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On a randomly accelerated particle

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The focus of this note is to learn more about the Kolmogorov equation describing the dynamics of a randomly accelerated particle. We first explore some existing results of the Kolmogorov equation from the stochastic and differential equation points of view and discuss its solvability with and without boundary conditions. More specifically, we introduce stochastic processes and Brownian motion and we present a connection between a stochastic process and a differential equation. After looking at stochastic processes, we introduce generalized functions and derive the fundamental solution to the heat equation and to the Fokker–Planck equation. The problem with a reflecting boundary condition is also studied by using various methods such as separation of variables, self-similarity, and the reflection method.

1. Introduction

In our studies of mathematics, we will often come across different types of processes, including the stochastic process. A stochastic process is one that changes randomly with time. Even if one starts at the same point, one cannot predict how the process will evolve in the future. We can use stochastic processes to model random fluctuations. The best known example of a stochastic process is Brownian motion, which is the continuous, random movement of particles. It derives its name from Robert Brown's study [1828] of pollen floating on water; he noticed that the pollen grains moved continuously, but he could not find a pattern to their movement. Brownian motion is also a Markov process, in which future behavior depends only on the current or previous state, and all other states are irrelevant [Ibe 2013].

Later, Einstein [1905; 1926] derived a diffusion equation for the density of Brownian particles, whereas Smoluchowski [1906] created a kinetic model to represent the collision of the particles.

When dealing with stochastic processes, in particular Markov processes, a useful tool is the Chapman–Kolmogorov equation. This equation is used to determine the

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transition density function for moving from one state to another. The Chapman–Kolmogorov equation is

$$p(x,t \mid y,s) = \int_{-\infty}^{+\infty} p(x,t \mid z,r)\, p(z,r \mid y,s)\, \mathrm{d}z \qquad \text{for } s < r < t. \tag{1-1}$$

This equation considers the fact that if you go from y at time s to x at time t, you must go through an intermediate point z at time r [van Kampen 1981]. In many stochastic processes, the Chapman–Kolmogorov equation is very helpful because again, stochastic processes are random processes. We cannot predict exactly where a particle will be at a given time; we can only predict the probability that the particle will be at a certain point in a given time. This applies directly when we look at Brownian motion. In the case of Brownian motion, the transition probability density function is

$$p(x,t \mid y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/(2t)} \qquad \text{for } t > 0. \tag{1-2}$$

It is easy to see that p satisfies the partial differential equation (the heat equation)

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}, \tag{1-3}$$

and the initial condition $p(x,0 \mid y) = \delta(x-y)$. Here δ is a generalized function, which we will discuss more in detail in Section 3.1. This example illustrates the connection between Brownian motion (stochastic process) and the heat equation (differential equation) via the Chapman–Kolmogorov equation.

A wider range of diffusion processes can yield diffusion equations, which are often called the Fokker–Planck equations. The Fokker–Planck equations have many different applications such as modeling Brownian motion in drift, finance, and physics [Risken 1984]. For this reason, it is worthwhile to learn about their many properties and characteristics. The focus of this note is to investigate some properties of the simplest kinetic Fokker–Planck equation, also known as the Kolmogorov equation, given by

$$\frac{\partial p}{\partial t} = -v \frac{\partial p}{\partial x} + k \frac{\partial^2 p}{\partial v^2}, \tag{1-4}$$

where

$$p = p(t,x,v) \qquad \text{for } x \in \mathbb{R},\ v \in \mathbb{R},\ t > 0, \qquad \text{and} \qquad k > 0.$$

Here k is a diffusion coefficient. In the Kolmogorov equation, we have t, x, and v as single variables, whereas the more complicated forms of the Fokker–Planck equation consist of vectors in both x and v. It is important to look at the Kolmogorov equation first because once the simplest form has been studied, similar techniques may be applied to other forms of the equation.

Because the Fokker–Planck equation is used to model the movement of particles, it is necessary to look at some of the ways in which particles behave. In this note we will look at the case in which a particle moves randomly in a given space. The particle is not free to move as it pleases though; there is a wall, and once the particle hits the wall it is bounced back to the original space. In previous works, researchers (such as Skorohod [1961]) solved similar problems using approximation methods. In this work, we attempt to do so using separation of variables, self-similarity, and the reflection method.

2. Stochastic process of Fokker–Planck equation

We start out by determining if, like Brownian motion, the Fokker–Planck equation (1-4) comes from a stochastic process. For simplicity, we will take $k = 1$. A general form of the Fokker–Planck equation is

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i p) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij} p), \tag{2-1}$$

where n is a positive integer, b_i is the drift coefficient and a_{ij} is the diffusion coefficient.

Let us first consider $n = 2$. Letting $x = x_1$ and $v = x_2$, we see that in (1-4), v is the same as b_1. Since x is not included in this term, we will form a vector $\vec{b}$ such that $\vec{b} = [x_2, 0]^T$. Notice also that in (2-1),

$$\frac{1}{2}\sum_{i,j=1}^{n=2} \frac{\partial^2}{\partial x_i x_j}(a_{ij} p)$$

is nonzero only when both i and j are equal to 2. Therefore $a_{11} = a_{12} = a_{21} = 0$ and $a_{22} = 2$, so we have a matrix

$$A = (a_{ij}) = \sigma\sigma^T = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

A stochastic differential equation for $\vec{X} = [x_1, x_2]^T$ has the form

$$\mathrm{d}\vec{X} = \vec{b}(\vec{X}, t)\,\mathrm{d}t + \sigma(\vec{X}, t)\,\mathrm{d}\vec{B}. \tag{2-2}$$

Plugging in our values, we have

$$\begin{bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \mathrm{d}\vec{B}.$$

Multiplying these out, we obtain

$$\mathrm{d}x_1 = x_2\,\mathrm{d}t \quad \text{and} \quad \mathrm{d}x_2 = \sqrt{2}\,\mathrm{d}B_2.$$

Recalling that $x = x_1$, $v = x_2$, and letting $\mathrm{d}B = \xi(t)\,\mathrm{d}t$ (white noise), we obtain

$$\mathrm{d}x = v\,\mathrm{d}t, \quad \mathrm{d}v = \sqrt{2}\xi(t)\,\mathrm{d}t.$$

We have found the stochastic differential equation for the Fokker–Planck equation.

Looking at the solution above, we see that

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \sqrt{2}\xi(t).$$

Therefore, the Kolmogorov equation models a randomly accelerated particle.

We can do the same with the multidimensional Kolmogorov equation with no external forces. For instance, (1-4) can be generalized as

$$\frac{\partial p}{\partial t} = -v \cdot \nabla_x p + \Delta_v p, \tag{2-3}$$

where $p = p(t, x, v)$ and $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$. Recall that

$$v \cdot \nabla_x p = v_1 \partial_{x_1} p + v_2 \partial_{x_2} p + v_3 \partial_{x_3} p. \tag{2-4}$$

Similar to the previous case, we will let $x = (x_1, x_2, x_3)$ and $v = (x_4, x_5, x_6)$. Notice $n = 6$ in this case. We see in (2-1), v_i is the same as b_i. Let

$$\vec{b} = [x_4, x_5, x_6, 0, 0, 0]^T.$$

Notice that in (2-3), the term

$$\frac{1}{2} \sum_{i,j=1}^{n=6} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij} p)$$

only exists when both i and j are equal to 4, 5, and 6. Therefore, we have a matrix A in which $a_{44} = a_{55} = a_{66} = 2$ and all other terms are equal to 0. This gives us degenerate diffusion, which is different from Brownian motion. Here, "degenerate" means that the diffusion coefficient matrix is nonnegative, but not positive definite. We also know that our vector $\vec{X} = [x_1, x_2, x_3, x_4, x_5, x_6]^T$. Recalling the general form of a stochastic process (2-2) and plugging in our vectors and multiplying them out, we obtain

$$\begin{aligned} \mathrm{d}x_1 &= x_4\,\mathrm{d}t, & \mathrm{d}x_2 &= x_5\,\mathrm{d}t, & \mathrm{d}x_3 &= x_6\,\mathrm{d}t, \\ \mathrm{d}x_4 &= \sqrt{2}\,\mathrm{d}B_4, & \mathrm{d}x_5 &= \sqrt{2}\,\mathrm{d}B_5, & \mathrm{d}x_6 &= \sqrt{2}\,\mathrm{d}B_6. \end{aligned}$$

We have once again found the stochastic differential equations, so we know that the kinetic Fokker–Planck equation (2-4) comes from a stochastic process. The result of this section is well-known and we refer to [van Kampen 1981] for more discussion on the stochastic processes and the Fokker–Planck equation.

For the rest of the note, we will study the properties of the solutions to (1-4) and (2-3) by using various methods.

3. Fundamental solutions of the Fokker–Planck equation

The fundamental solution is the solution of a particular equation with initial data at a single, concentrated point. The idea behind this is that if we have enough information about the solution of an equation at this infinitely dense point, we can draw enough information about the behavior of the equation at other points.

3.1. ***Delta function and fundamental solutions.*** We use the delta function (which is referred to as a generalized function) to represent the infinitely dense point. The delta function is formally defined by

$$\delta(x-\xi)=\begin{cases}0, & x\neq\xi,\\ +\infty, & x=\xi,\end{cases}$$

such that

$$\int_a^b \delta(x-\xi)\,\mathrm{d}x=1 \quad \text{as long as } a<\xi<b.$$

An interesting and very helpful property is that for any function $f(x)$,

$$\int_a^b f(x)\delta(x-\xi)\,\mathrm{d}x=f(\xi) \quad \text{if } a<\xi<b.$$

The above properties hold even if $a=-\infty$ and $b=+\infty$. Because of the information it yields, we often use the delta function as the initial condition when searching for fundamental solutions.

The definition of a fundamental solution for a linear differential operator L is

$$LF=0, \quad F_{(t=0)}=\delta. \tag{3-1}$$

3.2. ***Heat equation.*** In the introduction, we presented an example of the probability density function for Brownian motion when looking at stochastic processes. In this section, we show that we can also find a solution without considering a stochastic process. For instance, we can use the Fourier transform method to give rise to the fundamental solution of the heat equation [Olver 2014]. We denote the solution as $u(t,x)=F(t,x;\xi)$ and set the initial condition to be $F(0,x;\xi)=\delta(x-\xi)$. This must satisfy the heat equation (1-3), so we know

$$\frac{\partial F}{\partial t}=\frac{\partial^2 F}{\partial x^2}.$$

We must now reconstruct the equation using the properties of linearity and the Fourier transform method. After solving this, we take the inverse Fourier transform to obtain the fundamental solution of the heat equation (1-3).

We find that

$$F(t,x,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ik(x-\xi)-k^2 t}\, \mathrm{d}k = \frac{1}{2\sqrt{\pi t}} e^{-(x-\xi)^2/(4t)} \quad \text{for } t > 0. \tag{3-2}$$

Recall the probability density function (1-2). In this section we obtained the same result, except we are off by a multiple of $\frac{1}{2}$. The reason for this is that here, we started with the diffusion coefficient $k = 1$ instead of $k = \frac{1}{2}$.

Once we have the fundamental solution of a differential equation, we can find other solutions using the convolution

$$u(t,x) = (F * f)(t,x), \tag{3-3}$$

where

$$(F * f)(t,x) = \int_{\xi \in \mathbb{R}} F(t,x,\xi) f(\xi)$$

and with the initial condition $u(0,x) = f(x)$.

3.3. *Kolmogorov equation.* In this section, we are interested in constructing the fundamental solution to the Fokker–Planck equation (1-4) and (2-3). In fact, Kolmogoroff [1934] provided the formula for the fundamental solution to the Fokker–Planck equation, but did not give any details on the construction. After finding the solution for the Fokker–Planck equation, we will consider the case of the Kolmogorov equation.

Tanski [2004] found the fundamental solution of the Fokker–Planck equation

$$\frac{\partial n}{\partial t} + v_x \frac{\partial n}{\partial x} + v_y \frac{\partial n}{\partial y} + v_z \frac{\partial n}{\partial z} - \alpha\left(\frac{\partial}{\partial v_x}(v_x n) + \frac{\partial}{\partial v_y}(v_y n) + \frac{\partial}{\partial v_z}(v_z n)\right) = k\left(\frac{\partial^2 n}{\partial v_x^2} + \frac{\partial^2 n}{\partial v_y^2} + \frac{\partial^2 n}{\partial v_z^2}\right). \tag{3-4}$$

He used the method of characteristics to come up with the fundamental solution of the form

$$G = \frac{1}{(2\pi)^6}\left(\frac{\pi}{k\sqrt{D}}\right)^3 \exp\left\{\frac{-1}{4kD}\left[\frac{1}{2\alpha}(1-e^{-2\alpha t})(\hat{x}^2+\hat{y}^2+\hat{z}^2) - \left(\frac{2}{\alpha^2}(1-e^{-\alpha t}) - \frac{1}{\alpha^2}(1-e^{-\alpha t})\right)(\hat{x}\hat{v}_x + \hat{y}\hat{v}_y + \hat{z}\hat{v}_z) + \left(\frac{t}{\alpha^2} - \frac{2}{\alpha^3}(1-e^{-\alpha t}) + \frac{1}{2\alpha^3}(1-e^{-2\alpha t})\right)(\hat{v}_x^2 + \hat{v}_y^2 + \hat{v}_z^2)\right]\right\}, \tag{3-5}$$

where

$$\hat{x} = x - (x_0 + (v_{x0}/\alpha)(1 - e^{-\alpha t})),$$
$$\hat{y} = y - (y_0 + (v_{y0}/\alpha)(1 - e^{-\alpha t})),$$
$$\hat{z} = z - (z_0 + (v_{z0}/\alpha)(1 - e^{-\alpha t})),$$
$$D = \frac{\det(A)}{k^2},$$

and A is a matrix with

$$\det(A) = k^2 \frac{\alpha t(1 - e^{-2\alpha t}) - 2(1 - e^{-\alpha t})^2}{2\alpha^4}.$$

This matches the results of [Kolmogoroff 1934].

In our case, we would like to look at a slightly more specific equation. We look at the Fokker–Planck equation of the form

$$\partial_t p + v \cdot \nabla_x p = \Delta_v p, \tag{3-6}$$

which can be rewritten as

$$\partial_t p + v_1 \partial_{x_1} p + v_2 \partial_{x_2} p + v_3 \partial_{x_3} p = (\partial^2_{v_1} p + \partial^2_{v_2} p + \partial^2_{v_3} p).$$

We follow Tanski's method in order to find the fundamental solution of our equation. The result does not follow directly from [Tanski 2004]. We have that $x = x_1, y = x_2, z = x_3$, and $v_x = v_1, v_y = v_2, v_z = v_3$, and $k = 1$. We let $N = N(t, p_1, p_2, p_3, q_1, q_2, q_3)$ be the Fourier transformation in (x, v). It is equivalent to

$$\frac{1}{(2\pi)^6} \int_{R^6} e^{-i(x_1 p_{x_1} + x_2 p_{x_2} + x_3 p_{x_3} + v_1 q_1 + v_2 q_2 + v_3 q_3)} p \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \, \mathrm{d}v_1 \, \mathrm{d}v_2 \, \mathrm{d}v_3.$$

In terms of N, the Fourier transform equals

$$\partial_t N - p_1 \partial_{q_1} N - p_2 \partial_{q_2} N - p_3 \partial_{q_3} N = -(q_1^2 + q_2^2 + q_3^2) N.$$

We then come up with

$$\mathrm{d}t = \frac{\mathrm{d}p_1}{0} = \frac{\mathrm{d}p_2}{0} = \frac{\mathrm{d}p_3}{0} = \frac{\mathrm{d}q_1}{-p_1} = \frac{\mathrm{d}q_2}{-p_2} = \frac{\mathrm{d}q_3}{-p_3} = \frac{-\mathrm{d}N/N}{(q_1^2 + q_2^2 + q_3^2)}.$$

Solving this we find

$$p_1 = p_{10}, \qquad p_2 = p_{20}, \qquad p_3 = p_{30},$$
$$q_1 = -p_1 t + q_{10}, \quad q_2 = -p_2 t + q_{20}, \quad q_3 = -p_3 t + q_{30},$$
$$N = N_0 e^{-\frac{1}{2}\left((p_1^2 + p_2^2 + p_3^2)\frac{1}{3}t^3 - (p_1 q_{10} + p_2 q_{20} + p_3 q_{30})t^2 + (q_{10}^2 + q_{20}^2 + q_{30}^2)t\right)}.$$

Plugging in our values for q_{10}, q_{20}, and q_{30}, we obtain

$$N = N_0 \exp\Big\{-\tfrac{1}{2}\Big((p_1^2 + p_2^2 + p_3^2)\tfrac{1}{3}t^3 - \big(p_1(q_1 + p_1 t) + p_2(q_2 + p_2 t) + p_3(q_3 + p_3 t)\big)t^2 + \big((q_1 + p_1 t)^2 + (q_2 + p_2 t)^2 + (q_3 + p_3 t)^2\big)t\Big)\Big\}$$

which leaves us with

$$N = N_0 e^{-\frac{1}{2}\left((p_1^2+p_2^2+p_3^2)\frac{1}{3}t^3-(p_1q_1+p_2q_2+p_3q_3)t^2+(q_1^2+q_2^2+q_3^2)t\right)}.$$

We take the initial density value as

$$n_0 = \delta(x_1 - x_{10})\delta(x_2 - x_{20})\delta(x_3 - x_{30})\delta(v_1 - x_{10})\delta(v_2 - x_{20})\delta(v_3 - x_{30}).$$

The Fourier transform of the initial density becomes

$$N_0 = e^{-i(x_{10}p_1+x_{20}p_2+x_{30}p_3+v_{10}q_1+v_{20}q_2+v_{30}q_3)}.$$

Plugging in the initial values we obtain

$$\widehat{N_0} = e^{-i(x_{10}p_1+x_{20}p_2+x_{30}p_3+v_{10}(p_1t+q_{10})+v_{20}(p_2t+q_{20})+v_{30}(p_3t+q_{30}))},$$

which is the Fourier transform of

$$\widehat{n_0} = \delta(x_1 - (x_{10} + v_{10}t))\delta(x_2 - (x_{20} + v_{20}t)) \delta(x_3 - (x_{30} + v_{30}t))\delta(v_1 - v_{10})\delta(v_2 - v_{20})\delta(v_3 - v_{30}).$$

In our example, we get the matrix A to be

$$A = \begin{bmatrix} \frac{1}{3}t^3 & -\frac{1}{2}t^2 \\ -\frac{1}{2}t^2 & t \end{bmatrix}.$$

This matrix is created from the terms related to N, where a_{11} is the term coming from p_i^2, and the a_{12} and a_{21} terms are obtained by dividing the term for $p_i q_j$ in half. Finally, a_{22} is the term associated with q_i^2. Its determinant is

$$\det(A) = \tfrac{1}{12}t^4$$

and

$$D = \tfrac{1}{12}t^4, \quad \text{since } D = \frac{\det(A)}{k^2} \quad \text{and} \quad k = 1.$$

The inverse is given by

$$A^{-1} = \frac{12}{t^4}\begin{bmatrix} t & \frac{1}{2}t^2 \\ \frac{1}{2}t^2 & \frac{1}{3}t^3 \end{bmatrix}.$$

We now combine $\widehat{n_0}$ with A^{-1} to obtain

$$G=\frac{1}{(2\pi)^6}\left(\frac{\pi}{k\sqrt{D}}\right)^3\widehat{n_0}\exp\left\{-\frac{3}{t^4}\big(t(x_1^2+x_2^2+x_3^2)-t^2(x_1v_1+x_2v_2+x_3v_3)+\tfrac{1}{3}t^3(v_1^2+v_2^2+v_3^2)\big)\right\},$$

which gives us

$$G=\frac{1}{(2\pi)^6}\left(\frac{\pi}{k\sqrt{D}}\right)^3\exp\left\{-\frac{3}{t^4}\big(t(\widehat{x_1}^2+\widehat{x_2}^2+\widehat{x_3}^2)-t^2(\widehat{x_1}\widehat{v_1}+\widehat{x_2}\widehat{v_2}+\widehat{x_3}\widehat{v_3})+\tfrac{1}{3}t^3(\widehat{v_1}^2+\widehat{v_2}^2+\widehat{v_3}^2)\big)\right\}.$$

Plugging in $D=\frac{1}{12}t^4$, we have

$$G=\frac{1}{(2\pi)^6}\left(\frac{2\sqrt{3}\pi}{t^2}\right)^3\exp\left\{-\frac{3}{t^4}\big(t(\widehat{x_1}^2+\widehat{x_2}^2+\widehat{x_3}^2)-t^2(\widehat{x_1}\widehat{v_1}+\widehat{x_2}\widehat{v_2}+\widehat{x_3}\widehat{v_3})+\tfrac{1}{3}t^3(\widehat{v_1}^2+\widehat{v_2}^2+\widehat{v_3}^2)\big),\right\}$$

where

$$\widehat{x_1}=x_1-(x_{10}+v_{10}t),\quad \widehat{x_2}=x_2-(x_{20}+v_{20}t),\quad \widehat{x_3}=x_3-(x_{30}+v_{30}t),$$
$$\widehat{v_1}=v_1-v_{10},\quad \widehat{v_2}=v_2-v_{20},\quad \widehat{v_3}=v_3-v_{30}.$$

The same procedure can be performed for the Kolmogorov equation (1-4):

$$\partial_t p+v\partial_x p=\partial_v^2 p.$$

We obtain the fundamental solution

$$G=\frac{1}{(2\pi)^2}\left(\frac{2\sqrt{3}\pi}{t^2}\right)e^{-\frac{3}{t^4}\left(t\hat{x}^2-t^2\hat{x}\hat{v}+\frac{1}{3}t^3\hat{v}^2\right)},\tag{3-7}$$

where $\hat{x}=x-(x_0+v_0t)$ and $\hat{v}=v-v_0$.

If we want to solve a problem with general initial conditions, we can do so using

$$p(t,x,v)=\iint G(t,x,v,x_0,v_0)\,p(x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0.\tag{3-8}$$

This gives a representation formula for a solution to the Kolmogorov equation in the whole space.

Remark 3.1. After this work had been performed, we found out that Tanski [2008] solved the problem. We refer to [Tanski 2004; 2008] for more details on the construction of the fundamental solution of the general Fokker–Planck equations.

4. Reflecting boundary conditions

Oftentimes, particles are not free to move around as they please; they are influenced by their surroundings. This is the focus of this section. In particular, we are interested in the case where the particle is reflected back to the plane once it hits the boundary (or wall). Consider, for example, that $\{x = 0\}$ is the wall of the domain $\{x > 0,\ v \in \mathbb{R}\}$. We represent this behavior with the boundary condition

$$p(0, -v) = p(0, v) \quad \text{for all } v. \tag{4-1}$$

The first natural question is: are there any "simple solutions" of (1-4) satisfying this boundary condition? We first consider the possible stationary solutions. The equation to solve is

$$v\partial_x p = \partial_v^2 p, \tag{4-2}$$

with the condition (4-1).

4.1. ***Stationary solutions.*** Suppose the solution to (4-1)–(4-2) takes the form

$$p(x, v) = X(x)V(v). \tag{4-3}$$

Plugging this into (4-2), we get

$$vX'V = XV''.$$

Dividing both sides by vXV and letting this equal $-\lambda$, we get

$$\frac{X'}{X} = \frac{V''}{vV} = -\lambda.$$

Solving for X we find

$$X(x) = X_0 e^{-\lambda x},$$

where X_0 is some constant.

We now try to solve for V. Because of the boundary condition, we know that V must satisfy

$$V(v) = V(-v).$$

It will also satisfy

$$V''(v) = V''(-v).$$

Replacing these values we find

$$-\lambda v V(v) = \lambda v V(-v).$$

Using our boundary condition, we obtain

$$-\lambda v V(v) = \lambda v V(v).$$

Moving everything to one side we see that

$$-2\lambda v V(v) = 0.$$

We do not want v or $V(v)$ to equal 0; therefore $\lambda = 0$ must be true. This also means that $V'' = 0$. Integrating leads us to the solution $V(v) = av + b$. We need this equation to satisfy the boundary condition, which in turn leads us to the conclusion that $V(v) = b$.

Now that we know λ, let us solve for X. Plugging in our value of λ, we find that $X(x) = X_0$. Recall the form from (4-3). Therefore we get $p(x, v) = C$, where $C = X_0 b$. Hence we see that only constants will solve the problem.

In many cases, the total mass of particles is positive. If we view p as a probability density, then

$$\int_{x>0} \int_{v\in\mathbb{R}} p(x, v)\, \mathrm{d}x\, \mathrm{d}v = 1.$$

Since the domain is infinite, no constant will satisfy this criterion. There is no other interesting solution to the stationary problem by using separation of variables.

4.2. ***Kummer functions.*** We will try again to find a solution to (4-1)-(4-2)- by a different method. Because of the scaling invariance property of the equation, we want a solution of the form

$$p(x, v) = x^{\alpha} \phi(-v^3/(9x)).$$

When done this way, we get

$$\partial_x p = \alpha x^{\alpha-1}\phi + (v^3/(9x^2))x^{\alpha}\phi',$$
$$\partial_v^2 p = x^{\alpha}(-3v^2/(9x))^2\phi'' + x^{\alpha}(-6v/(9x))\phi'.$$

After some calculations, we obtain

$$z\phi'' + \left(\tfrac{2}{3} - z\right)\phi' + \alpha\phi = 0, \tag{4-4}$$

where $z = -v^3/(9x)$. This form satisfies the Kummer equations. Equation (4-4) has two independent solutions: M and U [Abramowitz and Stegun 1965].

We now examine the asymptotic behavior of the solutions to see whether the boundary conditions are satisfied by these solutions. Our boundary condition is given in (4-1): $p(0, v) = p(0, -v)$.

Taking the boundary condition into account, when x approaches 0 and $v > 0$, we notice z approaches $+\infty$, and when x approaches 0 and $v < 0$, we notice z approaches $-\infty$. Therefore, we will study the asymptotic behavior of the solution of (4-4) as z approaches $+\infty$ and $-\infty$ to match the boundary condition. We start with the first kind of solution M. From [Abramowitz and Stegun 1965, 13.1.5], we obtain

$$M\left(-\alpha, \tfrac{2}{3}, -z\right) \approx \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}+\alpha\right)} z^{\alpha} \quad \text{as } z \to +\infty, \tag{4-5}$$

and from [Abramowitz and Stegun 1965, 13.1.4],

$$M\left(-\alpha, \tfrac{2}{3}, z\right) \approx \frac{\Gamma\left(\frac{2}{3}\right) e^{z} z^{-\alpha-\frac{2}{3}}}{\Gamma(-\alpha)} \quad \text{as } z \to +\infty. \tag{4-6}$$

The behavior as z approaches $+\infty$ differs from when z approaches $-\infty$. Therefore, the first kind of solution does not satisfy the boundary condition.

Now we will look at our second independent solution, $U(-\alpha, \frac{2}{3}, z)$. Recall the solution from [Abramowitz and Stegun 1965, 13.5.2]:

$$U\left(-\alpha, \tfrac{2}{3}, z\right) = z^{\alpha}\left\{\sum_{n=0}^{R-1} \frac{(-\alpha)_n \left(1-\alpha-\frac{2}{3}\right)_n}{n!}(-z)^{-n} + O(|z|^{-R})\right\},$$

where $-\frac{3}{2}\pi < \arg(z) < \frac{3}{2}\pi$.

As z approaches $+\infty$, the defining behavior becomes

$$U\left(-\alpha, \tfrac{2}{3}, z\right) \approx z^{\alpha}.$$

Let us define a new variable S so that

$$z = -v^3/(9x) = -S^3 = (-S)^3 \quad \text{where } S \in \mathbb{R}, -\tfrac{1}{2}\pi < \arg(-S) < \tfrac{1}{2}\pi.$$

Therefore, we obtain

$$U\left(-\alpha, \tfrac{2}{3}, -S^3\right) \approx |S|^{3\alpha} \quad \text{as } S \to -\infty.$$

In order to examine the behavior as z approaches $-\infty$, we look at [Abramowitz and Stegun 1965, 13.1.3]:

$$U\left(-\alpha, \tfrac{2}{3}, z\right) = \frac{\pi}{\sin\left(\frac{2}{3}\pi\right)}\left\{\frac{M\left(-\alpha, \frac{2}{3}, z\right)}{\Gamma\left(1-\alpha-\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)} - z^{1-\frac{2}{3}}\frac{M\left(1-\alpha-\frac{2}{3}, 2-\frac{2}{3}, z\right)}{\Gamma(-\alpha)\Gamma\left(2-\frac{2}{3}\right)}\right\}. \tag{4-7}$$

Recall that $z = -S^3$ and the previously obtained formula (4-5). Plugging this into (4-7), we obtain

$$U\left(\alpha, \tfrac{2}{3}, -S^3\right) = \frac{\pi}{\sin(\frac{2}{3}\pi)}\left\{\frac{1}{\Gamma\left(\frac{1}{3}-\alpha\right)\Gamma\left(\frac{2}{3}+\alpha\right)} + \frac{1}{\Gamma(-\alpha)\Gamma(1+\alpha)}\right\} S^{3\alpha}.$$

We now use the following identity from [Abramowitz and Stegun 1965]:

$$\Gamma(-x)\Gamma(1+x) = -\frac{\pi}{\sin(\pi x)},$$

which gives us

$$\Gamma\big(\tfrac{1}{3}-\alpha\big)\Gamma\big(\tfrac{2}{3}+\alpha\big)=\frac{\pi}{\sin\big(\pi\big(\tfrac{2}{3}+\alpha\big)\big)} \quad \text{and} \quad \Gamma(-\alpha)\Gamma(1+\alpha)=-\frac{\pi}{\sin(\pi\alpha)}.$$

Recall the trigonometric identity

$$\frac{\sin\big(\pi\big(\alpha+\tfrac{2}{3}\big)\big)-\sin(\pi\alpha)}{\sin(\tfrac{2}{3}\pi)}=2\cos\big(\pi\big(\alpha+\tfrac{1}{3}\big)\big).$$

As a result,

$$U\big(-\alpha,\tfrac{2}{3},-S^3\big)\approx 2\cos\big(\pi\big(\alpha+\tfrac{1}{3}\big)\big)S^3 \quad \text{as } S\to+\infty.$$

If our boundary conditions are satisfied, then we have

$$2\cos\big(\pi\big(\alpha+\tfrac{1}{3}\big)\big)|S|^{3\alpha}=|S|^{3\alpha};$$

hence,

$$2\cos\big(\pi\big(\alpha+\tfrac{1}{3}\big)\big)=1.$$

Solving for α, we find that $\alpha=0$ or $\alpha=-\frac{2}{3}$.

In the case that $\alpha=0$, we would obtain a constant, which has been already found in the previous section by separation of variables. In the case of $\alpha=-\frac{2}{3}$, there is a singularity near the origin. However, it turns out that it is positive and integrable near the origin. The solution

$$p(x,v)=x^{-\frac{2}{3}}U\big(\tfrac{2}{3},\tfrac{2}{3},-v^3/(9x)\big)$$

to the stationary problem (4-1)–(4-2) could be useful in studying the behavior of the solution with the boundary condition near the boundary. We refer to [Hwang et al. 2015a] for more discussion on the Kummer functions and their applications to the Kolmogorov equation (1-4).

5. Reflection method

We will now try to solve (1-4),

$$\partial_t p+v\partial_x p=\partial_v^2 p,$$

where $x>0$, $v\in\mathbb{R}$ and $t>0$. We also require that $p(t,x,v)$ satisfies $p(t,0,v)=p(t,0,-v)$ and initial data $p(0,x,v)=p_0(x,v)$ satisfies the compatibility condition $p_0(0,v)=p_0(0,-v)$.

Although we do not know the solution of this problem yet, we do know the solution on the whole real line (when $x\in\mathbb{R}$). Therefore, we will attempt to use the reflection method to solve our problem.

The main result of this section is the following.

Theorem 5.1. *Define*

$$\bar{p}(t,x,v)=\int_{-\infty}^{+\infty}\int_{0}^{+\infty}[G(t,x,v,x_0,v_0)+G(t,x,v,-x_0,-v_0)]p_0(x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0$$

for $t>0$, $x>0$, $v\in\mathbb{R}$. *Here G is the fundamental solution obtained in Section 3.3 and* p_0 *is the given initial data for our problem. Then* $\bar{p}(t,x,v)$ *satisfies*:

(1) $\bar{p}_t+v\bar{p}_x=\bar{p}_{vv}$ *for* $t>0$, $x>0$, $v\in\mathbb{R}$.

(2) $\lim_{t\to 0}\bar{p}(t,x,v)=p_0(x,v)$ *for* $x>0$, $v\in\mathbb{R}$.

(3) $\bar{p}(t,0,v)=\bar{p}(t,0,-v)$ *for* $t>0$, $v\in\mathbb{R}$.

Proof. In order to prove the theorem, we first assume that p solves our problem and extend p to the whole space.

We let

$$\bar{q}(t,x,v)=\begin{cases}p(t,x,v), & x>0,\\ p(t,-x,-v), & x<0,\end{cases}$$

and let

$$\bar{q}_0(x_0,v_0)=\bar{q}(0,x,v)=\begin{cases}p_0(x,v), & x>0,\\ p_0(-x,-v), & x<0.\end{cases}$$

We see that $\bar{q}(t,x,v)$ satisfies our boundary conditions: plugging in 0 for x, we have

$$p(t,0,v)=p(t,0,-v)\quad \text{if } \bar{q}(t,x,v) \text{ is continuous.}$$

First, we check that $\bar{q}$ solves the problem in the whole space. We know that the equation satisfies the problem when $x>0$, since this is our original problem. However, we must check that the second half of our solution also satisfies the problem.

When $x<0$, we find that $\bar{q}=p(t,-x,-v)$ satisfies

$$\begin{aligned}\partial_t\bar{q}(t,x,v)&=\partial_t p(t,-x,-v),\\ \partial_x\bar{q}(t,x,v)&=-\partial_x p(t,-x,-v),\\ \partial_v^2\bar{q}(t,x,v)&=-\partial_v^2 p(t,-x,-v).\end{aligned}$$

On the other hand, since $-x>0$, we have that $p(t,-x,-v)$ satisfies the equation

$$\partial_t p(t,-x,-v)+(-v)(\partial_x p(t,-x,-v))-\partial_v^2 p(t,-x,-v)=0,$$

which is the same as

$$\partial_t p(t,-x,-v)+(v)(-\partial_x p(t,-x,-v))-\partial_v^2 p(t,-x,-v)=0.$$

Now by using the above relations for the derivatives of $\bar{q}$, we see that

$$\partial_t\bar{q}(t,x,v)+v(\partial_x\bar{q}(t,x,v))-\partial_v^2\bar{q}(t,x,v)=0$$

for $x < 0$. Since we have seen that $\bar{q}$ solves the whole space problem, we can obtain the solution $\bar{q}(t, x, v)$ with the extended initial data $\bar{q}_0(x_0, v_0)$ by using $G(t, x, v, x_0, v_0)$, where G is the fundamental solution we obtained earlier in (3-7),

$$\begin{aligned}\bar{q}(t,x,v) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(t,x,v,x_0,v_0)\bar{q}_0(x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0 \\ &= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} G(t,x,v,x_0,v_0)p_0(x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0 \\ &\qquad + \int_{-\infty}^{+\infty}\int_{-\infty}^{0} G(t,x,v,x_0,v_0)p_0(-x_0,-v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0.\end{aligned}$$

Let $\tilde{x} = -x_0$ and $\tilde{v} = -v_0$. We get

$$\begin{aligned}\bar{q}(t,x,v) = \int_{-\infty}^{+\infty}\int_{0}^{+\infty} & G(t,x,v,x_0,v_0)p_0(x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0 \\ &+ \int_{-\infty}^{+\infty}\int_{0}^{+\infty} G(t,x,v,-\tilde{x},-\tilde{v})p_0(\tilde{x},\tilde{v})\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{v}.\end{aligned}$$

We can now add the two parts and we obtain

$$\bar{q}(t,x,v)=\int_{-\infty}^{+\infty}\int_{0}^{+\infty}[G(t,x,v,x_0,v_0)+G(t,x,v,-x_0,-v_0)]p_0(x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0.$$

This is a solution to the whole space problem, but we are only looking for the solution to the half line. Therefore, we restrict the solution to $x > 0$, $v \in \mathbb{R}$, $t > 0$. It is now clear that the first two conditions in the theorem are satisfied. We must now check the third condition.

Recall our solution

$$\begin{aligned}&\bar{q}(t,x,v) \\ &= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} p_0(x_0,v_0)\Big[\frac{\sqrt{3}}{2\pi t^2}e^{-\frac{3}{t^4}\left(t(x-x_0-v_0t)^2-t^2(x-x_0-v_0t)(v-v_0)+\frac{t^3}{3}(v-v_0)^2\right)} \\ &\qquad + \frac{\sqrt{3}}{2\pi t^2}e^{-\frac{3}{t^4}\left(t(x+x_0+v_0t)^2-t^2(x+x_0+v_0t)(v+v_0)+\frac{t^3}{3}(v+v_0)^2\right)}\Big]\,\mathrm{d}x_0\,\mathrm{d}v_0.\end{aligned}$$

Let us check if our boundary conditions are satisfied:

$$\begin{aligned}&\bar{q}(t,0,v) \\ &= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} p_0(x_0,v_0)\Big[\frac{\sqrt{3}}{2\pi t^2}e^{-\frac{3}{t^4}\left(t(x_0+v_0t)^2-t^2(x_0+v_0t)(v_0-v)+\frac{1}{3}t^3(v-v_0)^2\right)} \\ &\qquad + \frac{\sqrt{3}}{2\pi t^2}e^{-\frac{3}{t^4}\left(t(x_0+v_0t)^2-t^2(x_0+v_0t)(v+v_0)+\frac{1}{3}t^3(v+v_0)^2\right)}\Big]\,\mathrm{d}x_0\,\mathrm{d}v_0,\end{aligned}$$

$$\bar{q}(t,0,-v)$$
$$=\int_{-\infty}^{+\infty}\int_{0}^{+\infty} p_0(x_0,v_0)\Big[\frac{\sqrt{3}}{2\pi t^2}e^{-\frac{3}{t^4}\left(t(x_0+v_0t)^2-t^2(x_0+v_0t)(v_0+v)+\frac{1}{3}t^3(v+v_0)^2\right)}$$
$$+\frac{\sqrt{3}}{2\pi t^2}e^{-\frac{3}{t^4}\left(t(x_0+v_0t)^2-t^2(x_0+v_0t)(v_0-v)+\frac{1}{3}t^3(v-v_0)^2\right)}\Big]\,dx_0\,dv_0.$$

As we can see, both of these are equal and therefore our solution meets all three conditions. We see that $\bar{q}=\bar{p}$ and find that when $\bar{q}$ is restricted to the half line, it is $\bar{p}$ defined in the statement of the theorem. □

6. Conclusion

Our note focuses on the Kolmogorov equation and teaches us some of its important properties. We first introduced stochastic processes including Brownian motion. Next, we searched for stationary solutions to our equation. We started off by looking for a solution of the form $p(x,v)=X(x)V(v)$. When looking at this case, we found that the result is a constant. Next, we searched for a solution of self-similar type, but this time one of the form $p(x,v)=x^{\alpha}\phi(-v^3/(9x))$, because of the scaling invariant property of the equation. In our attempt to solve this we found that with a reflecting boundary condition, a nonconstant solution exists when $\alpha=-\frac{2}{3}$. We also found the fundamental solution to the heat equation and the Kolmogorov equation. Once we had the fundamental solution, we were able to solve the differential equation with reflecting boundary condition. We first solved the problem on the whole space and then restricted it to the half line. Now that we have completed our investigations, it would be worthwhile to see the behavior of the Kolmogorov equation with different boundary conditions. In the case of absorbing boundary conditions, we refer to [Hwang et al. 2014; 2015b]. It would be interesting to investigate the long term behavior of the solutions, particularly whether the solution to the evolution problem would converge to the stationary solution. We leave this study for future projects. In addition, it would be useful to look at some of the many applications of this multifaceted equation. These investigations would be beneficial for many fields and could provide insight to some of the more obscure areas.

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