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Here, we consider a regularized mean-field game model that features a low-order regularization. We prove the existence of solutions with positive density. To do so, we combine a priori estimates with the continuation method. In contrast with high-order regularizations, the low-order regularizations are easier to implement numerically. Moreover, our methods give a theoretical foundation for this approach.

1. Prologue

On August 22, 2015, eighteen young mathematicians (B.Sc. and M.Sc. students) arrived at King Abdullah University of Science and Technology (KAUST) in Thuwal, Kingdom of Saudi Arabia. They were participants in the first KAUST summer camp in applied partial differential equations. Among them were Argentinians, Armenians, Chinese, Italians, Japanese, Mexicans, Portuguese, and Saudis. For many of them, this was their first time abroad. All were looking forward to the following three weeks.

We designed the summer camp to give an intense hands-on three-week Ph.D. experience. It comprised courses, seminars, a project, and a final presentation. The

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project was an essential component of the summer camp, and its main outcome is the present paper. Our objectives were to introduce students to an active research topic, teach effective paper writing techniques, and develop their presentation skills. Numerous challenges lay ahead. First, we had three weeks to achieve these goals. Second, students had distinct backgrounds. Third, we planned to study a research-level problem, not a simple exercise.

We selected a problem in mean-field games, a recent and active area of research. The primary goal was to prove the existence of solutions of a system of partial differential equations. To avoid unnecessary technicalities, we considered the one-dimensional case, where the partial differential equations become ordinary differential equations. The project involved partial differential equation methods that are usually taught in advanced courses: a priori estimate methods, the infinite-dimensional implicit function theorem, and the continuation method. In spite of the elementary nature of the proofs, the results presented here are a relevant and original contribution to the theory of mean-field games.

We divided the students into five groups and assigned tasks to each of them. Roughly, each of the sections of this paper corresponds to a task. The students were given a rough statement of the results to be proven, and their task was to figure out the appropriate assumptions, the precise statements, and the proofs. The work of the different groups had to be coordinated to make sure that the assumptions, results, and proofs fit nicely with each other and that duplicate work was avoided. Several KAUST graduate students and postdocs were of invaluable help in this regard.

This project would not have been possible within such a short time frame without the use of new technologies. The paper was written in a collaborative fashion using the platform Authorea that allowed all the groups to work simultaneously. In this way, all groups had access to the latest version of the assumptions and to the current statements of the theorems and propositions. Each group could easily comment and make corrections on other group's work.

This project illustrates how research in mathematics can be a collaborative experience even with a large number of participants. Moreover, it gave each of the students in the summer camp a glimpse of real research in mathematics. Finally, this was the first experience for the Ph.D. students and postdocs who helped in this project in mentoring and advising students. This summer camp was a unique and valuable experience for all participants whose results we share in this paper.

2. Introduction

Mean-field game (MFG) theory is the study of strategic decision making in large populations of small interacting individuals who are also called agents or players. The MFG framework was developed in the engineering community by Caines,

Huang, and Malhamé [Huang et al. 2006; 2007] and in the mathematical community by Lasry and Lions [2006a; 2006b; 2007]. These games model the behavior of rational agents who play symmetric differential games. In these problems, each player chooses their optimal strategy in view of global (or macroscopic) statistical information on the ensemble of players. This approach leads to novel problems in nonlinear equations. Current research topics are the applications of MFGs (including, for example, growth theory in economics and environmental policy), mathematical problems related to MFGs (existence, uniqueness, and regularity questions), and numerical methods in the MFGs framework (discretization, convergence, and efficient implementation).

Here, we consider the following problem:

Problem 1. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the one-dimensional torus, identified with the interval $[0, 1]$ whenever convenient. Fix a C^2 Hamiltonian, $H : \mathbb{R} \to \mathbb{R}$, and a continuous potential, $V : \mathbb{T} \to \mathbb{R}$. Let α and ϵ be positive numbers with $\epsilon \leq 1$ for definedness. Find $u, m \in C^2(\mathbb{T})$ satisfying $m > 0$ and

$$\begin{cases} u - u_{xx} + H(u_x) + V(x) = m^\alpha + \epsilon(m - m_{xx}), \\ m - m_{xx} - (H'(u_x)m)_x = 1 - \epsilon(u - u_{xx}). \end{cases} \tag{2-1}$$

In this problem, m is the distribution of players and $u(x)$ is the value function for a typical player in the state x. We stress that the condition $m > 0$ is an essential component of the problem. So, if (u, m) solves Problem 1, we require m to be strictly positive. We will show the existence of solutions to this problem under suitable assumptions on the Hamiltonian that are described in Section 3. An example that satisfies those assumptions is $H(p) = (1 + p^2)^{\gamma/2}$ with $1 < \gamma < 2$ and any $V : \mathbb{T} \to \mathbb{R}$ of class C^2.

When $\epsilon = 0$, (2-1) becomes

$$\begin{cases} u - u_{xx} + H(u_x) + V(x) = m^\alpha, \\ m - m_{xx} - (H'(u_x)m)_x = 1. \end{cases} \tag{2-2}$$

The system in (2-2) is a typical MFG model similar to the one introduced in [Lasry and Lions 2006a]. The Legendre transform of the Hamiltonian, H, given by $L(v) = \sup_p -pv - H(p)$ is the cost in units of time that an agent incurs by choosing to move with a drift v; the potential V accounts for spatial preferences of the agents; the term m^α encodes congestion effects.

The MFG models proposed in [Lasry and Lions 2006a; 2006b] consist of a system of partial differential equations that have (2-2) as a particular case. The current literature covers a broad range of problems, including stationary problems [Gomes et al. 2012; 2014; Gomes and Ribeiro 2013; Gomes and Sánchez Morgado 2014; Pimentel and Voskanyan 2015], heterogeneous populations [Cirant 2015], time-dependent models [Cardaliaguet et al. 2015; Gomes et al. 2015; 2016; Gomes

and Pimentel 2015; 2016; Porretta 2014; 2015], congestion problems [Gomes and Mitake 2015; Graber 2015], and obstacle-type problems [Gomes and Patrizi 2015]. For a recent account of the theory of MFGs, we suggest the survey paper [Gomes and Saúde 2014] and the course [Lions 2012].

The system in (2-1) arises as an approximation of (2-2) that preserves monotonicity properties. Monotonicity-preserving approximations to MFG systems were introduced in [Ferreira and Gomes 2015]. In that paper, the authors consider mean-field games in dimension $d \geq 1$, which include the following example:

$$\begin{cases} u - \Delta u + H(Du, x) + V(x) = m^{\alpha} + \epsilon(m + \Delta^{2q} m) + \beta_{\epsilon}(m), \\ m - \Delta m - \operatorname{div}(D_p H(Du, x)m) = 1 - \epsilon(u + \Delta^{2q} u), \end{cases} \tag{2-3}$$

where q is a large enough integer, and β_ϵ is a suitable penalization that satisfies $\beta_\epsilon(m) \to -\infty$ as $m \to 0$. Then, as $\epsilon \to 0$, the solutions of (2-3) converge to solutions of (2-2). Yet, from the perspective of numerical methods, both the high-order degree of (2-3) and the singularity caused by the penalty, β_ϵ, are unsatisfactory due to a poor conditioning of discretizations. Here, we investigate a low-order regularization that may be more suitable for computational problems.

A fundamental difficulty in the analysis of (2-1) is the nonnegativity of m. The Fokker–Planck equation in (2-2) has a maximum principle, and, consequently, $m \geq 0$ for any solution of (2-2). Due to the coupling, this property is not evident in the corresponding equation in (2-1). The previous regularization in (2-3) relies on a penalty that forces the positivity of m. This mechanism does not exist in (2-1), and we are not aware of any general method to prove the existence of positive solutions of (2-1).

Our main result is the following theorem.

Theorem 2.1. *Suppose that Assumptions 1–7 hold* (*see Section 3*). *Then, there exists* $\epsilon_0 > 0$ *such that for all* $0 < \epsilon < \epsilon_0$, *Problem 1 admits a* $C^{2,1/2}$ *solution* (u, m).

Theorem 2.1 introduces a low-order regularization procedure for (2-2) for which existence of solutions can be established without penalty terms. Because high-order regularization methods and penalty terms create serious difficulties in the numerical implementation, this result is relevant to the numerical approximation of (2-2). Moreover, we believe that the techniques we consider here can be extended to higher-dimensional problems.

To prove the main result, we use the continuation method. The first step is to establish a priori estimates for the solutions of (2-1). Then, we replace the potential V by λV for $0 \leq \lambda \leq 1$. For $\lambda = 0$, which corresponds to $V = 0$ in (2-1), we determine an explicit solution. The a priori estimates give that the set Λ of values λ for which (2-1) has a solution is a closed set. Finally, we apply an infinite-dimensional version of the implicit function theorem to show that Λ is relatively open in $[0, 1]$. This proves the existence of solutions.

The remainder of this paper is structured as follows. We discuss the main assumptions in Section 3. Next, in Section 4, we start our study of (2-1) by considering the case $V = 0$ and constructing an explicit solution. Sections 5–9 are devoted to a priori estimates for solutions of (2-1). These estimates include energy and second-order bounds, discussed respectively in Sections 5 and 6, Hölder and $C^{2,1/2}$ estimates, addressed respectively in Sections 7 and 8, and lower bounds on m, given in Section 9. Next, we lay out the main results needed for the implicit function theorem. We introduce the linearized operator in Section 10 and discuss its injectivity and surjectivity properties. Finally, the proof of Theorem 2.1 is presented in Section 11.

3. Main assumptions

We start by recalling that $C^{2,1/2}(\mathbb{T})$ is the space of all functions in $C^2(\mathbb{T})$ whose second derivative is $\frac{1}{2}$-Hölder continuous.

To prove Theorem 2.1, we need to introduce various assumptions that are natural in this class of problems. These encode distinct properties of the Hamiltonian in a convenient way. We begin by stating a polynomial growth condition for the Hamiltonian.

Assumption 1. There exist positive constants, C_1, C_2, C_3, and $\gamma > 1$, such that for all $p \in \mathbb{R}$, the Hamiltonian H satisfies

$$-C_1 + C_2|p|^\gamma \leq H(p) \leq C_1 + C_3|p|^\gamma.$$

For convex Hamiltonians, the expression $pH'(p) - H(p)$ is the Lagrangian written in momentum coordinates. The next assumption imposes polynomial growth in this quantity.

Assumption 2. There exist positive constants, $\tilde{C}_1$, $\tilde{C}_2$, and $\tilde{C}_3$, such that for all $p \in \mathbb{R}$, we have

$$-\tilde{C}_1 + \tilde{C}_2|p|^\gamma \leq pH'(p) - H(p) \leq \tilde{C}_1 + \tilde{C}_3|p|^\gamma.$$

Because we look for solutions $(u, m) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$ of Problem 1, we require in Assumptions 3 and 5 more regularity for V and H.

Assumption 3. The potential V is of class C^2.

Because the Hamilton–Jacobi equation in (2-2) arises from an optimal control problem, it is natural to suppose that the Hamiltonian H is convex.

Assumption 4. H is convex.

Assumption 5. The Hamiltonian H is of class C^4.

Here, we work with subquadratic Hamiltonians. Accordingly, we impose the following condition on γ.

Assumption 6. The constant γ satisfies $\gamma < 2$.

Finally, we state a growth condition on the derivative of the Hamiltonian. The exponent γ is the same as in Assumptions 1 and 2. This is a natural growth condition that the model $H(p) = (1+|p|^2)^{\gamma/2}$ satisfies.

Assumption 7. There exists a positive constant, $\bar{C}$, such that for all $p \in \mathbb{R}$, we have

$$|H'(p)| \leq \bar{C}(1+|p|^{\gamma-1}).$$

4. The $V = 0$ case

To prove Theorem 2.1, we use the continuation method. More precisely, we consider system (2-1) with V replaced by λV for $0 \leq \lambda \leq 1$. Next, we show the existence of the solution for all $0 \leq \lambda \leq 1$. As a starting point, we study the $\lambda = 0$ case; that is, $V = 0$. We show that (2-1) admits a solution in this particular instance.

Proposition 4.1. *Suppose that $V = 0$. Then, there exists an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, Problem 1 admits a solution (u, m).*

Proof. We look for constant solutions (u, m). In this case, we have $u_x = u_{xx} = m_x = m_{xx} = 0$. Accordingly, (2-1) reduces to

$$\begin{cases} u + H(0) = m^\alpha + \epsilon m, \\ m = 1 - \epsilon u. \end{cases}$$

In the previous system, solving the first equation for u and replacing the resulting expression into the second, we get

$$\epsilon m^\alpha + (1+\epsilon^2)m - 1 - \epsilon H(0) = 0. \tag{4-1}$$

We set $g(m) = \epsilon m^\alpha + (1+\epsilon^2)m - 1 - \epsilon H(0)$, so that (4-1) reads $g(m) = 0$. Next, we notice that $g(0) = -1 - \epsilon H(0)$. For small enough $\epsilon_0 > 0$ and for all $0 < \epsilon < \epsilon_0$, we have $g(0) < 0$. On the other hand, if we take a constant $C > |H(0)|$, we have

$$g(1+\epsilon C) > 1 + \epsilon C - 1 - \epsilon H(0) = \epsilon(C - H(0)) > 0.$$

Because $0 < 1 + \epsilon C$, by the intermediate value theorem, there exists a constant $m_0 \in]0, 1+\epsilon C[$ such that $g(m_0) = 0$. Then, setting $u_0 = (1-m_0)/\epsilon$, we conclude that the pair (u_0, m_0) satisfies the requirements. □

Remark 4.2. Note that if $H(0) > 0$, then $g(0) < 0$ and $g(1+\epsilon C) > 0$. In this case, the previous proposition holds for all $\epsilon > 0$.

5. Energy estimates

MFG systems such as (2-2) admit many a priori estimates. Among those, energy estimates stand out for their elementary proof — the multiplier method. Here, we apply this method to (2-1).

Proposition 5.1. *Suppose that Assumptions 1 and 2 hold. Let (u, m) solve Problem 1. Then,*

$$\int_0^1 m^{\alpha+1}\,dx + \int_0^1 |u_x|^\gamma (1+m)\,dx + \epsilon \int_0^1 (u^2+m^2+u_x^2+m_x^2)\,dx \le C, \quad (5\text{-}1)$$

where C is a universal positive constant depending only on the constants in Assumptions 1 and 2 and on $\|V\|_{L^\infty}$.

Proof. We begin by multiplying the first equation in (2-1) by $(1+\epsilon-m)$ and the second one by u. Adding the resulting expressions and integrating, we get

$$\begin{aligned}\int_0^1 \big[(1+\epsilon)H(u_x)+m(u_xH'(u_x)-H(u_x))\big]\,dx \\ +\int_0^1 m^{\alpha+1}\,dx+\epsilon\int_0^1 (u^2+m^2+u_x^2+m_x^2)\,dx \\ =-\epsilon\int_0^1 u\,dx+\int_0^1 (m-1-\epsilon)V(x)\,dx+(1+\epsilon)\int_0^1 m^\alpha\,dx+\epsilon(1+\epsilon)\int_0^1 m\,dx, \quad (5\text{-}2)\end{aligned}$$

where we used integration by parts and the periodicity of u and m to obtain

$$\int_0^1 mu_{xx}\,dx - \int_0^1 um_{xx}\,dx = 0,$$

$$\int_0^1 u_{xx}\,dx = u_x\big|_0^1 = 0, \qquad \int_0^1 m_{xx}\,dx = m_x\big|_0^1 = 0,$$

$$\int_0^1 mm_{xx}\,dx = -\int_0^1 m_x^2\,dx, \quad \int_0^1 uu_{xx}\,dx = -\int_0^1 u_x^2\,dx,$$

$$\int_0^1 u(H'(u_x)m)_x\,dx = -\int_0^1 u_xH'(u_x)m\,dx.$$

Next, we observe that by Assumptions 1 and 2, and using the fact that $0<\epsilon\le 1$, we have

$$\begin{aligned}\int_0^1 \big[(1+\epsilon)H(u_x)+m(H'(u_x)u_x-H(u_x))\big]\,dx \\ \ge \int_0^1 [-2C_1-\tilde{C}_1m+K_0|u_x|^\gamma(1+m)]\,dx, \quad (5\text{-}3)\end{aligned}$$

where $K_0 := \min\{C_2, \tilde{C}_2\}$.

From (5-2) and (5-3), it follows that

$$\int_0^1 K_0|u_x|^\gamma(1+m)\,dx+\int_0^1 m^{\alpha+1}\,dx+\epsilon\int_0^1(u^2+m^2+u_x^2+m_x^2)\,dx$$
$$\leq\frac{\epsilon}{2}\int_0^1 u^2\,dx+\tfrac{1}{2}+(\|V\|_\infty+2+\tilde{C}_1)\int_0^1 m\,dx+2\int_0^1 m^\alpha\,dx+2(\|V\|_\infty+C_1), \quad (5\text{-}4)$$

where we also used the estimates $2u\leq u^2+1$ and $0<\epsilon\leq 1$.

Finally, we observe that for every $\delta_1,\delta_2>0$, there exist constants, K_1 and K_2, such that

$$\int_0^1 m^\alpha\,dx\leq\delta_1\int_0^1 m^{\alpha+1}\,dx+K_1,\qquad \int_0^1 m\,dx\leq\delta_2\int_0^1 m^{\alpha+1}\,dx+K_2. \quad (5\text{-}5)$$

Consequently, taking $\delta_1=\frac{1}{8}$ and $\delta_2=1/(4(\|V\|_\infty+2+\tilde{C}_1))$ in (5-5) and using the resulting estimates in (5-4), we conclude that (5-1) holds. □

Corollary 5.2. *Suppose that Assumptions 1 and 2 hold. Let (u,m) solve Problem 1. Then,*

$$\int_0^1 m\,dx\leq C,$$

where C is a universal positive constant depending only on the constants in Assumptions 1 and 2 and on $\|V\|_{L^\infty}$.

Proof. Due to (5-1) and because m is positive,

$$\int_0^1 m^{\alpha+1}\,dx\leq C,$$

where C is a universal positive constant depending only on the constants in Assumptions 1 and 2 and on $\|V\|_{L^\infty}$. Consequently, using Young's inequality, we have

$$\int_0^1 m\,dx\leq\frac{1}{\alpha+1}\int_0^1 m^{\alpha+1}\,dx+\frac{\alpha}{\alpha+1}\leq\frac{C}{\alpha+1}+\frac{\alpha}{\alpha+1}.$$ □

6. Second-order estimates

We proceed in our study of (2-1) by examining another technique to obtain a priori estimates. These estimates give additional control over high-order norms of the solutions.

Proposition 6.1. *Suppose that Assumption 3 holds. Let (u,m) solve Problem 1. Then, we have*

$$\int_0^1(H''(u_x)u_{xx}^2m+\alpha m^{\alpha-1}m_x^2)\,dx+\epsilon\int_0^1(m_x^2+m_{xx}^2+u_x^2+u_{xx}^2)\,dx\leq C, \quad (6\text{-}1)$$

where $C > 0$ denotes a universal constant depending only on $\|V\|_{C^2}$. Moreover, under Assumption 4,

$$\int_0^1 \alpha m^{\alpha-1} m_x^2 \,\mathrm{d}x + \epsilon \int_0^1 (m_x^2 + m_{xx}^2 + u_x^2 + u_{xx}^2)\,\mathrm{d}x \le C. \tag{6-2}$$

Proof. To simplify the notation, we represent by C any positive constant that depends only on $\|V\|_{C^2}$ and whose value may change from one instance to another.

Multiplying the first equation in (2-1) by m_{xx} and the second one by u_{xx} yields

$$(u - u_{xx} + H(u_x) + V(x))m_{xx} = (m^\alpha + \epsilon(m - m_{xx}))m_{xx},$$
$$(m - m_{xx} - (H'(u_x)m)_x)u_{xx} = (1 - \epsilon(u - u_{xx}))u_{xx}.$$

Subtracting the above equations integrated over [0, 1] gives

$$\int_0^1 (um_{xx} - mu_{xx} + u_{xx})\,\mathrm{d}x + \int_0^1 [H(u_x)m_{xx} + (H'(u_x)m)_x u_{xx}]\,\mathrm{d}x$$
$$+ \int_0^1 V(x)m_{xx}\,\mathrm{d}x - \int_0^1 m^\alpha m_{xx}\,\mathrm{d}x + \epsilon \int_0^1 (-mm_{xx} + m_{xx}^2 - uu_{xx} + u_{xx}^2)\,\mathrm{d}x = 0. \tag{6-3}$$

Next, we evaluate each of the integrals above. Using the integration by parts formula and the periodicity of boundary conditions, we have

$$\int_0^1 (um_{xx} - mu_{xx} + u_{xx})\,\mathrm{d}x = 0. \tag{6-4}$$

In addition,

$$\int_0^1 \left[(H'(u_x)m)_x u_{xx} + H(u_x)m_{xx}\right]\mathrm{d}x$$
$$= \int_0^1 \left[H''(u_x)mu_{xx}^2 + (H(u_x))_x m_x + (H(u_x))m_{xx}\right]\mathrm{d}x$$
$$= \int_0^1 H''(u_x)mu_{xx}^2\,\mathrm{d}x. \tag{6-5}$$

Furthermore, we have

$$-\int_0^1 m^\alpha m_{xx}\,\mathrm{d}x = \int_0^1 \alpha m^{\alpha-1} m_x^2\,\mathrm{d}x \tag{6-6}$$

and

$$\int_0^1 -Vm_{xx}\,\mathrm{d}x = -\int_0^1 V_{xx}m\,\mathrm{d}x \le \int_0^1 |V_{xx}|m\,\mathrm{d}x \le C\int_0^1 m\,\mathrm{d}x \le C, \tag{6-7}$$

where we used Corollary 5.2.

Finally,

$$\epsilon \int_0^1 (-mm_{xx} + m_{xx}^2 - uu_{xx} + u_{xx}^2)\,\mathrm{d}x = \epsilon \int_0^1 (m_x^2 + m_{xx}^2 + u_x^2 + u_{xx}^2)\,\mathrm{d}x. \quad (6\text{-}8)$$

Using (6-3)–(6-8), we get

$$\int_0^1 H''(u_x) m u_{xx}^2\,\mathrm{d}x + \int_0^1 \alpha m^{\alpha-1} m_x^2\,\mathrm{d}x + \epsilon \int_0^1 (m_x^2 + m_{xx}^2 + u_x^2 + u_{xx}^2)\,\mathrm{d}x = -\int_0^1 V m_{xx} \le C.$$

This completes the proof of (6-1). To conclude the proof of Proposition 6.1, we observe that Assumption 4 implies that H'' is a nonnegative function, which together with (6-1) gives (6-2). □

7. Hölder continuity

We recall that Morrey's theorem in one dimension [Evans 1998] gives the following result.

Proposition 7.1. *Let $f \in C^1(\mathbb{T})$. Then,*

$$|f(x) - f(y)| \le \|f_x\|_{L^2} |x - y|^{1/2} \quad \forall\, x, y \in \mathbb{T}.$$

Proposition 7.2. *Suppose that Assumptions 1–4 hold. Let (u, m) solve Problem 1. Then, u, u_x, m, and m_x are $\frac{1}{2}$-Hölder continuous functions with L^∞-norms and Hölder constants bounded by $C/\sqrt{\epsilon}$, where C is a universal constant depending only on the constants in Assumptions 1 and 2 and on $\|V\|_{C^2}$.*

Proof. By Proposition 5.1, we have that

$$\epsilon \int_0^1 (m^2 + u^2 + m_x^2 + u_x^2)\,\mathrm{d}x \le C, \quad (7\text{-}1)$$

where C is a universal constant depending only on the constants in Assumptions 1 and 2 and on $\|V\|_{L^\infty}$.

According to Proposition 7.1, we have

$$|u(x) - u(y)| \le \|u_x\|_{L^2} |x - y|^{1/2} \quad \forall\, x, y \in \mathbb{T}. \quad (7\text{-}2)$$

Moreover, combining the bound on $\|u\|_{L^2}$ given by (7-1), the mean-value theorem for definite integrals, and the Hölder continuity given by (7-2), we get the L^∞ bound on u. A similar inequality holds for m. Next, we observe that Proposition 6.1 (see (6-2)) gives bounds for $\|u_{xx}\|_{L^2}$ and $\|m_{xx}\|_{L^2}$ of the same type as (7-1). Accordingly, the functions u_x and m_x are also $\frac{1}{2}$-Hölder continuous, and their L^∞ norms are

bounded by $C/\sqrt{\epsilon}$, where C depends only on the constants in Assumptions 1 and 2 and on $\|V\|_{C^2}$. □

Remark 7.3. Consider Problem 1 with V replaced by λV for some $\lambda \in [0, 1]$. By revisiting the proofs of Propositions 5.1 and 6.1, we can readily check that the bounds stated in these propositions are uniform with respect to $\lambda \in [0, 1]$. More precisely, (5-1), (6-1), and (6-2) are still valid for a universal positive constant C that depends only on the constants in Assumptions 1 and 2 and on $\|V\|_{C^2}$. In particular, Proposition 7.2 remains unchanged.

8. Higher regularity

The bounds in the previous section give Hölder regularity for any solution (u, m) of Problem 1 and for its derivatives (u_x, m_x). Here, we use (2-1) to improve this result and prove Hölder regularity for u_{xx} and m_{xx}.

Proposition 8.1. *Suppose that Assumptions 1–5 hold. Let (u, m) solve Problem 1. Then $(u, m) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$.*

Proof. Solving for $m - m_{xx}$ in the second equation of (2-1) and replacing the resulting expression in the first equation yields

$$[1+\epsilon^2+\epsilon H''(u_x)m]u_{xx} = (1+\epsilon^2)u + H(u_x) - \epsilon + V(x) - m^\alpha - \epsilon H'(u_x)m_x. \quad (8\text{-}1)$$

Because H is convex, we have $H''(u_x) \geq 0$. Consequently, $1 + \epsilon^2 + \epsilon H''(u_x)m \geq 1 > 0$. This allows us to rewrite (8-1) as

$$u_{xx} = \frac{(1+\epsilon^2)u + H(u_x) - \epsilon + V(x) - m^\alpha - \epsilon H'(u_x)m_x}{1+\epsilon^2+\epsilon H''(u_x)m}. \quad (8\text{-}2)$$

Because u, m, u_x, and m_x are $\frac{1}{2}$-Hölder continuous and because H and H' are locally Lipschitz functions, it follows that

$$(1+\epsilon^2)u + H(u_x) - \epsilon + V(x) - m^\alpha - \epsilon H'(u_x)m_x$$

is also $\frac{1}{2}$-Hölder continuous. Similarly, due to Assumption 5, $1+\epsilon^2+\epsilon H''(u_x)m$ is also $\frac{1}{2}$-Hölder continuous and bounded from below by 1. Therefore, u_{xx} is $\frac{1}{2}$-Hölder continuous; thus, $u \in C^{2,1/2}(\mathbb{T})$.

Finally, we observe that the second equation in (2-1) is equivalent to

$$m_{xx} = m + \epsilon(u - u_{xx}) - 1 - H''(u_x)mu_{xx} - H'(u_x)m_x. \quad (8\text{-}3)$$

Hence, analogous arguments to those used above yield that m_{xx} is also $\frac{1}{2}$-Hölder continuous. Thus, $m \in C^{2,1/2}(\mathbb{T})$. □

9. Lower bounds on m

Here, we establish our last a priori estimate, which gives lower bounds on m. We begin by proving an auxiliary result.

Lemma 9.1. *Suppose that Assumptions 1–4, 6, and 7 hold. Let (u, m) solve Problem 1. Then, $\|\epsilon(u - u_{xx})\|_\infty \le C\epsilon^{1-\gamma/2}$, where C is a universal positive constant depending only on the constants in Assumptions 1, 2, and 7 and on $\|V\|_{C^2}$.*

Proof. To simplify the notation, C represents a positive constant depending only on the constants in Assumptions 1, 2, and 7 and on $\|V\|_{C^2}$ and whose value may change from one instance to another.

Note that $\max\{\epsilon^{1/2}, \epsilon, \epsilon^{2-\gamma/2}, \epsilon^{3/2}, \epsilon^2\} \le \epsilon^{1-\gamma/2}$ because $0 < 1 - \frac{1}{2}\gamma < \frac{1}{2}$ (see Assumption 6).

By Proposition 7.2, we have that $\|u\|_\infty \le C/\sqrt{\epsilon}$. Thus,

$$\|\epsilon u\|_\infty \le C\epsilon^{1-\gamma/2}. \tag{9-1}$$

Next, we examine $\|\epsilon u_{xx}\|_\infty$. The identity (8-2) and the condition $1 + \epsilon^2 + \epsilon H''(u_x)m > 1$ give

$$\begin{aligned}\|\epsilon u_{xx}\|_\infty \le \|\epsilon(1+\epsilon^2)u\|_\infty + \|\epsilon H(u_x)\|_\infty \\ + \epsilon^2 + \|\epsilon V\|_\infty + \|\epsilon m^\alpha\|_\infty + \|\epsilon^2 H'(u_x)m_x\|_\infty.\end{aligned} \tag{9-2}$$

By (9-1) and by the boundedness of V, it follows that

$$\|\epsilon(1+\epsilon^2)u\|_\infty + \epsilon^2 + \|\epsilon V\|_\infty \le C\epsilon^{1-\gamma/2}.$$

According to Propositions 5.1 and 6.1, we have

$$\int_0^1 m^{\alpha+1}\,dx \le C \quad \text{and} \quad \int_0^1 \alpha m^{\alpha-1} m_x^2\,dx = \frac{4\alpha}{(\alpha+1)^2}\int_0^1 (m^{(\alpha+1)/2})_x^2\,dx \le C.$$

The first integral guarantees that there exists $x_0 \in \mathbb{T}$ such that $m^{(\alpha+1)/2}(x_0) \le C$. Then, because $m > 0$ and because $m \in C^1(\mathbb{T})$, the second integral together with Proposition 7.1 implies that for all $x \in \mathbb{T}$,

$$0 < m^\alpha(x) = (m^{\alpha/2}(x))^2 \le \left(m^{(\alpha+1)/2}(x) - m^{(\alpha+1)/2}(x_0) + m^{(\alpha+1)/2}(x_0) + 1\right)^2 \le C.$$

Hence, $\|\epsilon m^\alpha\|_\infty \le C\epsilon^{1-\gamma/2}$.

Assumption 1 and Proposition 7.2 give

$$|H(u_x)| \le C(1 + \epsilon^{-\gamma/2}).$$

This implies that $\|\epsilon H(u_x)\|_\infty \le C\epsilon^{1-\gamma/2}$.

Combining Assumption 7 with Proposition 7.2 gives the bound

$$|H'(u_x)| \le C(1 + \epsilon^{-(\gamma-1)/2}).$$

By Proposition 7.2, we have that $|m_x| \le C/\sqrt{\epsilon}$. Therefore,

$$\|\epsilon^2 H'(u_x) m_x\|_\infty \le C\epsilon^{1-\gamma/2}.$$

Collecting all the estimates proved above, we conclude from (9-1) and (9-2) that

$$\|\epsilon(u - u_{xx})\|_\infty \le C\epsilon^{1-\gamma/2}. \qquad \square$$

Proposition 9.2. *Suppose that Assumptions 1–4, 6, and 7 hold. Let*

$$\bar{\epsilon}_0 := \left(\frac{1}{2C}\right)^{2/(2-\gamma)},$$

where C is the constant given by Lemma 9.1. Let (u, m) solve Problem 1 with $0 < \epsilon < \min\{1, \bar{\epsilon}_0\}$. Then, there exists $\bar{m} > 0$ such that $m > \bar{m}$ on $\mathbb{T}$. Moreover, $\bar{m}$ is a universal constant depending only on the constants in Assumptions 1, 2, and 7, on $\|V\|_{C^2}$, and on ϵ.

Proof. Multiplying the second equation in (2-1) by $1/m$ and integrating with respect to x in $[0, 1]$, we obtain

$$\int_0^1 \left(1 - \frac{m_{xx}}{m} - \frac{(H'(u_x)m)_x}{m}\right) dx = \int_0^1 \left(\frac{1}{m} - \epsilon\frac{u - u_{xx}}{m}\right) dx. \tag{9-3}$$

Integration by parts and periodicity yield

$$\int_0^1 \frac{m_{xx}}{m}\, dx = \int_0^1 \frac{m_x^2}{m^2}\, dx.$$

Then, (9-3) can be rewritten as

$$\int_0^1 \left(\frac{1}{m} + \frac{m_x^2}{m^2}\right) dx = 1 + \int_0^1 \frac{\epsilon(u - u_{xx})}{m}\, dx - \int_0^1 \frac{(H'(u_x)m)_x}{m}\, dx.$$

Next, we estimate the right-hand side of this identity. By Lemma 9.1, for $0 < \epsilon < \bar{\epsilon}_0$, we have $\|\epsilon(u - u_{xx})\|_\infty < \frac{1}{2}$. Consequently,

$$\int_0^1 \left(\frac{1}{2m} + \frac{m_x^2}{m^2}\right) dx \le 1 + \left|\int_0^1 \frac{(H'(u_x)m)_x}{m}\, dx\right| = 1 + \left|\int_0^1 H'(u_x)\frac{m_x}{m}\, dx\right|, \tag{9-4}$$

where in the last equality we used the integration by parts formula and the periodicity of u_x. In view of Cauchy's inequality, we conclude that

$$\left|\int_0^1 H'(u_x)\frac{m_x}{m}\, dx\right| \le \int_0^1 \left|H'(u_x)\frac{m_x}{m}\right| dx \le \int_0^1 \left(\frac{(H'(u_x))^2}{2} + \frac{m_x^2}{2m^2}\right) dx. \tag{9-5}$$

Invoking Assumptions 6 and 7, we obtain the estimates

$$(H'(u_x))^2 \le \bar{C}^2(1 + |u_x|^{\gamma-1})^2 \le 2\bar{C}^2(1 + |u_x|^{2(\gamma-1)}) \le 2\bar{C}^2(2 + |u_x|^2)$$

in $\mathbb{T}$. These estimates, (9-4), (9-5), and Proposition 5.1 yield

$$\int_0^1 \left(\frac{1}{2m} + \frac{m_x^2}{2m^2}\right) dx \le 1 + \bar{C}^2(2 + C/\epsilon).$$

Consequently, for $\tilde{C} = 2 + 2\bar{C}^2(2 + C/\epsilon)$, we obtain the bounds

$$\int_0^1 \frac{1}{m}\, dx \le \tilde{C} \quad \text{and} \quad \int_0^1 \frac{m_x^2}{m^2}\, dx = \int_0^1 (\ln(m))_x^2\, dx \le \tilde{C}.$$

The first bound implies that there exists $x_0 \in \mathbb{T}$ such that $1/(m(x_0)) \le \tilde{C} + 1$; that is, $\ln(m(x_0)) \ge -\ln(\tilde{C} + 1)$. The second bound, together with Proposition 7.1, implies that for all $x \in \mathbb{T}$, the value of $|\ln(m(x)) - \ln(m(x_0))| \le \sqrt{\tilde{C}}$. Hence, for all $x \in \mathbb{T}$,

$$m(x) \ge e^{-\sqrt{\tilde{C}} - \ln(\tilde{C}+1)}. \qquad \square$$

Remark 9.3. As in Remark 7.3, the statement of Proposition 9.2 remains unchanged if we replace V by λV for some $\lambda \in [0, 1]$ in Problem 1.

10. The linearized operator

Consider the functional, F, defined for $(u, m, \lambda) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T};]0, \infty[) \times [0, 1]$ by

$$F(u, m, \lambda) = \begin{bmatrix} u - u_{xx} + H(u_x) + \lambda V - m^\alpha - \epsilon(m - m_{xx}) \\ m - m_{xx} - (H'(u_x)m)_x - 1 + \epsilon(u - u_{xx}) \end{bmatrix}. \tag{10-1}$$

Note that under Assumption 5, the functional F is a C^1 map between $C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T};]0, \infty[) \times [0, 1]$ and $C^{0,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$.

To prove Theorem 2.1, we use the continuation method and show that for every $\lambda \in [0, 1]$, the equation

$$F(u, m, \lambda) = 0 \tag{10-2}$$

has a solution, $(u, m) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T};]0, \infty[)$. Theorem 2.1 then follows by taking $\lambda = 1$ and by observing that system (2-1) is equivalent to $F(u, m, 1) = 0$.

The implicit function theorem plays a crucial role in proving the solvability of (10-2). To use this theorem, for each $\lambda \in [0, 1]$, we introduce the linearized operator L of $F(\cdot, \cdot, \lambda)$ at $(u, m) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T};]0, \infty[)$; that is,

$$\begin{aligned} L(f, v) &= \frac{\partial F}{\partial \mu}(u + \mu v, m + \mu f, \lambda)\Big|_{\mu=0} \\ &= \begin{bmatrix} v - v_{xx} + H'(u_x)v_x - \alpha m^{\alpha-1} f - \epsilon(f - f_{xx}) \\ f - f_{xx} - (H''(u_x)v_x m + H'(u_x) f)_x + \epsilon(v - v_{xx}) \end{bmatrix} \end{aligned} \tag{10-3}$$

for $(f, v) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$. Under Assumption 5 and because $(u, m) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T};]0, \infty[)$, the operator L defines a map from $C^{2,1/2}(\mathbb{T}) \times$

$C^{2,1/2}(\mathbb{T})$ into $C^{0,1/2}(\mathbb{T}) \times C^{0,1/2}(\mathbb{T})$. Moreover, this map is continuous and linear. Next, we show that it is also an isomorphism between $C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$ and $C^{0,1/2}(\mathbb{T}) \times C^{0,1/2}(\mathbb{T})$.

Proposition 10.1. *Suppose that Assumptions 4 and 5 hold. Fix $\lambda \in [0, 1]$ and assume that $(u, m) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T};]0, \infty[)$ satisfies $F(u, m, \lambda) = 0$. Then, the operator L given by (10-3) is an isomorphism between $C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$ and $C^{0,1/2}(\mathbb{T}) \times C^{0,1/2}(\mathbb{T})$.*

Proof. To prove the proposition, we begin by applying the Lax–Milgram theorem in $H^1(\mathbb{T}) \times H^1(\mathbb{T})$, after which we bootstrap additional regularity. Here, we endow $H^1(\mathbb{T}) \times H^1(\mathbb{T})$ with the inner product

$$\big\langle (\theta_1, \theta_2), (\bar\theta_1, \bar\theta_2) \big\rangle_{H^1(\mathbb{T})\times H^1(\mathbb{T})} = \int_0^1 (\theta_1\bar\theta_1 + \theta_2\bar\theta_2 + \theta_{1x}\bar\theta_{1x} + \theta_{2x}\bar\theta_{2x})\,\mathrm{d}x$$

for $(\theta_1, \theta_2), (\bar\theta_1, \bar\theta_2) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$.

Consider the bilinear form $B : (H^1(\mathbb{T}) \times H^1(\mathbb{T})) \times (H^1(\mathbb{T}) \times H^1(\mathbb{T})) \to \mathbb{R}$ defined for $(v, f), (w_1, w_2) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$ by

$$\begin{aligned} B\left(\begin{pmatrix} v \\ f \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) &= \int_0^1 (f + \epsilon v) w_1 \,\mathrm{d}x + \int_0^1 [f_x + H''(u_x) v_x m + H'(u_x) f + \epsilon v_x] w_{1x} \,\mathrm{d}x \\ &\quad - \int_0^1 [v + H'(u_x) v_x - \alpha m^{\alpha-1} f - \epsilon f] w_2 \,\mathrm{d}x + \int_0^1 (\epsilon f_x - v_x) w_{2x} \,\mathrm{d}x. \end{aligned}$$

Note that if $(v, f) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$, then

$$B\left(\begin{pmatrix} v \\ f \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \int_0^1 [-L_1(f, v) w_2 + L_2(f, v) w_1] \,\mathrm{d}x, \tag{10-4}$$

where L_1 and L_2 are the first and second components of L, respectively.

Next, we prove that B is coercive and bounded in $H^1(\mathbb{T}) \times H^1(\mathbb{T})$. Fix (v, f), $(w_1, w_2) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$. Using the integration by parts formula and the periodicity of v and f, we obtain

$$B\left(\begin{pmatrix} v \\ f \end{pmatrix}, \begin{pmatrix} v \\ f \end{pmatrix}\right) = \int_0^1 [\alpha m^{\alpha-1} f^2 + H''(u_x) v_x^2 m + \epsilon(v^2 + v_x^2 + f^2 + f_x^2)] \,\mathrm{d}x.$$

Because $H'' \geq 0$ by Assumption 4 and because $m > 0$, we have

$$B\left(\begin{pmatrix} v \\ f \end{pmatrix}, \begin{pmatrix} v \\ f \end{pmatrix}\right) \geq \epsilon \left\| \begin{pmatrix} v \\ f \end{pmatrix} \right\|^2_{H^1(\mathbb{T})\times H^1(\mathbb{T})},$$

which proves the coercivity of B.

Because m, u, and H are $C^{2,1/2}$-functions on the compact set $[0,1]$, we have that m, u, m_x, u_x, u_{xx}, H, $H'(u_x)$, and $H''(u_x)$ are bounded. Therefore, there exists a positive constant, C, that depends only on these bounds and for which

$$\left|B\left(\begin{pmatrix} v \\ f \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right)\right| \le C \left\|\begin{pmatrix} v \\ f \end{pmatrix}\right\|_{H^1(\mathbb{T})\times H^1(\mathbb{T})} \left\|\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right\|_{H^1(\mathbb{T})\times H^1(\mathbb{T})},$$

where we also used Hölder's inequality. This proves the boundedness of B.

Finally, we fix $b = (b_1, b_2) \in C^{0,1/2}(\mathbb{T}) \times C^{0,1/2}(\mathbb{T})$, and we consider the bounded and linear functional $G : H^1(\mathbb{T}) \times H^1(\mathbb{T}) \to \mathbb{R}$ defined for $(w_1, w_2) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$ by

$$G\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \int_0^1 (-b_1 w_2 + b_2 w_1)\, \mathrm{d}x.$$

By the Lax–Milgram theorem, there exists a unique $(v, f) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$ such that for all $(w_1, w_2) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$, we have

$$B\left(\begin{pmatrix} v \\ f \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = G\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

This is equivalent to saying that for all $(w_1, w_2) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$,

$$B\left(\begin{pmatrix} v \\ f \end{pmatrix}, \begin{pmatrix} -w_2 \\ w_1 \end{pmatrix}\right) = G\begin{pmatrix} -w_2 \\ w_1 \end{pmatrix} = \int_0^1 (-b_1 w_1 - b_2 w_2)\, \mathrm{d}x.$$

From this and (10-4), we conclude that $L(f, v) = b$ has a unique weak solution $(f, v) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$. Because $b \in C^{0,1/2}(\mathbb{T}) \times C^{0,1/2}(\mathbb{T})$ is arbitrary, L is injective. To prove surjectivity, it suffices to check that the weak solution of $L(f, v) = b$ is in $C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$. This higher regularity follows from a bootstrap argument.

Fix $b = (b_1, b_2) \in C^{0,1/2}(\mathbb{T}) \times C^{0,1/2}(\mathbb{T})$ and let $(f, v) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$ be the weak solution of $L(f, v) = b$ given by the Lax–Milgram theorem. Then, we have the following identity in the weak sense:

$$v_{xx} = \frac{g}{1 + \epsilon^2 + \epsilon H''(u_x)m}, \tag{10-5}$$

where

$$g = v(1+\epsilon^2) + H'(u_x)v_x - \alpha m^{\alpha-1} f - \epsilon v_x (H'(u_x)m)_x \\ - \epsilon (H'(u_x) f)_x - \epsilon b_2 - b_1 \in L^2(\mathbb{T}).$$

We recall that $1 + \epsilon^2 + \epsilon H''(u_x)m > 1$. Hence, $v_{xx} \in L^2(\mathbb{T})$, and so $v \in H^2(\mathbb{T})$. Moreover, because

$$f_{xx} = f - (H''(u_x)v_x m)_x - (H'(u_x) f)_x + \epsilon(v - v_{xx}) - b_2 \tag{10-6}$$

in the weak sense, similar arguments yield $f_{xx} \in L^2(\mathbb{T})$ and $f \in H^2(\mathbb{T})$.

So far, $(f, v) \in C^{1,1/2}(\mathbb{T}) \times C^{1,1/2}(\mathbb{T})$. This implies that $g \in C^{0,1/2}(\mathbb{T})$. Then, using the fact that $1 + \epsilon^2 + \epsilon H''(u_x)m$ also belongs to $C^{0,1/2}(\mathbb{T})$ and is bounded from below by 1, from (10-5) it follows that $v_{xx} \in C^{0,1/2}(\mathbb{T})$. Consequently, in view of (10-6), $f_{xx} \in C^{0,1/2}(\mathbb{T})$. Hence, $(f, v) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$. Therefore, the unique solution given by the Lax–Milgram theorem is a strong solution with $C^{2,1/2}$ regularity. Thus, L is surjective. Because L is injective and surjective, it is an isomorphism. □

11. Proof of the main theorem

In this last section, we prove Theorem 2.1. We assume that $\epsilon > 0$ satisfies $\epsilon < \min\{1, \epsilon_0, \bar{\epsilon}_0\}$, where ϵ_0 and $\bar{\epsilon}_0$ are given by Propositions 4.1 and 9.2, respectively.

Let F be the functional defined in (10-1). For each $\lambda \in [0, 1]$, consider the problem of finding $(u, m) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T};]0, \infty[)$ satisfying (10-2). From Propositions 4.1 and 8.1, such a pair (u, m) exists for $\lambda = 0$. Next, using the continuation method, we prove that this is true not only for $\lambda = 0$ but also for all $\lambda \in [0, 1]$.

More precisely, let Λ be the set of values $\lambda \in [0, 1]$ for which (10-2) has a solution $(u, m) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$ with $m \geq \bar{m}$ in $\mathbb{T}$, where $\bar{m} > 0$ is given by Proposition 9.2. Note that $\bar{m}$ does not depend on λ (see Remark 9.3). As we just argued, Λ is a nonempty set. In the subsequent two propositions, we show that Λ is a closed and open subset of $[0, 1]$. Consequently, $\Lambda = [0, 1]$.

Proposition 11.1. *Suppose that Assumptions 1–7 hold. Then, Λ is a closed subset of $[0, 1]$.*

Proof. Let $(\lambda^n)_{n\in\mathbb{N}} \subset \Lambda$ and $\lambda \in [0, 1]$ be such that $\lim_{n\to\infty} \lambda^n = \lambda$. We claim that $\lambda \in \Lambda$.

By definition of Λ, for each $n \in \mathbb{N}$, there exists $(u^n, m^n) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$ satisfying (10-2) and $m^n \geq \bar{m}$ in $\mathbb{T}$. Then, by Proposition 7.2 (also see Remark 7.3), $(u^n)_{n\in\mathbb{N}}$, $(m^n)_{n\in\mathbb{N}}$, $(u^n_x)_{n\in\mathbb{N}}$, and $(m^n_x)_{n\in\mathbb{N}}$ are uniformly bounded in $C^{0,1/2}(\mathbb{T})$. Consequently, by the Arzelà–Ascoli theorem, we can find $(u, m, \tilde{u}, \tilde{m}) \in C^{0,1/2}(\mathbb{T}) \times C^{0,1/2}(\mathbb{T}) \times C^{0,1/2}(\mathbb{T}) \times C^{0,1/2}(\mathbb{T})$ such that, up to a subsequence that we do not relabel,

$$\lim_{n\to\infty} \|(u^n, m^n, u^n_x, m^n_x) - (u, m, \tilde{u}, \tilde{m})\|_\infty = 0. \tag{11-1}$$

We now recall that if $(w^n)_{n\in\mathbb{N}}$ is a sequence of differentiable functions on $[0, 1]$ such that $(w^n)_{n\in\mathbb{N}}$ converges uniformly to some w on $[0, 1]$ and such that $(w^n_x)_{n\in\mathbb{N}}$ converges uniformly on $[0, 1]$, then $w_x = \lim_{n\to\infty} w^n_x$ on $[0, 1]$. Consequently, by (11-1), we have $\tilde{u} = u_x$ and $\tilde{m} = m_x$.

Next, we show that $(u^n_{xx})_{n\in\mathbb{N}}$ and $(m^n_{xx})_{n\in\mathbb{N}}$ are also uniformly convergent sequences on $[0, 1]$. In view of (8-2), we have for every $n \in \mathbb{N}$,

$$u^n_{xx} = \frac{(1+\epsilon^2)u^n + H(u^n_x) - \epsilon + \lambda^n V(x) - (m^n)^\alpha - \epsilon H'(u^n_x)m^n_x}{1+\epsilon^2+\epsilon H''(u^n_x)m^n}. \tag{11-2}$$

By Assumption 5 and by the uniform convergence of $(u^n, m^n, \lambda^n, u^n_x, m^n_x)_{n\in\mathbb{N}}$ to $(u, m, \lambda, u_x, m_x)$ on $[0, 1]$, it follows from (11-2) that $(u^n_{xx})_{n\in\mathbb{N}}$ converges uniformly on $[0, 1]$. Then, the limit of $(u^n_{xx})_{n\in\mathbb{N}}$ is necessarily u_{xx}. Analogous arguments (see (8-3)) give that $(m^n_{xx})_{n\in\mathbb{N}}$ converges uniformly to m_{xx} on $[0, 1]$. Thus, $(u, m) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T};]0, \infty[)$. Moreover, $\lim_{n\to\infty} F(u^n, m^n, \lambda^n) = F(u, m, \lambda)$. Finally, because for all $n \in \mathbb{N}$, the functional $F(u^n, m^n, \lambda^n) = 0$ and $m^n \geq \bar{m}$ in $\mathbb{T}$, we have that $F(u, m, \lambda) = 0$ and $m \geq \bar{m}$ in $\mathbb{T}$. Hence, $\lambda \in \Lambda$. □

Proposition 11.2. *Suppose that Assumptions 1–7 hold. Then,* Λ *is an open subset of* $[0, 1]$.

Proof. Let $\lambda_0 \in \Lambda$. Then, there exists $(u_0, m_0) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$ satisfying $F(u_0, m_0, \lambda_0) = 0$ and $m_0 \geq \bar{m}$ in $\mathbb{T}$. By Proposition 10.1 and by the implicit function theorem in Banach spaces (see, for example, [Dieudonné 1960]), we can find $\delta > 0$ such that, for every $\lambda^* \in]\lambda - \lambda_0, \lambda + \lambda_0[$, there exists $(u^*, m^*) \in C^{2,1/2}(\mathbb{T}) \times C^{2,1/2}(\mathbb{T})$ satisfying $F(u^*, m^*, \lambda^*) = 0$ and $m^* \geq \bar{m}$ in $\mathbb{T}$. Moreover, the implicit function theorem also guarantees that the map $\lambda^* \mapsto m^*$ is continuous. Hence, if δ is small enough, we have $m^* > 0$ in $\mathbb{T}$. Then, Proposition 9.2 gives $m^* > \bar{m}$ in $\mathbb{T}$. Therefore, $\lambda^* \in \Lambda$ and, consequently, Λ is open. □

Finally, we sum up the proof of our main result.

Proof of Theorem 2.1. Let $\epsilon > 0$ be such that $\epsilon < \min\{1, \epsilon_0, \bar{\epsilon}_0\}$, where ϵ_0 is given by Proposition 4.1 and where $\bar{\epsilon}_0$ is given by Proposition 9.2.

Propositions 11.1 and 11.2 give that Λ is a relatively open and closed set in $[0, 1]$. It is a nonempty set due to Propositions 4.1, 8.1, and 9.2. Hence, $\Lambda = [0, 1]$. Finally, we observe that Theorem 2.1 corresponds to the $\lambda = 1$ case. □

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