 2.5. *Let $u, v \in D_\infty$ and assume there is a chain from u to v of degree $\boldsymbol{d}$. Then $\boldsymbol{d} = d(u^{-1}v) + 2(r, s)$, where $r, s \in \mathbb{Z}_{\geq 0}$. In particular, $\boldsymbol{d} \geq d(u^{-1}v)$.*

Proof. Let $\beta_0 := (a_0, b_0),\ \beta_1 := (a_1, b_1),\ \ldots,\ \beta_{n-1} := (a_{n-1}, b_{n-1})$ be the labels of the edges of the chain π. Then $v = u s_{\beta_0} \cdots s_{\beta_{n-1}}$ and $\boldsymbol{d} = \beta_0 + \cdots + \beta_{n-1} = (a_0 + \cdots + a_{n-1},\ b_0 + \cdots + b_{n-1})$. Now $d(u^{-1}v) = d(s_{\beta_0} \cdots s_{\beta_{n-1}})$. If $s_{\beta_0} \cdots s_{\beta_{n-1}}$ is nonreduced, one needs to perform some cancellations of the form $s_0^2 = 1$ or $s_1^2 = 1$. Each of these result in a decrease by 2 of the number of s_0's, respectively s_1's, in an expression for $s_{\beta_0} \cdots s_{\beta_{n-1}}$. Thus $d(u^{-1}v) = \boldsymbol{d} - 2(r, s)$ as claimed. □

3. Calculation of the curve neighborhoods

Let $\boldsymbol{d} = (d_1, d_2)$ be a degree such that $d_1 \neq d_2$. We denote by $\alpha(\boldsymbol{d})$ the maximal root α such that $\alpha \leq \boldsymbol{d}$. Clearly there is exactly one such root, and it is easy to find the following explicit formula for it:

$$\alpha(\boldsymbol{d}) = \begin{cases} (d_1, d_1 + 1) & \text{if } d_1 < d_2, \\ (d_2 + 1, d_2) & \text{if } d_1 > d_2. \end{cases} \tag{1}$$

In order to find the curve neighborhoods of an element $u \in D_\infty$, we need the following key result.

Lemma 3.1. *Let $u \in D_\infty$ and $\boldsymbol{d} = (d_1, d_2)$ be a degree. Consider the set*

$$\mathcal{A}_{\boldsymbol{d}}(u) := \{v \in D_\infty : \ell(uv) = \ell(u) + \ell(v), d(v) \leq \boldsymbol{d}\}. \tag{2}$$

Then the following hold:

- $\mathcal{A}_{\boldsymbol{d}}(u)$ *has a unique maximal element if* $u \neq 1$ *or if* $u = 1$ *and* $\boldsymbol{d} \neq (a, a)$ *for some nonnegative integer* a.
- *If* $\boldsymbol{d} = (a, a)$ *and* $u = 1$ *then the maximal elements of* $\mathcal{A}_{\boldsymbol{d}}(u)$ *are* $(s_0 s_1)^a$ *and* $(s_1 s_0)^a$.

Proof. Clearly, $1 \in \mathcal{A}_{\boldsymbol{d}}(u)$ so $\mathcal{A}_{\boldsymbol{d}}(u) \neq \varnothing$. For any $v \in \mathcal{A}_{\boldsymbol{d}}(u)$, we have $d(v) \leq \boldsymbol{d}$. Hence the set $\mathcal{A}_{\boldsymbol{d}}(u)$ is finite, and so it must contain a maximal element. Lemma 2.3 implies there can be at most two maximal elements v_1 and v_2 and they must have the same length. We consider each of the situations in the statement:

Case 1: $u \neq 1$. Assume there are two maximal elements v_1, v_2. Since $u \neq 1$, either uv_1 or uv_2 is not reduced, say uv_1. Then $\ell(uv_1) < \ell(u) + \ell(v_1)$, and this contradicts that $v_1 \in \mathcal{A}_{\boldsymbol{d}}(u)$.

Case 2: $u = 1$. In this case, the set $\mathcal{A}_{\boldsymbol{d}}(u)$ coincides with the set of all $v \in D_\infty$ such that $d(v) \leq \boldsymbol{d}$. From the description of D_∞, it follows that $d(v) = (a, a)$ or $d(v) = (a, a+1)$ or $d(v) = (a+1, a)$ for some nonnegative integer a. Further, the reduced decomposition of v is known in each case: there are two possibilities for v if $d(v) = (a, a)$, and there is exactly one (in fact, $v = s_{\alpha(\boldsymbol{d})}$) in the other two cases. The claim follows from this. $\square$

In what follows, we will denote by $\max \mathcal{A}_{\boldsymbol{d}}(u)$ the set of maximal elements in the (finite) partially ordered set $\mathcal{A}_{\boldsymbol{d}}(u)$. Our main result is:

Theorem 3.2. *Let* $u \in D_\infty$ *and* $\boldsymbol{d} = (d_1, d_2)$ *be a degree. Then*

$$\Gamma_{\boldsymbol{d}}(u) = \{uw : w \in \max \mathcal{A}_{\boldsymbol{d}}(u)\}.$$

We will prove this theorem in the next two sections, which correspond to the cases $u = 1$ and $u \neq 1$. For now, notice that the proof of Lemma 3.1, and some easy arguments based on reduced decompositions in D_∞, imply that if $u = 1$ then the set of maximal elements of $\mathcal{A}_{\boldsymbol{d}}(1)$ is

$$\max \mathcal{A}_{\boldsymbol{d}}(1) = \begin{cases} \{s_{\alpha(\boldsymbol{d})}\} & \text{if } \boldsymbol{d} = (d_1, d_2) \text{ and } d_1 \neq d_2, \\ \{(s_0 s_1)^a, (s_1 s_0)^a\} & \text{if } \boldsymbol{d} = (a, a). \end{cases} \tag{3}$$

If $u \neq 1$, we assume for simplicity that last simple reflection in the reduced word for u is s_0, i.e., $u = \cdots s_0$. (The other situation will be symmetric). Then

$$\max \mathcal{A}_{\boldsymbol{d}}(u) = \begin{cases} \{s_1 s_{\alpha(\boldsymbol{d}-(0,1))}\} & \text{if } d_0 = d_1, \\ \{s_{\alpha(\boldsymbol{d})}\} & \text{if } d_1 > d_0, \\ \{s_0 s_{\alpha(\boldsymbol{d})}\} & \text{if } d_1 < d_0. \end{cases} \tag{4}$$

The two formulas give explicit combinatorial rules to determine the curve neighborhood $\Gamma_{\boldsymbol{d}}(u)$. See Section 3C below for several examples.

3A. ***Curve neighborhoods for $u = 1$.***

Theorem 3.3. *Let $\boldsymbol{d} = (d_1, d_2)$ be a degree. Then the curve neighborhood of the identity can be calculated in the following way*:

$$\Gamma_{\boldsymbol{d}}(1) = \max \mathcal{A}_{\boldsymbol{d}}(1) = \begin{cases} \{s_{\alpha(\boldsymbol{d})}\} & \text{if } d_1 \neq d_2, \\ \{(s_0 s_1)^a, (s_1 s_0)^a\} & \text{if } \boldsymbol{d} = (a, a). \end{cases}$$

Proof. If $v \in \Gamma_{\boldsymbol{d}}(1)$ then there exists a chain of degree $\leq \boldsymbol{d}$ joining 1 to v. Then by Lemma 2.5, $\boldsymbol{d} \geq d(v)$. In particular, $v \in \mathcal{A}_{\boldsymbol{d}}(1)$; thus $\Gamma_{\boldsymbol{d}}(1) \subset \mathcal{A}_{\boldsymbol{d}}(1)$, and the inclusion is compatible with the partial order $<$. Conversely, if v is any element in $\mathcal{A}_{\boldsymbol{d}}(1)$ then there exists a chain of degree $d(v) \leq \boldsymbol{d}$ joining 1 to v. If v is maximal in $\mathcal{A}_{\boldsymbol{d}}(1)$, and because $\Gamma_{\boldsymbol{d}}(1) \subset \mathcal{A}_{\boldsymbol{d}}(1)$, it follows that $v \in \Gamma_{\boldsymbol{d}}(1)$. □

3B. ***General curve neighborhoods.*** The goal of this section is to find a formula for the curve neighborhoods $\Gamma_d(u)$ for $u \neq 1$ and $\boldsymbol{d} \neq (0, 0)$. First we need some preparatory lemmas.

Lemma 3.4. *Let $u \in D_\infty$, $z \in \mathcal{A}_{\boldsymbol{d}}(1)$ and $v \in \Gamma_{\boldsymbol{d}}(u)$. Then*:

(a) *$\ell(uz) \leq \ell(v)$ and $d(u^{-1}v) \leq \boldsymbol{d}$.*

(b) *If $z \in \Gamma_{\boldsymbol{d}}(1)$ (i.e., z is maximal in $\mathcal{A}_{\boldsymbol{d}}(1)$), then $\ell(u^{-1}v) \leq \ell(z)$.*

Proof. Since $z \in \mathcal{A}_{\boldsymbol{d}}(1)$, there exists a chain of degree $d(z) \leq \boldsymbol{d}$ joining 1 to z. Multiplying this chain by u on the left gives a chain between u and uz of the same degree. The first statement in (a) follows by the maximality of v. To prove the second statement in (a), notice that since $v \in \Gamma_{\boldsymbol{d}}(u)$, there exists a chain from u to v of degree $\leq \boldsymbol{d}$. If we multiply each element of this chain on the left by u^{-1}, we obtain a chain from 1 to $u^{-1}v$ of the same degree. The fact that $d(u^{-1}v) \leq \boldsymbol{d}$ follows from Lemma 2.5. Finally, (b) follows from the maximality of z, using also that maximal elements in $\mathcal{A}_{\boldsymbol{d}}(1)$ have the same length, by (3). □

The following lemma gives a strong constraint on the possible elements in $\Gamma_{\boldsymbol{d}}(u)$.

Lemma 3.5. *Let $v \in \Gamma_{\boldsymbol{d}}(u)$. Then $u^{-1}v \in \mathcal{A}_{\boldsymbol{d}}(u)$.*

Proof. We have seen in Lemma 3.4 that $d(u^{-1}v) \leq \boldsymbol{d}$. It remains to show that $\ell(v) = \ell(u) + \ell(u^{-1}v)$. This clearly holds for $u = 1$ and from now on we assume $u \neq 1$. From Lemma 2.1(c) it follows that $\ell(v) = \ell(uu^{-1}v) \leq \ell(u) + \ell(u^{-1}v)$. If the inequality is strict then $\ell(u^{-1}v) > \ell(v) - \ell(u) = \ell(v) - \ell(u^{-1})$. But $\ell(u) \leq \ell(v)$, thus by Lemma 2.1(d) it follows that

$$\ell(u^{-1}v) = \ell(u) + \ell(v).$$

Consider now an element $z \in \Gamma_{\boldsymbol{d}}(1) = \max \mathcal{A}_{\boldsymbol{d}}(1)$ (by Theorem 3.3). We invoke Lemma 3.4 to obtain

$$\ell(uz) \leq \ell(v) < \ell(u) + \ell(u^{-1}v) \leq \ell(u) + \ell(z).$$

This implies the expression uz is not reduced. But since $u \neq 1$, we can eliminate the first simple reflection from the reduced expression for z to define $z' < z$ such that $\ell(z') = \ell(z) - 1$ and $\ell(uz') = \ell(u) + \ell(z')$. Notice that $d(z') < d(z) \leq \boldsymbol{d}$; thus $z' \in \mathcal{A}_{\boldsymbol{d}}(1)$. Then we have the inequalities

$$\ell(v) \geq \ell(uz') = \ell(u) + \ell(z) - 1 \geq \ell(u) + \ell(u^{-1}v) - 1 = \ell(u) + \ell(u) + \ell(v) - 1,$$

where the first inequality follows from Lemma 3.4(a) and the last inequality follows from Lemma 3.4(b). Taking the extreme sides and subtracting $\ell(v)$, we obtain $0 \geq 2\ell(u) - 1$, which is impossible since $\ell(u) \geq 1$. Thus $\ell(v) = \ell(u) + \ell(u^{-1}v)$ and this finishes the proof. □

We are ready to prove our main result.

Theorem 3.6. *Let $\boldsymbol{d} = (d_1, d_2)$ be a nonzero degree and $u \in D_\infty$. Then*

$$\Gamma_{\boldsymbol{d}}(u) = \{uw : w \in \max \mathcal{A}_{\boldsymbol{d}}(u)\}.$$

Proof. Let $v \in \Gamma_{\boldsymbol{d}}(u)$. Then Lemma 3.5 implies $u^{-1}v \in \mathcal{A}_{\boldsymbol{d}}(u)$. From Lemma 3.1 (or (4)), there exists a unique maximal element of $\mathcal{A}_{\boldsymbol{d}}(u)$, call it w. Then $u^{-1}v \leq w$ and clearly w is also in $\mathcal{A}_{\boldsymbol{d}}(1)$. By Lemma 3.4(a), we deduce $\ell(uw) \leq \ell(v)$. Then

$$\ell(u) + \ell(u^{-1}v) = \ell(v) \geq \ell(uw) = \ell(u) + \ell(w).$$

This implies $\ell(u^{-1}v) \geq \ell(w)$. Together with $u^{-1}v \leq w$, this forces $u^{-1}v = w$; i.e., $v = uw$ as claimed. □

3C. *Examples.* We provide several examples determining $\Gamma_{\boldsymbol{d}}(u)$.

- Let $u = 1$ and $\boldsymbol{d} = (9, 4)$. From (1) we obtain $\alpha(\boldsymbol{d}) = (5, 4)$; thus

  $$\Gamma_{(9,4)}(1) = \{s_{(5,4)}\} = \{s_0 s_1 s_0 s_1 s_0 s_1 s_0 s_1 s_0\}.$$

- Let $u = 1$ and $\boldsymbol{d} = (4, 4)$. By (3) the two maximal elements in $\mathcal{A}_{(4,4)}(1)$ are $s_0 s_1 s_0 s_1 s_0 s_1 s_0 s_1$ and $s_1 s_0 s_1 s_0 s_1 s_0 s_1 s_0$. Then

  $$\Gamma_{(4,4)}(\mathrm{id}) = \{s_0 s_1 s_0 s_1 s_0 s_1 s_0 s_1,\ s_1 s_0 s_1 s_0 s_1 s_0 s_1 s_0\}.$$

- Let $u = s_0 s_1 s_0$ and $\boldsymbol{d} = (3, 3)$. From (4),

  $$\max \mathcal{A}_{(3,3)}(u) = \{s_1 s_{\alpha((3,3)-(0,1))}\} = \{s_1 s_0 s_1 s_0 s_1 s_0\}.$$

  Thus $\Gamma_{(3,3)}(s_0 s_1 s_0) = \{(s_0 s_1 s_0)(s_1 s_0 s_1 s_0 s_1 s_0)\}$.

- Let $u = s_1 s_0 s_1$ and $\boldsymbol{d} = (3, 3)$. From the symmetric version of (4),

  $$\max \mathcal{A}_{(3,3)}(u) = \{s_0 s_{\alpha((3,3)-(1,0))}\} = \{s_0 s_1 s_0 s_1 s_0 s_1\}.$$

  Thus $\Gamma_{(3,3)}(s_1 s_0 s_1) = \{(s_1 s_0 s_1)(s_0 s_1 s_0 s_1 s_0 s_1)\}$.

- Let $u = s_0 s_1 s_0$ and $\boldsymbol{d} = (9,4)$. Then $\alpha((9,4)) = (5,4)$ and using (4) again, $\max \mathcal{A}_{\boldsymbol{d}}(u) = \{s_1 s_0 s_1 s_0 s_1 s_0 s_1 s_0\}$. Then

$$\Gamma_{(9,4)}(s_0 s_1 s_0) = \{(s_0 s_1 s_0)(s_1 s_0 s_1 s_0 s_1 s_0 s_1 s_0)\}.$$

- Let $u = s_0 s_1 s_0$ and $d = (4,9)$. Then $\alpha((4,9)) = (4,5)$ and $\max \mathcal{A}_{\boldsymbol{d}}(u) = \{s_1 s_0 s_1 s_0 s_1 s_0 s_1 s_0 s_1\}$. From this we obtain

$$\Gamma_{(9,4)}(s_0 s_1 s_0) = \{(s_0 s_1 s_0)(s_1 s_0 s_1 s_0 s_1 s_0 s_1 s_0 s_1)\}.$$

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