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Total variation based denoising methods for speckle noise images

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In this paper, we introduce a new algorithm based on total variation for denoising speckle noise images. Total variation was introduced by Rudin, Osher, and Fatemi in 1992 for regularizing images. Chambolle proposed a faster algorithm based on the duality of convex functions for minimizing the total variation, but his algorithm was built for Gaussian noise removal. Unlike Gaussian noise, which is additive, speckle noise is multiplicative. We modify the original Chambolle algorithm for speckle noise images using the first noise equation for speckle denoising, proposed by Krissian, Kikinis, Westin and Vosburgh in 2005. We apply the Chambolle algorithm to the Krissian et al. speckle denoising model to develop a faster algorithm for speckle noise images.

1. Introduction

Image restoration, especially image denoising, is a very important process and is often necessary as preprocessing for other imaging techniques such as segmentation and compression. For the last two decades, various partial differential equation (PDE) based models have been developed for this purpose [Rudin et al. 1992; Perona and Malik 1990; Kornprobst et al. 1997; Catté et al. 1992; Alvarez et al. 1992; Chan and Vese 1997; Vese and Chan 1997; Marquina and Osher 2000; Chan et al. 1999; Joo and Kim 2003a; 2003b; Kim 2004; Kim and Lim 2007]. In general, an observed image f, corrupted by Gaussian noise n, is represented by the equation

$$f = u + n, \tag{1}$$

where u is the original noise-free image. Here $u,\ f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$. For any denoising model, the main objective is to reconstruct u from an observed image f.

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Rudin, Osher, and Fatemi [Rudin et al. 1992] proposed the total variation (TV) denoising model as the minimization problem

$$\min_u \int_\Omega |\nabla u| \, \mathrm{d}\vec{x} \tag{2}$$

subject to the constraints,

$$\int_\Omega f \, \mathrm{d}\vec{x} = \int_\Omega u \, \mathrm{d}\vec{x}, \tag{3}$$

$$\int_\Omega \tfrac{1}{2}(f-u)^2 \, \mathrm{d}\vec{x} = \sigma^2, \tag{4}$$

where σ is the standard deviation of the noise n. These constraints ensure that the resulting image and the observed image are close to each other.

Combining the above constraints, the TV functional is obtained by

$$F(u) = \int_\Omega |\nabla u| \, \mathrm{d}\vec{x} + \frac{\lambda}{2} \int_\Omega (f-u)^2 \, \mathrm{d}\vec{x}. \tag{5}$$

Here, λ is a constraint parameter. The equivalent Euler–Lagrange equation gives the TV denoising model as

$$\frac{\partial u}{\partial t} - \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = \lambda(f-u). \tag{6}$$

To avoid singularities, it was regularized by using $|\nabla u| \approx |\nabla^\varepsilon u| = (u_x^2 + u_y^2 + \varepsilon^2)^{1/2}$.

In this paper, we introduce a faster denoising method, compared to TV model, for speckle noise images. Unlike Gaussian noise, speckle noise is multiplicative and requires a model separate from those for Gaussian noise images. The first effective speckle denoising model was developed by Krissian, Kikinis, Westin, and Vosburgh [Krissian et al. 2005], and they proposed a new noise equation for speckle denoising. For our new model, we modify the original Chambolle algorithm designed for the TV model (6), and develop a fast and accurate speckle denoising method based on the noise equation proposed by Krissian et al.

2. The Chambolle algorithm: dual approach

In this section, we provide a brief description of the Chambolle algorithm for the TV model. Chambolle [2004] provided a fast algorithm for minimizing the total variation. Detailed background and development of the algorithm can be found in his lecture notes [Chambolle et al. 2010]. The work is based on the dual formulation of Chan, Golub, and Mulet [Chan et al. 1999] and of Carter [2001]. To avoid the staircasing effect, he derived the Euler–Lagrange equation in the sense of convex analysis. His paper also contains the proof of the convergence of his algorithm.

The Chambolle algorithm. Chambolle [2004] started with the Rudin, Osher, and Fatemi (ROF) minimization functional [Rudin et al. 1992]

$$\min_u \left[\lambda J(u) + \int_\Omega \tfrac{1}{2}|u-f|^2 \, d\vec{x} \right], \tag{7}$$

where $J(u) = \int_\Omega |\nabla u| \, d\vec{x}$. He proved that u is a minimizer of (7) if and only if

$$\frac{f-u}{\lambda} \in \partial J(u),$$

where ∂F denotes the subdifferential of a convex function F. Hence, the Euler–Lagrange equation obtained by Chambolle is given as

$$\lambda \partial J(u) + u - f \ni 0. \tag{8}$$

Note that $u = (u_{ij})$, where $i, j = 1, \dots, N$, is the discrete image. Thus $u \in X = \mathbb{R}^{N \times N}$. Applying the Legendre–Fenchel identity, he obtained the dual problem as

$$\min_{|p_{i,j}| \le 1} \frac{1}{2} \left\| \operatorname{div} p - \frac{f}{\lambda} \right\|^2, \tag{9}$$

where $\operatorname{div} p = w$ for $p = (p_{i,j})$ with $i, j = 1, \dots, N \in Y = X \times X$ and $w = (f-u)/\lambda$. Here $\|\cdot\|$ is the Euclidean norm, which is defined similarly to (6) in [Chambolle 2004]. We can recover u by

$$u = f - \lambda w = f - \lambda \operatorname{div} p. \tag{10}$$

Hence, to find the denoised image u, the following problem must be solved for p:

$$\min \left\{ \|\lambda \operatorname{div} p - f\|^2 : p \in Y, \ |p_{i,j}| \le 1 \ \forall i, j = 1, \dots, N \right\}. \tag{11}$$

Chambolle proposed the following algorithm to solve for p. Choosing $\tau > 0$ and taking $p^0 = 0$ we derive p^n for any $n > 0$, by

$$p_{i,j}^{n+1} = \frac{p_{i,j}^n + \tau(\nabla(\operatorname{div} p^n - f/\lambda))_{i,j}}{1 + \tau \left| (\nabla(\operatorname{div} p^n - f/\lambda))_{i,j} \right|}. \tag{12}$$

The next theorem proved the convergence of the algorithm in [Chambolle 2004]:

Theorem 2.1. *If $\tau < \frac{1}{8}$, then $f - \lambda \operatorname{div} p^n$ converges to u as $n \to \infty$.*

Image denoising with the Chambolle algorithm. In general, for Gaussian noise images, Chambolle proposed updating λ at each iteration by the formula

$$\lambda^{n+1} = \frac{N\sigma}{g_n} \lambda^n = \frac{N\sigma}{\|\operatorname{div} p^{n+1}\|} = \frac{\|f - u_c\|}{\|\operatorname{div} p^{n+1}\|}, \tag{13}$$

where u_c is the noise-free clear image and the variance σ^2 of the noise is known. For $s > 0$, he defined $g(s) = \|s \operatorname{div} p\|$. Here, the starting value $\lambda_0 > 0$ is chosen arbitrarily. Thus the Chambolle algorithm for TV denoising [Chambolle 2004; Chambolle et al. 2010] is given by

$$p_{i,j}^{n+1} = \frac{p_{i,j}^{n} + \tau(\nabla(\operatorname{div} p^n - f/\lambda))_{i,j}}{1 + \tau|(\nabla(\operatorname{div} p^n - f/\lambda))_{i,j}|},$$
$$\lambda^{n+1} = \frac{\|f - u_c\|}{\|\operatorname{div} p^{n+1}\|},$$
$$u^{n+1} = f - \lambda^{n+1} \operatorname{div} p^{n+1}$$

for any $n \geq 0$. This algorithm converges almost twice as fast as the regular TV model (6).

3. Speckle noise

Speckle noise is mostly present in ultrasound images, synthetic aperture radar (SAR) images, and acoustic images. It is granular in nature, and it exists inherently in the image. Unlike Gaussian noise, which affects single pixels of an image, speckle noise affects multiple pixels. The noise is multiplicative, whereas Gaussian noise is additive. Hence, it is not possible to remove speckle noise with the traditional Gaussian denoising models.

Speckle denoising model by Krissian et al. This model was proposed by Krissian, Kikinis, Westin, and Vosburgh [Krissian et al. 2005], where they mainly dealt with speckle noise present in ultrasound images. They considered the speckle noise equation as

$$f = u + \sqrt{u}n, \tag{14}$$

where u is the desired image to find, n is Gaussian noise, and f is the observed image. Hence using $n = (f - u)/\sqrt{u}$, the general regularized minimization functional is given as

$$\min_u F(u) = \min_u \left(\int_\Omega \left[|\nabla u| + \frac{\lambda}{2} \left(\frac{f - u}{\sqrt{u}} \right)^2 \right] d\vec{x} \right).$$

Finally using the Euler–Lagrange equation of this functional, the TV based speckle denoising model [Marquina and Osher 2000; Kim and Lim 2007] is derived as

$$\frac{\partial u}{\partial t} - \frac{u^2}{f + u} |\nabla u| \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = \lambda\, |\nabla u|\, (f - u). \tag{15}$$

4. The modified Chambolle for speckle denoising (MCSD) model

The Chambolle algorithm gives faster results than the regular TV model. Unfortunately, the model is formulated to work only for synthetic Gaussian noises.

We modify it to obtain the modified Chambolle for speckle denoising (MCSD) model for natural speckle noise. In [Wen et al. 2016] the authors used the primal-dual approach of Chambolle [Chambolle and Pock 2011] to develop a TV based denoising model for Poisson noise images. Similar approaches were proposed for multiplicative-noise images based on the Chambolle primal-dual algorithm in [Chan et al. 2014; Huang et al. 2012; 2013a; 2013b; Dong and Zeng 2013] for image segmentation, denoising and deblurring.

The MCSD model is based on the Chambolle algorithm for faster TV denoising. We apply the Chambolle-TV algorithm on the Krissian et al. speckle model to obtain a faster speckle denoising model. We start with developing the Euler–Lagrange equation for MCSD and then build the algorithm based on it.

The Euler–Lagrange equation for MCSD. We start by developing the Euler–Lagrange equation based on the speckle noise equation (14) introduced by Krissian et al. The minimization functional will be given by

$$\min_u \left[\lambda J(u) + \int_\Omega \frac{1}{2} \frac{|u-f|^2}{u} \, d\vec{x} \right]. \tag{16}$$

A similar model has also been discussed in [Jin and Yang 2011]. In this paper, the authors also developed the denoising functional (16) for speckle noise images motivated by the ROF model [Rudin et al. 1992] and the speckle noise model by Krissian et al. [2005]. They proved the existence and uniqueness of the minimizer for the functional (16). The existence and uniqueness of weak solutions for the associated evolution equation were also derived. For numerical computation, they directly used the finite difference scheme for the evolution equation, based on the schemes introduced in [Rudin et al. 1992]. However, in our paper, we adopt a dual formation suggested by Chambolle [2004] (see also [Chambolle et al. 2010]) for (16) to produce a faster algorithm. This is the main difference between the results of [Jin and Yang 2011] and ours. Moreover, for the purpose of development of the Euler–Lagrange equation, we consider the following slightly modified functional by using the fact $u \approx f$:

$$\min_u \left[\lambda J(u) + \int_\Omega \frac{1}{2} \frac{|u-f|^2}{f} \, d\vec{x} \right]. \tag{17}$$

The following Theorem 4.1 provides us with the formulation of the Euler–Lagrange equation for our new model.

Theorem 4.1. *The Euler–Lagrange equation for the minimizing functional* (16) *is*

$$\partial J(u) + \frac{u-f}{\lambda u} \ni 0. \tag{18}$$

Proof. If u is the solution of the functional (16), since $u \approx f$, we have for any $v \in L^2(\Omega)$,

$$\begin{aligned}\lambda J(v) + \frac{1}{2}\int_\Omega \frac{|v-f|^2}{f}\,\mathrm{d}\vec{x} \geq \lambda J(u) + \frac{1}{2}\int_\Omega \frac{|u-f|^2}{f}\,\mathrm{d}\vec{x} \\ \Rightarrow \lambda J(v) \geq \lambda J(u) + \frac{1}{2}\int_\Omega \frac{1}{f}[(u-f)^2 - (v-f)^2]\,\mathrm{d}\vec{x} \\ = \lambda J(u) + \int_\Omega \frac{u-v}{f}\Big(\frac{v-u}{2} - (f-u)\Big)\,\mathrm{d}\vec{x} \\ = \lambda J(u) - \int_\Omega \frac{(u-v)^2}{2f}\,\mathrm{d}\vec{x} + \int_\Omega (v-u)\,\frac{f-u}{f}\,\mathrm{d}\vec{x}.\end{aligned}$$

Now for any $t \in \mathbb{R}$, we get

$$\lambda\big(J(u+t(v-u)) - J(u)\big) - t\int_\Omega (v-u)\,\frac{f-u}{f}\,\mathrm{d}\vec{x} \geq -\frac{t^2}{2}\int_\Omega \frac{(v-u)^2}{f}\,\mathrm{d}\vec{x}.$$

In the above inequality, the left-hand side is a convex function of $t \in \mathbb{R}$ and the right-hand side is a concave parabola (as a function of t). The maximum point of the parabola is at $t = 0$, and it meets the convex function at this point. Thus, we can easily conclude that the convex function on the left-hand side will be larger than the maximum of the parabola, which is zero here, at every point. Hence,

$$\lambda\big(J(u+t(v-u)) - J(u)\big) - t\int_\Omega (v-u)\,\frac{f-u}{f}\,\mathrm{d}\vec{x} \geq 0.$$

Since this is true for any $t \in \mathbb{R}$, considering $f \approx u$ and $t = 1$ gives us, for all $v \in L^2(\Omega)$,

$$J(v) \geq J(u) + \int_\Omega (v-u)\,\frac{f-u}{\lambda u}\,\mathrm{d}\vec{x}.$$

Using the definition of subdifferential [Chambolle 2004],

$$\frac{f-u}{\lambda u} \in \partial J(u).$$

Conversely, if this is true, then we see that (18) holds.

Thus u is a minimizer of (16), and (18) gives the required Euler–Lagrange equation. □

The MCSD algorithm. From the Euler–Lagrange equation (18) and the Legendre–Fenchel identity property,

$$u \in \partial J^*\Big(\frac{f-u}{\lambda u}\Big).$$

Setting $w = (f - u)/(\lambda u)$, we have

$$u \in \partial J^*(w) \Rightarrow \frac{f}{\lambda u} \in \frac{f-u}{\lambda u} + \frac{\partial J^*(w)}{\lambda u}$$

$$\Rightarrow 0 \in w - \frac{f}{\lambda u} + \frac{\partial J^*(w)}{\lambda u} \tag{19}$$

$$\Rightarrow 0 \in uw - \frac{f}{\lambda} + \frac{\partial J^*(w)}{\lambda}. \tag{20}$$

If the minimizing functional, where w is the minimizer, is given by

$$\frac{\left\| \sqrt{u} w - f/(\lambda \sqrt{u}) \right\|^2}{2} + \frac{1}{\lambda} J^*(w), \tag{21}$$

then similar analysis as in the proof of Theorem 4.1 yields that the corresponding Euler–Lagrange equation is (20).

Since $w \in K$, where $K = \{\operatorname{div} p : p \in Y, \|p_{i,j}\| \le 1 \ \forall i, j\}$, and $J^* = H$, where H is defined as

$$H(w) = \begin{cases} 0 & \text{if } w \in K, \\ +\infty & \text{if } w \notin K, \end{cases} \tag{22}$$

we get $J^*(w) = 0$. Therefore, finding a minimizer w for (21) is equivalent to solving the problem

$$\min_p \left\{ \left\| \sqrt{u} \operatorname{div} p - \frac{f}{\lambda \sqrt{u}} \right\|^2 : p \in Y, |p_{i,j}| \le 1 \ \forall i, j = 1, \ldots, N \right\}. \tag{23}$$

We need to find p for (23) and then recover u by

$$u = f - \lambda\, u \operatorname{div} p.$$

Now, to minimize (23), we consider

$$-\left[\nabla \left(\sqrt{u} \operatorname{div} p - \frac{f}{\lambda \sqrt{u}} \right) \right]_{i,j} + \alpha_{ij} p_{ij} = 0,$$

where $\alpha_{ij} \ge 0$ is a Lagrange multiplier. One can verify that $\alpha_{ij} > 0$ and $|p_{ij}| = 1$, or $|p_{ij}| < 1$ and $\alpha_{ij} = 0$. In any case,

$$\alpha_{ij} = \left| \left(\nabla \left(\sqrt{u} \operatorname{div} p^n - \frac{f}{\lambda \sqrt{u}} \right) \right)_{i,j} \right|.$$

Applying gradient descent, we can obtain the solution iteratively by the semi-implicit algorithm

$$p_{i,j}^{n+1} = p_{i,j}^n + \tau \left(\left(\nabla \left(\sqrt{u} \operatorname{div} p^n - \frac{f}{\lambda \sqrt{u}} \right) \right)_{i,j} - \left| \left(\nabla \left(\sqrt{u} \operatorname{div} p^n - \frac{f}{\lambda \sqrt{u}} \right) \right)_{i,j} \right| p_{i,j}^{n+1} \right) \tag{24}$$

for $n \geq 0$, $p^0 = 0$, and for an iterative time-step size $\tau > 0$. Then (24) gives

$$p_{i,j}^{n+1} = \frac{p_{i,j}^n + \tau\big(\nabla\big(\sqrt{u}\operatorname{div} p^n - f/(\lambda\sqrt{u})\big)\big)_{i,j}}{1 + \tau\big|\big(\nabla\big(\sqrt{u}\operatorname{div} p^n - f/(\lambda\sqrt{u})\big)\big)_{i,j}\big|}.$$

For the TV model, Chambolle proposed updating λ for denoising purposes using (13):

$$\lambda^{n+1} = \frac{\|f - u_c\|}{\|\operatorname{div} p^{n+1}\|}.$$

Here, u_c is the noise-free clear image. But this can only be obtained for synthetic images. For speckle noise images, we change (13) to

$$\lambda^{n+1} = \frac{\|f - f_s\|}{\|u^n \operatorname{div} p^{n+1}\|},$$

where f_s is the smoother version of the original or given image. For any (i, j), we obtain $f_s(i, j)$ by considering the average of the four surrounding pixels. Hence, the iterative algorithm for MCSD is given for $n \geq 0$ as

$$p_{i,j}^{n+1} = \frac{p_{i,j}^n + \tau\big(\nabla\big(\sqrt{u^n}\operatorname{div} p^n - f/(\lambda\sqrt{u^n})\big)\big)_{i,j}}{1 + \tau\big|\big(\nabla\big(\sqrt{u^n}\operatorname{div} p^n - f/(\lambda\sqrt{u^n})\big)\big)_{i,j}\big|}, \tag{25}$$

$$\lambda^{n+1} = \frac{\|f - f_s\|}{\|u^n \operatorname{div} p^{n+1}\|}, \tag{26}$$

$$u^{n+1} = f - \lambda^{n+1} u^n \operatorname{div} p^{n+1}. \tag{27}$$

Note that the problem satisfies the zero Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \tag{28}$$

5. Numerical procedure: the MCSD model

We now describe the numerical procedure used for the MCSD model. In [Chambolle 2004], the discrete gradient and divergence were defined using forward and backward differences respectively. Also, Chambolle used separate definitions for boundary points. In this model, we use central differences to define both the gradient and divergence, as we have seen that this gives more accurate results. Also, we introduce 1-pixel-wide ghost grids on each side to avoid separate definitions for boundary points and also to satisfy the zero Neumann boundary condition (28). The ghost grid values are obtained by

$$u(0, :) = u(2, :), \quad u(N+1, :) = u(N-1, :),$$
$$u(:, 0) = u(:, 2), \quad u(:, N+1) = u(:, N-1).$$

For $u \in X = \mathbb{R}^{N\times N}$, we have

$$(\nabla u)_{i,j} = \big((\nabla u)^1_{i,j}, (\nabla u)^2_{i,j}\big),$$

where

$$(\nabla u)^1_{i,j} = \tfrac{1}{2}(u_{i+1,j} - u_{i-1,j}), \quad (\nabla u)^2_{i,j} = \tfrac{1}{2}(u_{i,j+1} - u_{i,j-1})$$

for $1 \le i, j \le N$. For any vector $p = (p^1, p^2) \in Y = X \times X$, we define

$$(\operatorname{div} p)_{i,j} = \tfrac{1}{2}(p^1_{i+1,j} - p^1_{i-1,j}) + \tfrac{1}{2}(p^2_{i,j+1} - p^2_{i,j-1}) \tag{29}$$

for $1 \le i, j \le N$. Applying these definitions, we solve the iterative algorithm provided in (25)–(27).

Convergence analysis. For purposes of convergence analysis of the MCSD algorithm developed on page 332, we slightly modify the algorithm by using (19) for the Euler–Lagrange equation instead of (20). We do not iterate u in the algorithm by replacing it by f_s, where f_s is the smoother version of the original or given image. Thus the modified Euler–Lagrange equation is given by

$$0 \in w - \frac{f}{\lambda f_s} + \frac{\partial J^*(w)}{\lambda f_s}.$$

The corresponding minimizing functional is therefore

$$\frac{1}{2}\left\| w - \frac{f}{\lambda f_s}\right\|^2 + \frac{1}{\lambda f_s} J^*(w), \tag{30}$$

where $w = (f - u)/(\lambda u)$ is the minimizer. Thus we obtain the semi-implicit algorithm

$$p^{n+1}_{i,j} = p^n_{i,j} + \tau\left(\left(\nabla\left(\operatorname{div} p^n - \frac{f}{\lambda f_s}\right)\right)_{i,j} - \left|\left(\nabla\left(\operatorname{div} p^n - \frac{f}{\lambda f_s}\right)\right)_{i,j}\right| p^{n+1}_{i,j}\right) \tag{31}$$

and that

$$p^{n+1}_{i,j} = \frac{p^n_{i,j} + \tau\big(\nabla(\operatorname{div} p^n - f/(\lambda f_s))\big)_{i,j}}{1 + \tau\big|\big(\nabla(\operatorname{div} p^n - f/(\lambda f_s))\big)_{i,j}\big|} \tag{32}$$

for $n \ge 0$, $p^0 = 0$, and for an iterative time-step size $\tau > 0$. Then the following convergence theorem holds.

Theorem 5.1. *If $\tau \le \frac{1}{2}$, then $f - \lambda f_s \operatorname{div} p^n$, where p^n is obtained by* (32), *converges to u as $n \to \infty$.*

Proof. Fix $n \ge 0$ and let $\eta = (p^{n+1} - p^n)/\tau$. Since

$$\operatorname{div} p^{n+1} - \frac{f}{\lambda f_s} = \operatorname{div} p^n - \frac{f}{\lambda f_s} + \tau \operatorname{div} \eta,$$

we derive the following:

$$\begin{aligned}\left\|\operatorname{div} p^{n+1}-\frac{f}{\lambda f_s}\right\|^2 &= \left\|\operatorname{div} p^{n}-\frac{f}{\lambda f_s}+\tau \operatorname{div}\eta\right\|^2 \\ &= \left\|\operatorname{div} p^{n}-\frac{f}{\lambda f_s}\right\|^2 + 2\tau\left\langle \operatorname{div}\eta, \operatorname{div} p^n-\frac{f}{\lambda f_s}\right\rangle + \tau^2\|\operatorname{div}\eta\|^2.\end{aligned}$$

Let κ be the norm of $\operatorname{div}: Y \to X$. That is, $\kappa = \sup_{\|p\|_Y\le 1}\|\operatorname{div} p\|_X$. Then, we get $\|\operatorname{div}\eta\|^2 \le \kappa^2\|\eta\|_Y^2$. Using this and also by the property $\langle \operatorname{div} p, u\rangle = -\langle p, \nabla u\rangle$, we obtain the inequality

$$\left\|\operatorname{div} p^{n+1}-\frac{f}{\lambda f_s}\right\|^2 \le \left\|\operatorname{div} p^{n}-\frac{f}{\lambda f_s}\right\|^2 - \tau\left(2\left\langle \eta, \nabla\left(\operatorname{div} p^n-\frac{f}{\lambda f_s}\right)\right\rangle - \kappa^2\tau\|\eta\|_Y^2\right).$$

Note that

$$2\left\langle \eta, \nabla\left(\operatorname{div} p^n-\frac{f}{\lambda f_s}\right)\right\rangle - \kappa^2\tau\|\eta\|_Y^2 = \sum_{i,j=1}^{N} 2\eta_{i,j}\left(\nabla\left(\operatorname{div} p^n-\frac{f}{\lambda f_s}\right)\right)_{i,j} - \kappa^2\tau\eta_{i,j}^2$$

and by (31),

$$\eta_{i,j} = \left(\nabla\left(\operatorname{div} p^n-\frac{f}{\lambda f_s}\right)\right)_{i,j} - \rho_{i,j},$$

where $\rho_{i,j} = \left|\left(\nabla(\operatorname{div} p^n - f/(\lambda f_s))\right)_{i,j}\right| p_{i,j}^{n+1}$. Let $a_{i,j} := \left(\nabla(\operatorname{div} p^n - f/(\lambda f_s))\right)_{i,j}$. Then

$$\begin{aligned}2\eta_{i,j}\left(\nabla\left(\operatorname{div} p^n-\frac{f}{\lambda f_s}\right)\right)_{i,j} - \kappa^2\tau\eta_{i,j}^2 &= 2\eta_{i,j}a_{i,j} - 2\eta_{i,j}\rho_{i,j} + 2\eta_{i,j}\rho_{i,j} - \kappa^2\tau\eta_{i,j}^2 \\ &= 2\eta_{i,j}^2 + 2\eta_{i,j}\rho_{i,j} - \kappa^2\tau\eta_{i,j}^2 \\ &= (1-\kappa^2\tau)\eta_{i,j}^2 + \eta_{i,j}^2 + 2\eta_{i,j}\rho_{i,j} \\ &= (1-\kappa^2\tau)\eta_{i,j}^2 + a_{i,j}^2 - \rho_{i,j}^2.\end{aligned}$$

Since $p^0 = 0$, we can easily prove by induction that $|p_{i,j}^n| \le 1\ \forall i, j$ for all $n \ge 0$. This implies $\rho_{i,j} \le |a_{i,j}|$. Therefore, if $\tau \le 1/\kappa^2$, then

$$2\left\langle \eta, \nabla\left(\operatorname{div} p^n-\frac{f}{\lambda f_s}\right)\right\rangle - \kappa^2\tau\|\eta\|_Y^2 \ge 0,$$

which implies $\|\operatorname{div} p^n - f/(\lambda f_s)\|^2$ is decreasing with n. Hence, there exists a limit of $\|\operatorname{div} p^n - f/(\lambda f_s)\|^2$ as $n \to \infty$ and we can conclude $f - \lambda f_s \operatorname{div} p^n$ converges to a solution of the simplified version of the MCSD minimizing functional (21).

Now we need to show that $\tau \le 1/\kappa^2$ if $\tau \le \frac{1}{2}$. By (29),

$$\begin{aligned}\|\operatorname{div} p\|^2 &= \sum_{1\le i,j\le N} \Big(\tfrac{1}{2}(p^1_{i+1,j} - p^1_{i-1,j}) + \tfrac{1}{2}(p^2_{i,j+1} - p^2_{i,j-1})\Big)^2 \\ &\le \sum_{1\le i,j\le N} (p^1_{i+1,j})^2 + (p^1_{i-1,j})^2 + (p^2_{i,j+1})^2 + (p^2_{i,j-1})^2 \\ &\le 2\|p\|^2_Y.\end{aligned}$$

This proves that $\kappa^2 \le 2$. Since we assumed $\tau \le \frac{1}{2}$, we finally get $\tau \le 1/\kappa^2$. □

6. Numerical results for the MCSD model

The resulting images shown here are obtained using C++ programs which were compiled and run on Linux. Other than comparing the visual results, we also use peak-signal-to-noise ratio (PSNR) to measure image quality. The definition of PSNR is given as follows:

Definition 6.1 (PSNR). Let g be a noise-free clean image and u be the restored image obtained by denoising a noisy version of g. The PSNR is measured by

$$\text{PSNR} = 10 \log_{10}\bigg(\frac{\sum_{ij} 255^2}{\sum_{ij}(g_{ij} - u_{ij})^2}\bigg).$$

Note that if the denoised image is very close to the clean image, the denominator will be very small, thus providing a higher PSNR for a cleaner image. Also, the PSNR can be obtained for images with synthetically added noise only. Images with natural noise cannot have PSNR values since g is not available.

First we show the results for a Gaussian noise image (synthetic Lena image). From Figure 1, we can compare the results obtained from different denoising models. The MCSD model does not produce a nice result, as it is meant for speckle denoising. The Chambolle with central difference scheme provides the best denoised image. This fact is also supported by the PSNR values in Table 1, where the central difference Chambolle model has the highest PSNR and MCSD has the lowest, but the MCSD model is still faster than the TV model (6).

Next we have the results together with the residuals for the MCSD (25)–(27) and the Krissian et al. (15) models for speckle noise images. In Figure 2 (the speckle Lena image), we see that MCSD has comparable results to the Krissian et al. model. The residuals in Figure 3 show that MCSD has picked up more noise and less detail than the Krissian et al. model. From Table 2, we see that MCSD has a higher PSNR value than Krissian et al. and it also takes less time.

Figure 4 shows the results for an ultrasound image (liver image). Here also we can see MCSD performs better than Krissian et al. The residual image (Figure 5)

noise-free image — Gaussian noise image

TV — Chambolle

Chambolle (cdm) — MCSD

Figure 1. Results of TV, Chambolle and MCSD models (Gaussian Lena image).

noise-free image　　speckle noise image

Krissian et al.　　MCSD

Figure 2. Results of Krissian et al. and MCSD models (speckle Lena image).

TV		Chambolle		Chambolle (cdm)		MCSD	
time	PSNR	time	PSNR	time	PSNR	time	PSNR
13.71	27.78	4.54	27.71	4.64	28.86	8.98	26.06

Table 1. Model comparison for the Lena image with Gaussian noise and PSNR = 24.30, where time is measured in seconds.

shows MCSD picked up more noise but preserved more edges compared to the Krissian et al. model. We are unable to compare PSNR values here since they are images having natural noise. But Table 2 does show us that MCSD is faster than the Krissian et al. model.

Krissian et al. absolute residual

MCSD absolute residual

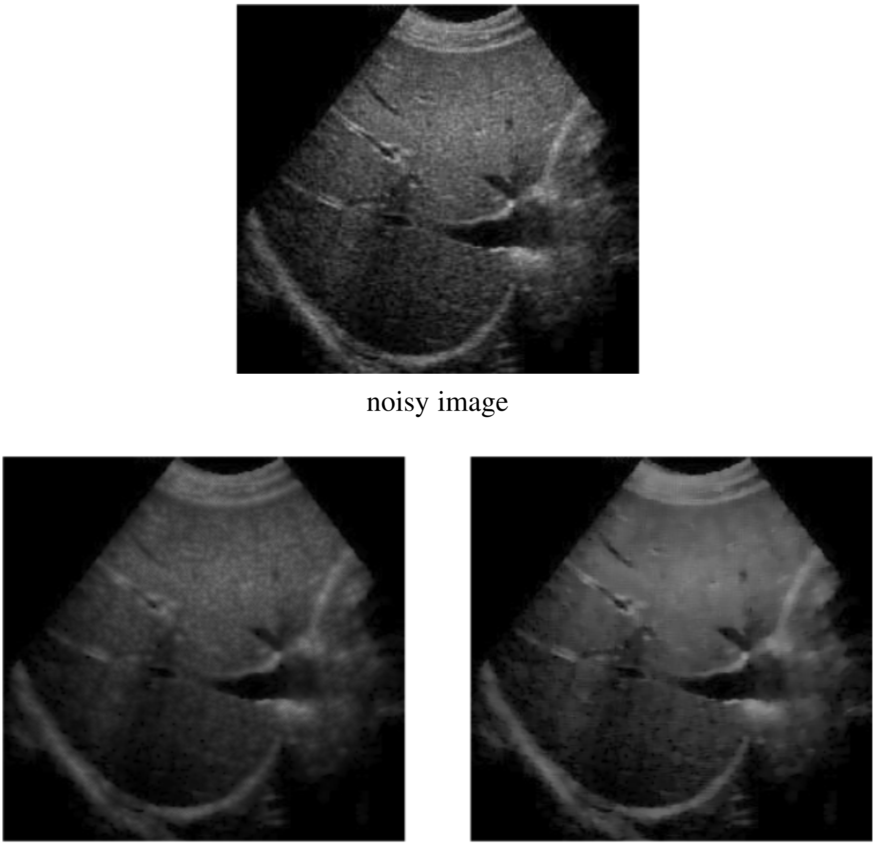
Krissian et al. noise residual

MCSD noise residual

Figure 3. Residuals of Krissian et al. and MCSD models (speckle Lena image).

images	Krissian et al.		MCSD	
	time	PSNR	time	PSNR
Lena (PSNR= 25.70)	2.83	27.02	0.14	28.52
liver	2.35	–	0.13	–

Table 2. Model comparison for speckle noise, where time is measured in seconds.



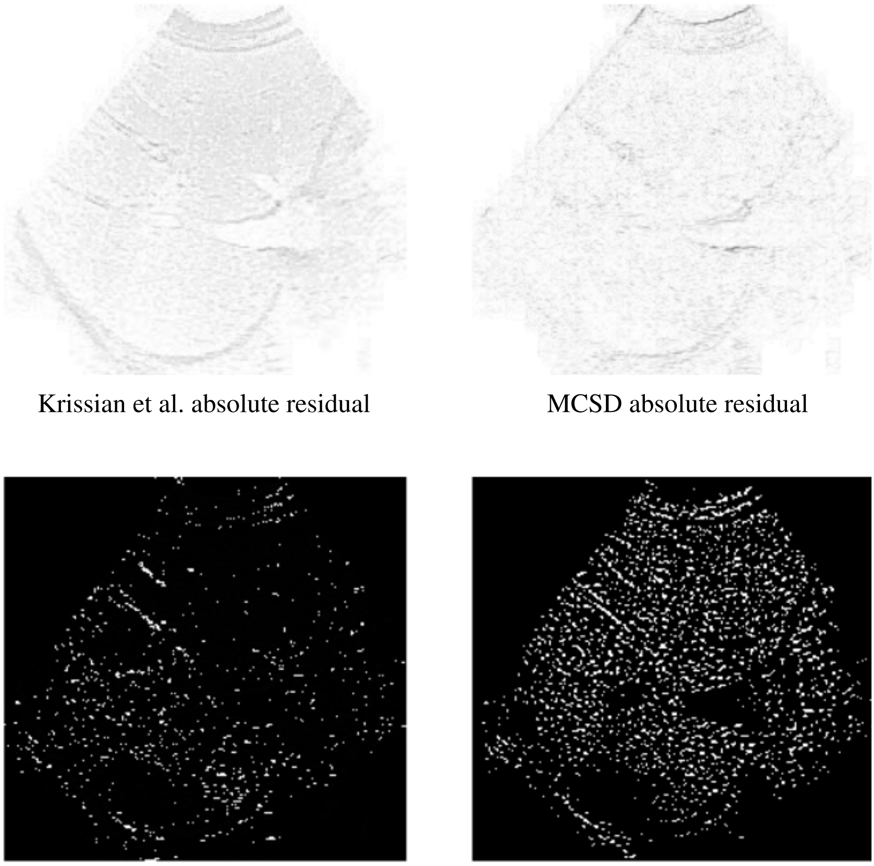
noisy image

Krissian et al. MCSD

Figure 4. Results of Krissian et al. and MCSD models (ultrasound liver image).

7. Conclusion

In this paper, we introduced our new TV based denoising model for speckle noise images. The new model provides a speckle noise version for the Chambolle algorithm, which was originally designed for faster solution of the ROF model. The results show a significant amount of improvement compared to the conventional TV based speckle denoising model. Based on a dual formation, the solution is updated directly from the dual space. The new method is therefore much more efficient than the method by Krissian et al. It is also numerically shown that the new method is more accurate than the Krissian et al. method. Under certain conditions on the time-step size, it is proved that the solution from the new algorithm converges to the minimizer of the new speckle denoising model.



Krissian et al. absolute residual — MCSD absolute residual

Krissian et al. noise residual — MCSD noise residual

Figure 5. Residuals of Krissian et al. and MCSD models (ultrasound liver image).

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