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Tiling annular regions with skew and T-tetrominoes

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(Communicated by Arthur T. Benjamin)

In this paper, we study tilings of annular regions in the integer lattice by skew and T-tetrominoes. We demonstrate the tileability of most annular regions by the given tile set, enumerate the tilings of width-2 annuli, and determine the tile counting group associated to this tile set and the family of all width-2 annuli.

1. Introduction

The first question in the mathematics of tilings is this: can a given region be tiled by a given set of tiles? By tiled, we mean that the region can be covered without gaps or overlaps by copies of the tiles in the tile set. If the answer is “yes”, the proof is often a single picture, which is satisfying to be sure. However, if the answer is “no”, the proof is often more interesting mathematically. Over the last 25 years, mathematical tools drawing on subjects in the undergraduate mathematics curriculum have been developed to answer in the negative the tileability question in many interesting cases (see, for instance, [Conway and Lagarias 1990; Korn 2004; Pak 2000; Thurston 1990]).

Other tiling questions have received attention as well, such as enumeration questions (how many different tilings are possible?) and connectivity questions (how must any two tilings of a region be related?). In 2000, an abelian group called the tile counting group was introduced in [Pak 2000] to encode information about such relationships, and this group has been found for several tile sets and families of regions (see, e.g., [Moore and Pak 2002; Muchnik and Pak 1999; Pak 2000; Korn 2004]).

We consider the tile set $\mathcal{T}$ in Figure 1 consisting of four T-tetrominoes (tiles t_1 through t_4) and four skew tetrominoes (tiles t_5 through t_8). We refer to the first four tiles as T-tiles, and the others as skew tiles. The regions we consider are annular regions in the integer lattice. For positive integers a, b, and n we define the annular region $A_n(a, b)$ to be the region in the integer lattice obtained from an

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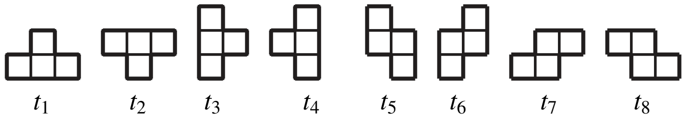

Figure 1. The tile set $\mathcal{T}$ consisting of T- and skew tetrominoes.

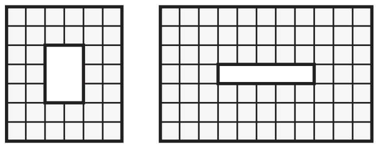

Figure 2. The annular regions $A_2(3, 2)$ and $A_3(1, 5)$.

$(a+2n) \times (b+2n)$ rectangle by removing the central $a \times b$ rectangle. We may think of $A_n(a, b)$ as an annulus of width-n units wrapped around an $a \times b$ rectangle. For instance, $A_2(3, 2)$ and $A_3(1, 5)$ are pictured in Figure 2. Certainly no annulus with width-$n = 1$ may be tiled by $\mathcal{T}$, so we assume $n \geq 2$. For an integer $n \geq 2$, let $\mathcal{A}_n$ represent all width-n annuli, and $\mathcal{A} = \bigcup_{n=2}^{\infty} \mathcal{A}_n$.

With respect to this tile set and family of regions, we prove three main results. We solve the tileability question for most annular regions with Theorem 9, and we enumerate tilings of width-2 annuli in Theorem 5. In Section 3 we address the question of how tilings of a given width-2 annulus must be related. As noted above, the tile counting group is an abelian group that gives information about such relations, and we determine the tile counting group associated to $\mathcal{T}$ and width-2 annuli in Theorem 8. We define the tile counting group in its generality and provide some illustration of it in Section 3 prior to the proof of Theorem 8.

The tile set $\mathcal{T}$ has been considered in other papers. For instance, [Lester 2012] solves the tileability question for rectangles with respect to $\mathcal{T}$, and [Korn 2004] looks at the tile counting group for a subset of $\mathcal{T}$ with respect to rectangles. Much is known about tile invariants and the tile counting group for tile sets over simply connected regions (see, for instance, [Conway and Lagarias 1990; Korn 2004; Moore and Pak 2002; Muchnik and Pak 1999; Pak 2000; Sheffield 2002; Thurston 1990]), but less is known for families of multiply connected regions, and this motivates our decision to study annular regions. The annular regions offer some control over the additional variation in possible tiling patterns that emerge beyond those found in rectangular regions. Finally, we note that our proofs are somewhat ad hoc, making use of the geometry of the annuli.

2. Tiling width-2 annular regions

Notice that the tile set $\mathcal{T}$ contains all rotations of each tile in the set, where by rotation we mean rotation by an integer multiple of $\pi/2$ radians, a rotation that

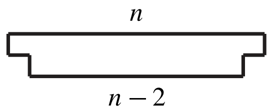

Figure 3. An extended-T of length n, denoted by X_n.

Figure 4. Tiling extended-Ts of odd length.

keeps the tile in the integer lattice. Any rotation of an annular region produces an annular region, and the set $\mathcal{T}$ tiles $A_n(a, b)$ if and only if it tiles $A_n(b, a)$. Further, note that horizontal or vertical reflection of a tiling of the annulus $A_n(a, b)$ will produce a distinct tiling of the same annulus $A_n(a, b)$.

It turns out that all width-2 annuli are tileable by $\mathcal{T}$, a fact we prove en route to enumerating the tilings of a given $A_2(a, b)$. To make this count it is first helpful to consider the extended-T.

Definition 1. Let $n \geq 3$. An *extended-T of length n*, denoted X_n, is any rotation of a region formed by removing the two corner squares from the bottom row of a $2 \times n$ rectangle.

We note that an extended-T has area $2n - 2$, so if n is even, the area of X_n is not divisible by 4, and hence X_n is not tileable by $\mathcal{T}$. However, an extended-T with odd length is more interesting with respect to the tile set $\mathcal{T}$.

Lemma 2. *Suppose $n \geq 3$ is odd. The following hold for the extended-T X_n:*

(i) *X_n is tileable by $\mathcal{T}$.*

(ii) *Any tiling of X_n by $\mathcal{T}$ uses an odd number of T-tiles.*

(iii) *The number of ways in which $\mathcal{T}$ can tile X_n is $2^{(n-3)/2}$.*

Proof. (i): For odd $n \geq 3$, the extended-T X_n as oriented in Figure 4 (left) may be tiled by placing the T-tile t_2 followed by $(n - 3)/2$ copies of the skew tile t_7.

(ii): We proceed by strong induction. X_3 can only be tiled by a single T-tile of the same shape as X_3. Now suppose any tiling of X_k uses an odd number of T-tiles for all odd $3 \leq k \leq n$. We show that any tiling of X_{n+2} also requires an odd number of T-tiles. In a given tiling of X_{n+2}, which we assume for the sake of argument is oriented as in Figure 4 (right), the left-most square may be covered with either the skew tile t_8 or the T-tile t_2. If it is covered by t_8 then the remaining region is an extended-T of length n, which requires an odd number of T-tiles by the inductive hypothesis. It follows that the tiling of X_{n+2} uses an odd number of T-tiles as well.

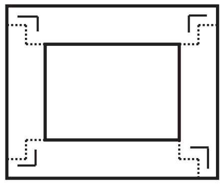

Figure 5. Any tiling of $A_2(a, b)$ can be decomposed into four extended-Ts, its T-structure.

Now suppose the left-most square of X_{n+2} is covered by t_2 instead. If no other T-tiles are present, then we're done. Otherwise, we proceed from left to right in the tiling until the next T-tile is found. Notice that as we proceed from left to right, if the next tile is not a T it must be the skew t_7. Notice further that the next T-tile placed will have to be the horizontal T-tile t_1. At this point, the shape of the untiled portion of the region is an extended-T of the form X_k for some odd $k < n$, as suggested in Figure 4 (right). Any tiling of the remaining portion requires an odd number of T-tiles by the inductive hypothesis, and it follows that the tiling of X_{n+2} itself uses an odd number of T-tiles.

(iii): This enumeration problem boils down to first picking the number of T-tiles used, which must be an odd number by (ii), and next picking the order in which skew and T-tiles are placed from left to right in the tiling of X_n. Once the number of T-tiles has been chosen, and the order of their placement has been chosen, the resulting tiling of X_n is uniquely determined. Thus the number of ways of tiling X_n is

$$\sum_{\substack{k=1\\ k \text{ is odd}}}^{m} \binom{m}{k} = 2^{m-1}.$$

Here, $m = (n-1)/2$, the number of total tiles needed to tile X_n. □

Lemma 3. *If α is a tiling of $A_2(a, b)$ by $\mathcal{T}$ then α may be viewed as the disjoint union of tilings of four extended-Ts.*

Proof. Note that in any attempt to tile the annular region $A_2(a, b)$, the corners must be covered by a tile. Both skew and T-tiles partially fill a corner in an L shape. Not all such configurations of these L-shapes can lead to valid tilings; however, it is necessary for any complete tiling of the region to have this structure. No matter how these L-shapes are arranged they allow us to uniquely decompose the region into four extended-Ts, as suggested in Figure 5. □

We call such a decomposition of an annulus into four extended-Ts a *T-structure* for that annulus.

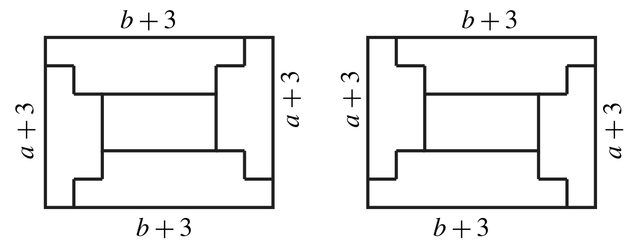

Figure 6. T-structures in the case a, b are even.

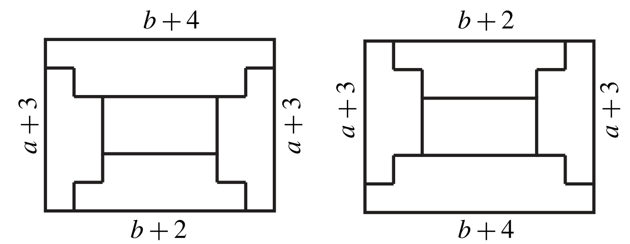

Figure 7. T-structures in the case a is even, b is odd.

Lemma 4. *There are exactly two T-structures for the annulus $A_2(a, b)$ in which each extended-T has odd length.*

Proof. We will consider three cases, according to the parities of a and b.

Case 1: Suppose a and b are both even. In this case, the use of an extended-T of length $a+1$ or $b+1$ would leave uncovered squares in the region, so we must use extended-Ts of length $a+3$ and $b+3$. There are only two possible ways to arrange extended-Ts of this length to cover the region; see Figure 6.

Case 2: Suppose a is even and b is odd (the case a odd and b even is handled by rotational symmetry). In this case, we must use two extended-Ts of length $a+3$ in our T-structure. This forces us to use one extended-T of length $b+4$ and one of length $b+2$ in order to cover the annulus and obtain the correct parity for the extended-Ts. Figure 7 depicts the two possible T-structures.

Case 3: Suppose a and b are odd. To obtain the correct parity for each extended-T, we must use lengths of $a+2$, $a+4$, $b+2$, or $b+4$. Note that if we pick our vertical extended-Ts such that one has length $a+2$ and the other has length $a+4$, then this would force each horizontal extended-T to have length $b+3$. However, this would be an extended-T of even length and therefore untileable. Thus the vertical (and therefore horizontal) extended-Ts must have the same length. This leads us to two possible configurations, as in Figure 8. □

We observe that because any width-2 annulus may be decomposed into odd length extended Ts, it follows that any width-2 annulus is tileable by $\mathcal{T}$. We can count

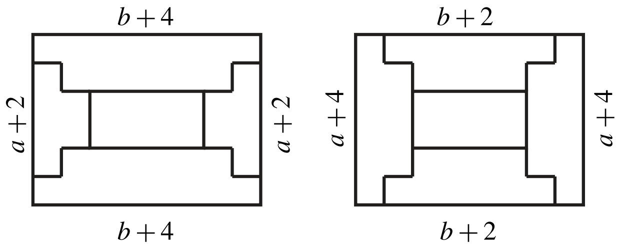

Figure 8. T-structures in the case a, b are odd.

the number of possible tilings of $A_2(a, b)$, thanks to the restrictions on possible T-structures.

Theorem 5. *The number of ways of tiling $A_2(a, b)$ by $\mathcal{T}$ is 2^{a+b+1}.*

Proof. We may consider three cases, based on the parities of a and b. We will show the calculations for the case where a and b are even. The other cases are analogous.

By Lemma 4, each horizontal extended-T in an allowable T-structure has length $b+3$. By Lemma 2(iii), the number of ways to tile a horizontal extended-T of this length is $2^{b/2}$. Similarly, the number of ways of tiling one of the vertical extended-Ts is $2^{a/2}$. Thus the total number of ways to tile $A_2(a, b)$ is

$$2 \cdot 2^{b/2} \cdot 2^{b/2} \cdot 2^{a/2} \cdot 2^{a/2} = 2^{a+b+1},$$

since we have two T-structures and two horizontal and two vertical extended-Ts. □

3. The tile counting group for width-2 annuli

We now turn to the question of how any two tilings of an annular region $A_2(a, b)$ by $\mathcal{T}$ must be related. Some mathematical machinery is necessary to address this question, and we take time here to develop this machinery for the reader's convenience.

Suppose a region Γ can be tiled by a tile set $\mathcal{T}$. It may be that Γ can be tiled in more than one way, and it is reasonable to ask how these tilings, or indeed any two tilings of Γ, must be related.

Suppose the tile set $\mathcal{T} = \{\tau_1, \tau_2, \dots, \tau_n\}$ consists of n tiles, and each tile in $\mathcal{T}$ has the same area (that is, it is comprised of the same number of squares). If α represents a particular tiling by $\mathcal{T}$ of a region Γ, we let $a_i(\alpha)$ equal the number of copies of tile τ_i that appears in the tiling. There are certain relations among the $a_i(\alpha)$ that hold for all tilings of a given region, and any such relation is called a *tile invariant* [Pak 2000].

One relation is an area invariant: since all tiles in $\mathcal{T}$ have the same area, for any tiling α of any region Γ, the linear combination $\sum_{i=1}^{n} a_i(\alpha)$ is constant. The value of the constant depends only on the region, not the particular tiling, and its value equals the total number of tiles needed to tile the region. A typical tile invariant has the form

$$\sum_{i=1}^{n} k_i a_i(\alpha) = c(\Gamma),$$

where $c(\Gamma)$ is a constant, each k_i is an integer, and the equality may be taken (mod m) for some n. As the value of the constant is independent of the particular tiling, and depends only on the region Γ, it is often cleaner to drop the α from the notation, in which case a tile invariant might be written as

$$2a_1 + a_2 - a_3 \equiv c(\Gamma) \pmod 4,$$

which in this case would mean that for any tiling of Γ, twice the number of copies of τ_1 plus the number of copies of τ_2 minus the number of copies of τ_3 is constant modulo 4.

The tile counting group $G(\mathcal{T}, \mathcal{R})$ associated to a tile set $\mathcal{T}$ and a collection of regions $\mathcal{R}$ was introduced in [Pak 2000] as a way to record in the form of a group the different tile invariants associated to a tile set and a family of regions. This group is defined as follows. To any tiling α of a region $\Gamma \in \mathcal{R}$ we associate an element w_α in the abelian group $\mathbb{Z}^n$ given by $w_\alpha = (a_1(\alpha), a_2(\alpha), \ldots, a_n(\alpha))$. We call w_α a *tile vector*. Now, if α and β are two tilings of the same region $\Gamma \in \mathcal{R}$, we call $w_\alpha - w_\beta$ a *difference vector*. In this setting we may view a tile invariant as a linear function from $\mathbb{Z}^n$ to $\mathbb{Z}$ (or possibly $\mathbb{Z}_m$) that maps each difference vector to 0. Let H denote the normal subgroup of $\mathbb{Z}^n$ generated by all possible difference vectors obtainable from our family of regions $\mathcal{R}$ and our tile set $\mathcal{T}$.

The *tile counting group* is then the quotient group

$$G(\mathcal{T}, \mathcal{R}) = \mathbb{Z}^n / H.$$

It seems that as we include more regions in our family $\mathcal{R}$, thus allowing for more difference vectors (from more regions that can be tiled in more than one way), the size of H will grow, and thus the size of the tile counting group will shrink. However, the tile counting group can stabilize rather quickly as you grow the number of regions in the family. Computations of tile counting groups can be difficult in general. Some computations can be found in [Hitchman 2015; Korn 2004; Moore and Pak 2002; Muchnik and Pak 1999; Pak 2000]. Before computing $G(\mathcal{T}, \mathcal{A}_2)$, we consider an example.

Suppose $\mathcal{T}_3$ consists of the ribbon tile trominoes as pictured in Figure 9, and $\mathcal{R}$ consists of a single region, the 3×3 square. This square has six tilings by $\mathcal{T}_3$, given in Figure 9. Tilings 3 and 1 give us difference vector $(1, 1, 1, 0) - (3, 0, 0, 0) = (-2, 1, 1, 0)$, and tilings 5 and 3 give us difference vector $(0, 1, 1, 1) - (1, 1, 1, 0) = (-1, 0, 0, 1)$. One can check that all other difference vectors from pairs of tilings

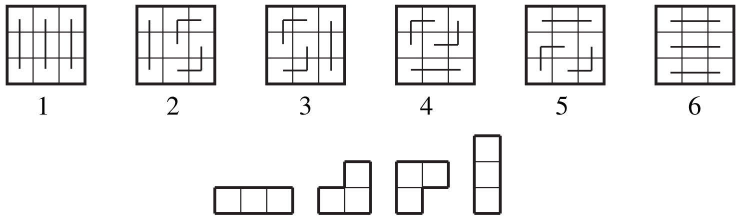

Figure 9. Tilings of a 3×3 square by ribbon tile trominoes.

taken from this collection are integer linear combinations of these two. Thus H is generated by $v_1 = (-2, 1, 1, 0)$ and $v_2 = (-1, 0, 0, 1)$.

We use two tile invariants to calculate the tile counting group. First, we have the area invariant: in any of the six tilings, we have that $a_1 + a_2 + a_3 + a_4$ is constant (equal to 3); second, we note that $a_2 - a_3$ is constant in all six tilings. Alternatively, these are tile invariants because for any difference vector $w_\alpha - w_\beta = (c_1, c_2, c_3, c_4)$, we know $c_1 + c_2 + c_3 + c_4 = 0$ and $c_2 - c_3 = 0$. The latter invariant is the Conway–Lagarias invariant [1990].

We claim the tile counting group $G(\mathcal{T}_3, \{[3 \times 3]\})$ is isomorphic to $\mathbb{Z}^2$. To see this, let $\phi : \mathbb{Z}^4 \to \mathbb{Z}^2$ be defined by $\phi(a, b, c, d) = (a+b+c+d, b-c)$. First note that ϕ is a group homomorphism and it is a surjection since we can map onto the generators of $\mathbb{Z}^2$: $\phi(1, 0, 0, 0) = (1, 0)$ and $\phi(-1, 1, 0, 0) = (0, 1)$. Second, we show $\ker\phi \subseteq H$: if $g = (a, b, c, d) \in \ker\phi$, then $b = c$, so $g = (a, b, b, d)$ where $a = -2b - d$ so

$$\begin{aligned} g &= (-2b - d, b, b, d) \\ &= (-2b, b, b, 0) + (-d, 0, 0, d) \\ &= b(-2, 1, 1, 0) + d(-1, 0, 0, 1) \\ &= bv_1 + dv_2. \end{aligned}$$

It follows that $g \in H$. Third, $H \subseteq \ker\phi$ since $\phi(v_1) = (0, 0)$ and $\phi(v_2) = (0, 0)$. Thus, by the first isomorphism theorem $G(\mathcal{T}_3, \mathcal{R}) = \mathbb{Z}^4/H \simeq \mathbb{Z}^2$.

Remarkably, the tile counting group here does not shrink if $\mathcal{R}$ grows to include all simply connected regions. That is, the two tile invariants used as the coordinate functions of ϕ persist as we expand $\mathcal{R}$. Conway and Lagarias [1990] introduced combinatorial group theoretic methods for deriving their tile invariant. Their inventive methods have motivated much research in tiling problems, including the development of the tile counting group itself. The Conway–Lagarias invariant is also revisited from a topological perspective in [Hitchman 2015], as is the tile counting group itself.

We now turn to the computation of $G(\mathcal{T}, \mathcal{A}_2)$. We begin again with extended-Ts. Figure 10 shows so-called “local moves” one may perform on a tiling of a horizontal

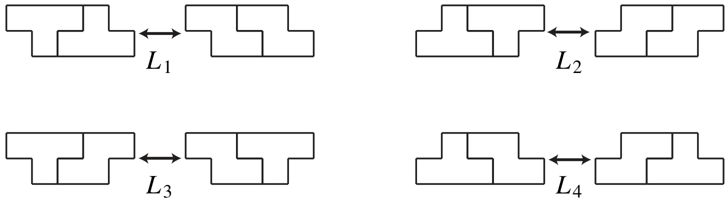

Figure 10. The set $\mathcal{L}$ of local moves on horizontal extended-T regions.

extended-T to produce a new tiling. That is, in each case, we may replace a local, two-tile configuration with a different configuration to generate a new tiling of the extended-T.

In general, a set of regions $\mathcal{R}$ has a *local move property* with respect to a tile set $\mathcal{T}$ if there exists a set of local moves, $\mathcal{L}$, such that every region $\Gamma \in \mathcal{R}$ has the feature that given any two tilings of Γ by $\mathcal{T}$, one can be made to match the other by a finite sequence of local moves.

Lemma 6. *The family of horizontal extended-Ts has a local move property with respect to the tile set $\mathcal{T}$, using the four local moves in the set $\mathcal{L} = \{L_1, L_2, L_3, L_4\}$ in Figure 10.*

Proof. Suppose $n \geq 3$ is odd, and α is a tiling of X_n, a horizontal extended-T oriented as in Figure 3. We show that α, by a finite number of local moves from $\mathcal{L}$, can be transformed into the tiling of X_n consisting of a single T-tile followed by $(n-3)/2$ copies of the skew t_7, as suggested in Figure 4 (left). It will then follow that any two tilings of X_n can be made to match by making local moves from $\mathcal{L}$. First, note that moves L_3 and L_4 tell us that T-and skew tiles "commute". By application of moves L_3 and L_4, we may convert the tiling α to one that consists of some number of T-tiles followed by some number of skew tiles. As observed in Lemma 2(ii), the number of T-tiles used in the tiling must be odd. Moves L_1 and L_2 may then be made to reduce the number of T-tiles two at a time until the number of T-tiles used is one. The $(n-3)/2$ skews now present in the tiling must all be copies of tile t_7 in order to have a valid tiling. $\square$

We note that an analogous result holds for vertical extended-T regions: for tilings by $\mathcal{T}$, the family of vertical extended-Ts has a local move property with respect to four local moves, which correspond to the moves in Figure 11.

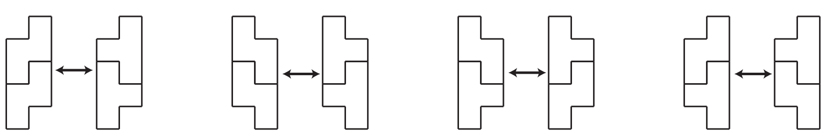

Figure 11. Local moves on vertical extended-T regions.

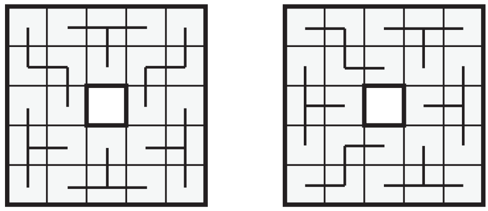

Figure 12. Two tilings of $A_2(1, 1)$ generate $v_5 = (0, 0, 0, 0, 1, 1, -1, -1)$.

One can show that the collection $\mathcal{A}_2$ does not have a local move property, essentially due to the fact that local moves cannot account for different T-structures among tilings.

The local moves on horizontal extended-Ts produce two distinct difference vectors in the subgroup H, the subgroup of $\mathbb{Z}^8$ generated by all difference vectors. Consider two tilings of $A_2(a, b)$ that differ by a single application of an L_1-move. We let v_1 denote the difference vector in this case, and note

$$v_1 = (1, 1, 0, 0, 0, 0, 0, -2).$$

Two tilings of $A_2(a, b)$ that differ by a single L_3-move or by a single L_4-move will generate the difference vector

$$v_2 = (0, 0, 0, 0, 0, 0, 1, -1).$$

Two tilings that differ by an L_2-move will produce the difference vector

$$(1, 1, 0, 0, 0, 0, -2, 0),$$

which equals $v_1 - 2v_2$, so it is a consequence of v_1 and v_2.

By a similar argument, two tilings that differ by some combination of the four vertical local moves in Figure 11 will have a difference vector that is a consequence of these two:

$$v_3 = (0, 0, 1, 1, -2, 0, 0, 0) \quad \text{and} \quad v_4 = (0, 0, 0, 0, 1, -1, 0, 0).$$

These four vectors do not quite generate H. For instance,

$$v_5 = (0, 0, 0, 0, 1, 1, -1, -1),$$

the difference vector determined by the two tilings of $A_2(1, 1)$ in Figure 12 is not a linear combination of the first four. This difference vector arises from tilings having distinct T-structures. It turns out that these five difference vectors generate H.

Lemma 7. *The subgroup H is generated by the five difference vectors*

$$\begin{aligned} v_1 &= (1, 1, 0, 0, 0, 0, 0, -2), \\ v_2 &= (0, 0, 0, 0, 0, 0, 1, -1), \\ v_3 &= (0, 0, 1, 1, -2, 0, 0, 0), \\ v_4 &= (0, 0, 0, 0, 1, -1, 0, 0), \\ v_5 &= (0, 0, 0, 0, 1, 1, -1, -1). \end{aligned}$$

Proof. Let H' be the normal subgroup of $\mathbb{Z}^8$ generated by the five difference vectors v_i. We show $H = H'$, where H is the normal subgroup generated by *all* difference vectors stated in the lemma. Clearly $H' \subseteq H$, and it remains to show that $H \subseteq H'$. Suppose α and β are two tilings of $A_2(a, b)$, and consider the difference vector $w_\alpha - w_\beta$. If these tilings have the same T-structures, then each tiling may be viewed as the tiling of a disjoint union of four extended-Ts of identical sizes, so one can be made to look like the other by a sequence of our local moves on extended-Ts. Thus, the difference vector $w_\alpha - w_\beta$ is in H'.

Now suppose α and β have distinct T-structures. Again, we consider cases based on the parities of a and b.

Case 1: Assume a and b are even, and α and β are tilings with distinct T-structures, given in Figure 6. Notice that in both T-structures, extended-Ts of the same orientation have the same size. This ensures that the difference vector $w_\alpha - w_\beta$ is a consequence of v_1, v_2, v_3, v_4, so it is in H'.

Case 2: Assume a is even and b is odd, and α and β represent tilings having distinct T-structures (the case a is odd and b is even is handled analogously). The tiling α can be transformed to a tiling α' with the same T-structure as α but having just a single T-tetromino in each extended-T as indicated in Figure 13. In particular, the single T-tetromino is either the bottommost or leftmost tile in the tiling of the extended-T, depending on its orientation. The rest of each extended-T is tiled by some number of copies of a single skew. Since α' was obtained from α by local moves within extended-Ts, $w_\alpha - w_{\alpha'} \in H'$. Similarly, β can be transformed to a tiling β' having the same T-structure but consisting of just one T-tetromino in each extended-T (at left or bottom), and $w_\beta - w_{\beta'} \in H'$. Furthermore, $w_{\alpha'} - w_{\beta'} = (0, 0, 0, 0, 0, 0, -1, 1) = -v_2$, which is also in H'. It follows that $w_\alpha - w_\beta$ is in H'.

Case 3: Assume a and b are odd, and α and β represent tilings having distinct T-structures. We may convert to tilings α' and β' as we did in Case 2, resulting in tilings with just one T-tile in each extended-T (see Figure 14). Then

$$w_{\alpha'} - w_{\beta'} = (0, 0, 0, 0, 1, 1, -1, -1) = v_5.$$

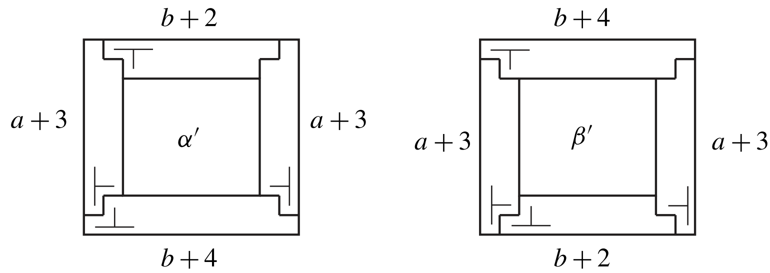

Figure 13. The difference vector for tilings with different T-structures in the (even) × (odd) case.

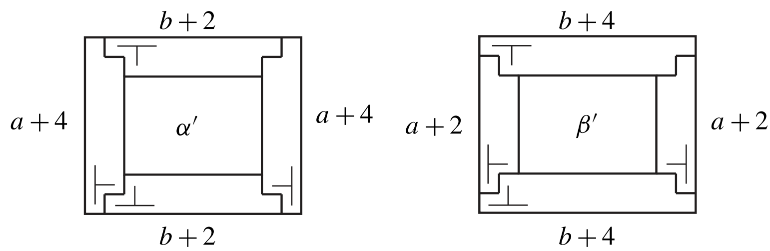

Figure 14. The difference vector for tilings with different T-structures in the (odd) × (odd) case.

Since

$$w_\alpha - w_\beta = (w_\alpha - w_{\alpha'}) + (w_{\alpha'} - w_{\beta'}) + (w_{\beta'} - w_\beta)$$

is the sum of three elements in H', it follows that $w_\alpha - w_\beta \in H'$.

Thus, $H' = H$. That is, the normal subgroup H is generated by the difference vectors v_1, v_2, v_3, v_4, and v_5. □

With a local move property on extended-T regions and the subgroup H in hand, we can write down various tile invariants. Of course, we have the area invariant: for any tiling α of a region Γ in $\mathcal{A}_2$ we have

$$\sum_{i=1}^{8} a_i = c(\Gamma).$$

Two other tile invariants arise by focusing on the horizontal and vertical T-tetrominoes present in any tiling of $A_2(a, b)$. In particular, we have

$$a_2 - a_1 = d(\Gamma),$$

$$a_4 - a_3 = e(\Gamma).$$

The first of these invariants says that the difference in the number of horizontal T-tiles used in any tiling of a given annulus is constant; the second says the same for the difference of vertical T-tiles used.

Also, the total number of horizontal tiles used in any tiling of a given $A_2(a, b)$ must be constant, modulo 2. That is,

$$a_1 + a_2 + a_7 + a_8 \equiv k(\Gamma) \pmod 2.$$

We now prove that all tile invariants are consequences of these four.

Theorem 8. *The tile counting group $G(\mathcal{T}, \mathcal{A}_2)$ is isomorphic to $\mathbb{Z}^3 \times \mathbb{Z}_2$.*

Proof. We use the tile invariants above to define $\phi : \mathbb{Z}^8 \to \mathbb{Z}^3 \times \mathbb{Z}_2$ as

$$\phi(c_1, c_2, \ldots, c_8) = \left(\sum_{i=1}^{8} c_i, c_2 - c_1, c_4 - c_3, [c_1 + c_2 + c_7 + c_8]_2 \right),$$

where $[k]_n$ represents the residue of k modulo n

The reader can check that ϕ is a homomorphism. To see that ϕ is a surjection, suppose $h \in \mathbb{Z}^3 \times \mathbb{Z}_2$. We want to show that there exists some $g \in \mathbb{Z}^8$ such that $\phi(g) = h$. Since $h \in \mathbb{Z}^3 \times \mathbb{Z}_2$, we know $h = (w, x, y, z)$, where $w, x, y \in \mathbb{Z}$ and $z \in \mathbb{Z}_2$.

Further suppose that $z = x + b$, where $b \in \mathbb{Z}$. In other words, $h = (w, x, y, [x+b]_2)$. Let $g = (0, x, 0, y, 0, w - (x + b + y), 0, b) \in \mathbb{Z}^8$. We have that $\phi(g) = h$ as desired, and it follows that ϕ is surjective.

Next we show $\ker \phi = H$. To see that $H \subseteq \ker \phi$, suppose $g \in H$. Observe that each $v_i \in \ker \phi$ for $i = 1, \ldots, 5$. It follows directly that $g \in \ker \phi$ since ϕ is a homomorphism.

Now suppose that $g = (c_1, c_2, \ldots, c_8) \in \ker \phi$. Then $c_2 - c_1 = 0$ and $c_4 - c_3 = 0$. Hence, $c_1 = c_2$ and $c_3 = c_4$. We also know that

$$0 \equiv c_1 + c_2 + c_7 + c_8 \equiv 2c_1 + c_7 + c_8 \equiv c_7 + c_8 \pmod 2.$$

That is, $c_7 + c_8$ is even, and $c_7 \equiv c_8 \pmod 2$. Furthermore, we know that

$$\sum_{i=1}^{8} c_i = 2c_1 + 2c_3 + c_5 + c_6 + c_7 + c_8 = 0,$$

from which it follows that $c_5 \equiv c_6 \pmod 2$ as well. In other words, if $g \in \ker \phi$, then g has the form $g = (c_1, c_1, c_3, c_3, c_5, c_6, c_7, c_8)$, where $c_5 \equiv c_6 \pmod 2$ and $c_7 \equiv c_8 \pmod 2$.

With g expressed in such a way, it is possible to express g as a linear combination of the difference vectors $v_1, \ldots, v_5$. Indeed, if we let $m = c_3 + \frac{1}{2}(c_5 + c_6)$ (which is an integer since $c_5 + c_6$ is even), then

$$g = c_1 v_1 + (c_7 + m) v_2 + c_3 v_3 + (m - c_6) v_4 + m v_5.$$

Thus, $g \in H$, as desired. $\square$

4. Extensions and remarks

In this section we consider width-n annuli for general n.

Theorem 9. *Let $A_n(a, b)$ be an annular region with $n \geq 2$. Then $\mathcal{T}$ tiles $A_n(a, b)$ if one of these conditions holds*:

(i) *n is even.*

(ii) *$n = 3$, with $a \equiv b \pmod 2$, and a and b are not both divisible by four.*

(iii) *$n \geq 5$ is odd and $a \equiv b \pmod 2$.*

Proof. (i): We have already seen that $\mathcal{T}$ tiles any width-2 annulus, and if $A_n(a, b)$ can be tiled by $\mathcal{T}$ then so can $A_{n+2}(a, b)$, since $A_{n+2}(a, b)$ may be viewed as the disjoint union of $A_n(a, b)$ and $A_2(a + 2n, b + 2n)$. It follows inductively that $A_n(a, b)$ can be tiled by $\mathcal{T}$ for any even $n \geq 2$, and we have tilings for all the regions for (i).

(ii): If $n \geq 3$ is odd, observe that $A_n(a, b)$ has area divisible by 4 if and only if $a \equiv b \pmod 2$. Figure 15 shows tilings of $A_3(1, 1)$, $A_3(1, 3)$, $A_3(3, 3)$, $A_3(2, 2)$, and $A_3(2, 4)$. One can then use reflections and rotations of the width-3 "expander" region in Figure 16 to effectively increase the a-dimension and the b-dimension of any of the tilings in Figure 15 by an integer multiple of 4 units. This can be achieved by inserting the expander regions as needed along the bold face seams given in the tilings of Figure 15. For instance, Figure 17 demonstrates how to extend the tiling of $A_3(1, 3)$ in Figure 15 to a tiling of the annulus $A_3(5, 7)$. Along each side of the annulus we may insert a width-3 expander region at the bold faced seam, indicated by an arrow to increase the length and width dimensions of the annulus by 4. Of course, we could have chosen to insert expander regions in just the vertical sides to obtain a tiling of $A_3(5, 3)$ or just the horizontal sides to obtain a tiling of $A_3(1, 7)$. In this way, we may generate tilings for all the regions for (ii).

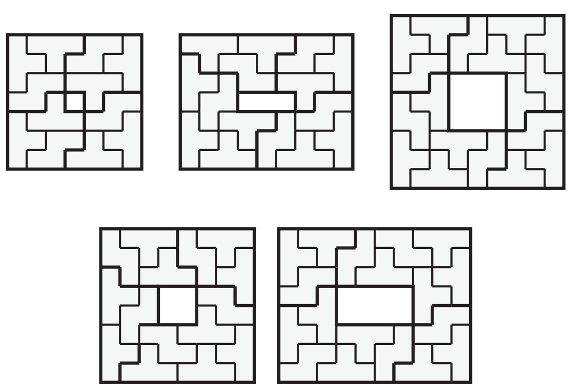

Figure 15. Tilings of $A_3(1, 1)$, $A_3(1, 3)$, $A_3(3, 3)$, $A_3(2, 2)$, and $A_3(2, 4)$.

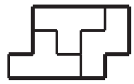

Figure 16. Tiling a width-3 expander region.

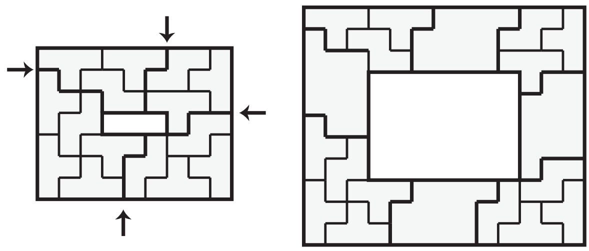

Figure 17. Extending a tiling of $A_3(1, 3)$ to a tiling of $A_3(5, 7)$.

(iii): Figure 18 shows a tiling of $A_5(4, 4)$ and width-5 expander regions. From these pieces, we may construct a tiling of $A_5(a, b)$ when a and b are both multiples of 4. Other regions $A_5(a, b)$ in which $a \equiv b \pmod 2$ may be viewed as the union of an $A_3(a, b)$ region and an $A_2(a+6, b+6)$ region, both of which may be tiled, so all $A_5(a, b)$ in which $a \equiv b \pmod 2$ may be tiled by $\mathcal{T}$. Finally, for odd $n \geq 7$, the region $A_n(a, b)$ may be viewed as the disjoint union of annuli $A_5(a, b)$ and $A_k(a+10, b+10)$, where $k \geq 2$ is even. If $a \equiv b \pmod 2$ then both these regions can be tiled by $\mathcal{T}$ so $A_n(a, b)$ can be tiled by $\mathcal{T}$ as well. Thus we have tilings of all the regions for (iii). □

We believe the converse to Theorem 9 holds as well. That is, if we suppose $A_n(a, b)$ has area divisible by 4, then we claim that $\mathcal{T}$ fails to tile $A_n(a, b)$ if and only if $n = 3$ and $a \equiv b \equiv 0 \pmod 4$. At the time of this writing, the proof that $A_3(4, 4)$ cannot be tiled by $\mathcal{T}$ is a brute force effort that involves tracking down all the scenarios for placing tiles, an argument comparable to the proof in [Lester

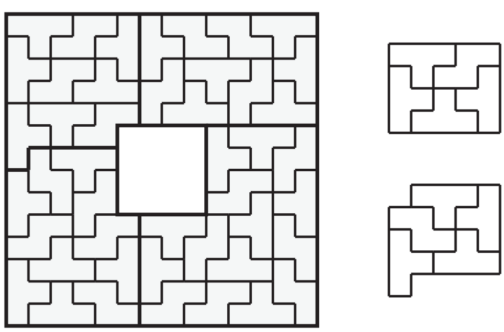

Figure 18. Tiling $A_5(4, 4)$, and width-5 expander regions.

2012] that $\mathcal{T}$ does not tile the 6×6 rectangle. That $\mathcal{T}$ tiles none of the regions $A_3(a, b)$ for $a, b \equiv 0 \pmod 4$ follows again by a brute force argument appealing to the geometry of the width-3 annuli. An elegant nonexistence proof using tile invariants remains elusive. For instance, there exists a signed tiling of $A_3(4, 4)$ by $\mathcal{T}$, which means no coloring argument exists to demonstrate the nontileability of $A_3(4, 4)$ by $\mathcal{T}$.

Enumeration and connectivity questions remain open for width-3 annuli. In fact, except for the area invariant, none of the tile invariants that hold for $\mathcal{A}_2$ persist when we pass to $\mathcal{A}_3$. Consider the invariant $a_2 - a_1$ over $\mathcal{A}_2$, and look again at the tiling of $A_3(1, 3)$ in Figure 15. In this tiling $a_2 - a_1 = 4 - 2 = 2$, but if we reflect this tiling about a horizontal axis we obtain a second tiling of $A_3(1, 3)$ in which $a_2 - a_1 = 2 - 4 = -2$. So, $a_2 - a_1$ is no longer a tile invariant if we pass to width-3 annuli. These two tilings of $A_3(1, 3)$ may be rotated by $\pi/2$ to show that $a_4 - a_3$ is no longer invariant over $\mathcal{A}_3$. Finally, consider the tile invariant that $a_1 + a_2 + a_7 + a_8$ is constant modulo 2, for tilings of regions in $\mathcal{A}_2$. The tiling of $A_3(2, 2)$ in Figure 15 uses eight horizontal tiles and seven vertical tiles. So, the given tiling gives $a_1 + a_2 + a_7 + a_8 \equiv 0 \pmod 2$. But if we rotate this tiling by $\pi/2$ we obtain a new tiling of the same annulus, $A_3(2, 2)$, that now has seven horizontal tiles so that $a_1 + a_2 + a_7 + a_8 \equiv 1 \pmod 2$ in this second tiling.

Finally, determining $G(\mathcal{T}, \mathcal{A}_n)$ for $n > 2$ remains open.

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