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Bases for the global Weyl modules of $\mathfrak{sl}_n$ of highest weight $m\omega_1$

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We utilize a theorem of B. Feigin and S. Loktev to give explicit bases for the global Weyl modules for the map algebras of the form $\mathfrak{sl}_n \otimes A$ of highest weight $m\omega_1$. These bases are given in terms of specific elements of the universal enveloping algebra, $\boldsymbol{U}(\mathfrak{sl}_n \otimes A)$, acting on the highest weight vector.

1. Introduction

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra. For the loop algebras $\mathfrak{g}\otimes\mathbb{C}[t, t^{-1}]$, the global Weyl modules were introduced by Chari and Pressley [2001]. Feigin and Loktev [2004] extended these global Weyl modules to the case where the Laurent polynomials above were replaced by the coordinate ring of a complex affine variety. Chari, Fourier and Khandai [Chari et al. 2010] then generalized this definition to the map algebras, $\mathfrak{g} \otimes A$, where A is a commutative, associative, complex unital algebra. Feigin and Loktev [2004] also gave an isomorphism which explicitly determines the structure of the global Weyl modules for the map algebras of $\mathfrak{sl}_n$ of highest weight $m\omega_1$.

The goal of this work is to use the structure isomorphism given by Feigin and Loktev to give nice bases for the global Weyl modules for the map algebras $\mathfrak{sl}_n \otimes A$ of $\mathfrak{sl}_n$ of highest weight $m\omega_1$. These bases will be given in terms of specific elements of $\boldsymbol{U}(\mathfrak{sl}_n \otimes A)$ acting on the highest weight vector. This was done in [Chamberlin 2011] in the case $n = 2$, but the case $n > 2$ has not previously appeared in the literature.

2. Preliminaries

2.1. ***The structure of $\mathfrak{sl}_n$.*** Recall that $\mathfrak{sl}_n$ is the Lie algebra of all complex traceless matrices. The Lie bracket is the commutator bracket given by $[A, B] = AB - BA$.

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Given any matrix $[b_{i,j}]$, define $\varepsilon_k([b_{i,j}]) := b_{k,k}$. For $i \in \{1, \dots, n-1\}$, define $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. Define

$$R^{\pm} := \{\pm(\alpha_i + \cdots + \alpha_j) \mid 1 \le i < j \le n-1\}$$

to be the positive and negative roots respectively, and define $R = R^+ \cup R^-$ to be the set of roots. Let $e_{i,j}$ be the $n \times n$ matrix with a one in the i-th row and j-th column and zeros in every other position. Define $h_i := h_{\alpha_i} = e_{i,i} - e_{i+1,i+1}$, for $i \in \{1, \dots, n-1\}$. Then $\mathfrak{h} := \operatorname{span}\{h_i \mid 1 \le i \le n\}$ is a Cartan subalgebra of $\mathfrak{sl}_n$. Given $\alpha = \alpha_i + \cdots + \alpha_j \in R^+$, define $x_\alpha := e_{i,j}$ and $x_{-\alpha} := e_{j,i}$. Then $\{h_i,\ x_{\pm\alpha} \mid 1 \le i \le n-1,\ \alpha \in R\}$ is a Chevalley basis for $\mathfrak{sl}_n$. Given $i \in \{1, \dots, n-1\}$, define $x_i := x_{\alpha_i} = e_{i,i+1}$ and $x_{-i} := x_{-\alpha_i} = e_{i+1,i}$. Note that, for all $1 \le i \le n-1$, $\operatorname{span}\{x_{-i}, h_i, x_i\} \cong \mathfrak{sl}_2$.

Define nilpotent subsuperalgebras $\mathfrak{n}^{\pm} := \operatorname{span}\{x_\alpha | \alpha \in R^{\pm}\}$ and note that $\mathfrak{sl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Define the set of fundamental weights $\{\omega_1, \dots, \omega_{n-1}\} \subset \mathfrak{h}^*$ by $\omega_i(h_j) = \delta_{i,j}$ for all $i, j \in \{1, \dots, n-1\}$. Define $P^+ := \operatorname{span}_{\mathbb{Z}_{\ge 0}}\{\omega_1, \dots, \omega_{n-1}\}$ to be the set of dominant integral weights.

2.2. ***Map algebras and Weyl modules.*** For the remainder of this work fix a commutative, associative, complex unital algebra A. Define the map algebra of $\mathfrak{sl}_n$ to be $\mathfrak{sl}_n \otimes A$ with Lie bracket given by linearly extending the bracket

$$[z \otimes a, w \otimes b] = [z, w] \otimes ab$$

for all $z, w \in \mathfrak{sl}_n$ and $a, b \in A$.

Define $\boldsymbol{U}(\mathfrak{sl}_n \otimes A)$ to be the universal enveloping algebra of $\mathfrak{sl}_n \otimes A$.

As in [Chari et al. 2010] we define the global Weyl model for $\mathfrak{sl}_n \otimes A$ of highest weight $\lambda \in P^+$ to be the module generated by a vector w_λ, called the highest weight vector, with relations

$$(x \otimes a)w_\lambda = 0, \quad (h \otimes 1)w_\lambda = \lambda(h)w_\lambda, \quad (x_{-i} \otimes 1)^{\lambda(h_i)+1} w_\lambda = 0,$$

for all $a \in A,\ x \in \mathfrak{n}^+,\ h \in \mathfrak{h}$, and $1 \le i \le n-1$.

2.3. ***Multisets.*** Given any set S, define a multiset of elements of S to be a multiplicity function $\chi : S \to \mathbb{Z}_{\ge 0}$. Define $\mathcal{F}(S) := \{\chi : S \to \mathbb{Z}_{\ge 0} : |\operatorname{supp} \chi| < \infty\}$. For $\chi \in \mathcal{F}(S)$, define $|\chi| := \sum_{s \in S} \chi(s)$. Notice that $\mathcal{F}(S)$ is an abelian monoid under function addition. For $\psi, \chi \in \mathcal{F}(S)$, we write $\psi \subseteq \chi$ if $\psi(s) \le \chi(s)$ for all $s \in S$. Define $\mathcal{F}(\chi)(S) := \{\psi \in \mathcal{F}(S) \mid \psi \subseteq \chi\}$. In the case $S = A$, the S will be omitted from the notation, so that $\mathcal{F} := \mathcal{F}(A)$ and $\mathcal{F}(\chi) = \mathcal{F}(\chi)(A)$.

If $\psi, \chi \in \mathcal{F}$ with $\psi \in \mathcal{F}(\chi)$ we define $\chi - \psi$ by standard function subtraction.

Also define $\pi : \mathcal{F} - \{0\} \to A$ by

$$\pi(\psi) := \prod_{a \in A} a^{\psi(a)},$$

and extend π to $\mathcal{F}$ by setting $\pi(0) = 1$. Define $\mathcal{M} : \mathcal{F} \to \mathbb{Z}$ by

$$\mathcal{M}(\psi) := \frac{|\psi|!}{\prod_{a \in A} \psi(a)!}.$$

$\mathcal{M}(\psi) \in \mathbb{Z}$ for all $\psi \in \mathcal{F}$ because if $\operatorname{supp} \psi = \{a_1, \dots, a_k\}$ then $\mathcal{M}(\psi)$ is the multinomial coefficient

$$\binom{|\psi|}{\psi(a_1), \dots, \psi(a_k)}.$$

For $s \in S$, define χ_s to be the characteristic function of the set $\{s\}$. Then, for all $\chi \in \mathcal{F}(S)$,

$$\chi = \sum_{s \in S} \chi(s) \chi_s.$$

2.4. ***The symmetric tensor space.*** Given any vector space W, there is an action of the symmetric group S_k on $W^{\otimes k} = W \otimes W \otimes \cdots \otimes W$ (k times) given by

$$\sigma(w_1 \otimes w_2 \otimes \cdots \otimes w_k) = w_{\sigma^{-1}(1)} \otimes w_{\sigma^{-1}(2)} \otimes \cdots \otimes w_{\sigma^{-1}(k)}, \quad \text{where } w_1, \dots, w_k \in W.$$

For any vector space W, define its k-th symmetric tensor space

$$S^k(W) = \operatorname{span}\left\{ \sum_{\sigma \in S_k} \sigma(w_1 \otimes \cdots \otimes w_k) \;\middle|\; w_1, \dots, w_k \in W \right\}.$$

Define $V \cong \mathbb{C}^n$ to be an $\mathfrak{sl}_n$-module via left matrix multiplication and write the basis as $v_1 := (1, 0, \dots, 0)$, and for $i \in \{1, \dots, n+m-1\}$, set $v_{i+1} := x_{-i} v_i$. Then $V \otimes A$ is an $\mathfrak{sl}_n \otimes A$-module under the action $(z \otimes a)(w \otimes b) = zw \otimes ab$.

Given $\varphi_1, \dots, \varphi_n \in \mathcal{F}$ with $k := \sum_{i=1}^n |\varphi_i|$, define

$$w(\varphi_1, \dots, \varphi_n) := \bigotimes_{a_1 \in \operatorname{supp} \varphi_1} (v_1 \otimes a_1)^{\otimes \varphi_1(a_1)} \otimes \cdots \otimes \bigotimes_{a_n \in \operatorname{supp} \varphi_n} (v_n \otimes a_n)^{\otimes \varphi_n(a_n)} \in (V \otimes A)^{\otimes k}$$

and

$$v(\varphi_1, \dots, \varphi_n) := \sum_{\sigma \in S_k} \sigma\big(w(\varphi_1, \dots, \varphi_n)\big) \in S^k(V \otimes A).$$

We will need the following theorem:

Theorem 2.4.1 [Feigin and Loktev 2004, Theorem 6]. *For all $m \in \mathbb{N}$, $W_A(m\omega_1) \cong S^m(V \otimes A)$ via the map given by*

$$w_{m\omega_1} \mapsto (v_1 \otimes 1)^{\otimes m}.$$

We will also need the following lemma:

Lemma 2.4.2. *Let $\boldsymbol{B}$ be a basis for A. Then the set*

$$\mathfrak{B} := \left\{ v(\varphi_1, \dots, \varphi_n) \;\middle|\; \varphi_1, \dots, \varphi_n \in \mathcal{F}(\boldsymbol{B}),\ \sum_{i=1}^{n} |\varphi_i| = m \right\}$$

is a basis for $S^m(V \otimes A)$.

Proof. $\mathfrak{B}$ spans $S^m(V \otimes A)$ because $\boldsymbol{B}$ spans A and $v_1, \dots, v_n$ span V. $\mathfrak{B}$ is linearly independent because the set

$$\left\{ (v_{j_1} \otimes b_1) \otimes \cdots \otimes (v_{j_m} \otimes b_m) \;\middle|\; j_1, \dots, j_m \in \{1, \dots, n\},\ b_1, \dots, b_m \in \boldsymbol{B} \right\}$$

is a basis for $(V \otimes A)^{\otimes m}$ and hence is linearly independent. □

Given $k \in \mathbb{N}$, define $\Delta^{k-1} : \boldsymbol{U}(\mathfrak{sl}_n \otimes A) \to \boldsymbol{U}(\mathfrak{sl}_n \otimes A)^{\otimes k}$ by extending the map $\mathfrak{sl}_n \otimes A \to \boldsymbol{U}(\mathfrak{sl}_n \otimes A)^{\otimes k}$ given by

$$\Delta^{k-1}(z \otimes a) = \sum_{j=0}^{k-1} 1^{\otimes j} \otimes (z \otimes a) \otimes 1^{\otimes k-1-j}.$$

Note that $\Delta^{k-1}(1) = 1^{\otimes k}$, not $k1^{\otimes k}$.

Since $V \otimes A$ is a $\boldsymbol{U}(\mathfrak{sl}_n \otimes A)$-module, $(V \otimes A)^{\otimes m}$ is a left $\boldsymbol{U}(\mathfrak{sl}_n \otimes A)$-module with u acting as $\Delta^{m-1}(u)$ followed by coordinatewise module actions. Moreover $S^m(V \otimes A)$ is a submodule under this action. Thus $S^m(V \otimes A)$ is a left $\boldsymbol{U}(\mathfrak{sl}_n \otimes A)$-module under this Δ^{m-1} action.

2.5. For all $i = 1, \dots, n-1$ and $\chi, \varphi \in \mathcal{F}$, recursively define $q_i(\varphi, \chi) \in \boldsymbol{U}(\mathfrak{sl}_n \otimes A)$ as follows:

$$\begin{aligned}
q_i(0, 0) &:= 1, \\
q_i(0, \chi) &:= -\frac{1}{|\chi|} \sum_{0 \neq \psi \in \mathcal{F}(\chi)} \mathcal{M}(\psi)\big(h_i \otimes \pi(\psi)\big) q_i(0, \chi - \psi), \\
q_i(\varphi, \chi) &:= -\frac{1}{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{d \in \operatorname{supp} \varphi} \mathcal{M}(\psi)\big(x_{-i} \otimes d\pi(\psi)\big) q_i(\varphi - \chi_d, \chi - \psi).
\end{aligned}$$

Given $\varphi_n, \dots, \varphi_n \in \mathcal{F}$, define

$$\begin{aligned}
q(\varphi_1, \dots, \varphi_n) := q_{n-1}(\varphi_n, \varphi_{n-1}) q_{n-2}\big((|\varphi_n| + |\varphi_{n-1}|)\chi_1, \varphi_{n-2}\big) \\
\times \cdots \times q_2\Big(\Big(\sum_{j=3}^{n} |\varphi_j|\Big)\chi_1, \varphi_2\Big) q_1\Big(\Big(\sum_{k=2}^{n} |\varphi_j|\Big)\chi_1, \varphi_1\Big).
\end{aligned}$$

Remark. The $q_i(0, \chi)$ coincide with the $p_i(\chi)$ defined in [Bagci and Chamberlin 2014].

3. Main theorem

The main result of this work is the theorem stated below.

Theorem 3.0.1. *Given a basis $\boldsymbol{B}$ for A and $m \in \mathbb{Z}_{>0}$, the set*

$$\left\{ q(\varphi_1, \dots, \varphi_n) w_{m\omega_1} \;\middle|\; \varphi_1, \dots, \varphi_n \in \mathcal{F}(\boldsymbol{B}),\ \sum_{i=1}^{n} |\varphi_i| = m \right\}$$

is a basis for $W_A(m\omega_1)$.

The proof of this theorem will be given after several lemmas and propositions.

3.1. *Necessary lemmas and propositions.*

Proposition 3.1.1. *For all $k \in \mathbb{N}$ $\Delta^k = (1^{\otimes k-1} \otimes \Delta^1) \circ \Delta^{k-1}$.*

Proof. The case $k = 1$ is trivial. For $k \geq 2$ and $u \in \boldsymbol{U}(\mathfrak{sl}_n \otimes A)$ we have

$$\begin{aligned}
(1^{\otimes k-1} \otimes \Delta^1)\big(\Delta^{k-1}(u)\big) &= (1^{\otimes k-1} \otimes \Delta^1)\left(\sum_{j=0}^{k-1} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-1-j}\right) \\
&= (1^{\otimes k-1} \otimes \Delta^1)\left(\sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-1-j} + 1^{\otimes k-1} \otimes u\right) \\
&= \sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-2-j} \otimes \Delta^1(1) + 1^{\otimes k-1} \otimes \Delta^1(u) \\
&= \sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-2-j} \otimes 1 \otimes 1 + 1^{\otimes k-1} \otimes (u \otimes 1 + 1 \otimes u) \\
&= \sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-j} + 1^{\otimes k-1} \otimes u \otimes 1 + 1^{\otimes k-1} \otimes 1 \otimes u \\
&= \sum_{j=0}^{k} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-j} = \Delta^k(u).
\end{aligned}$$

□

Given $\chi \in \mathcal{F}$ and $k \in \mathbb{N}$, define

$$\circ_k(\chi) = \left\{ \psi : \{1, \dots, k\} \to \mathcal{F}(\chi) \;\middle|\; \sum_{j=1}^{k} \psi(j) = \chi \right\}.$$

Lemma 3.1.2. *For all $i \in \{1, \dots, n-1\}$,*

$$\Delta^{k-1}\big(q_i(\varphi, \chi)\big) = \sum_{\substack{\psi \in \circ_k(\chi) \\ \phi \in \circ_k(\varphi)}} q_i\big(\phi(1), \psi(1)\big) \otimes \cdots \otimes q_i\big(\phi(k), \psi(k)\big)$$

Proof. This can be proven by induction on k. The case $k = 1$ is trivial. In the case $k = 2$ the lemma becomes

$$\Delta^1\big(q_i(\varphi, \chi)\big) = \sum_{\substack{\psi \in \mathcal{F}(\chi) \\ \phi \in \mathcal{F}(\varphi)}} q_i(\phi, \psi) \otimes q_i(\varphi - \phi, \chi - \psi).$$

This can be proven by induction on $|\varphi|$. For $k > 2$ use Proposition 3.1.1. The details in the $\mathfrak{sl}_2$ case can be found in [Chamberlin 2011]. This can be extended to the $\mathfrak{sl}_n$ case via the injection $\Omega_i : \mathfrak{sl}_2 \otimes A \to \mathfrak{sl}_n \otimes A$ given by

$$\Omega_i(x^- \otimes a) = x_{-i} \otimes a, \quad \Omega_i(h \otimes a) = h_i \otimes a, \quad \Omega_i(x^+ \otimes a) = x_i \otimes a,$$

for all $i \in \{1, \dots, n-1\}$ and $a \in A$. □

Lemma 3.1.3. *For all $\varphi, \chi \in \mathcal{F}$ with $|\varphi| + |\chi| > 1$ and all $i \in \{1, \dots, n-1\}$, $q_i(\varphi, \chi)(v_i \otimes 1) = 0$.*

Proof. Assume that $\varphi = 0$. This case will proceed by induction on $|\chi| > 1$. If $|\chi| = 2$ (so that $\chi = \{a, b\}$ for some $a, b \in A$) we have

$$\begin{aligned} q_i(0, \{a, b\})(v_i \otimes 1) &= [(h_i \otimes a) \otimes (h_i \otimes b) - (h_i \otimes ab)](v_i \otimes 1) \\ &= (h_i \otimes a) \otimes (v_i \otimes b) - (v_i \otimes ab) \\ &= (v_i \otimes ab) - (v_i \otimes ab) \\ &= 0. \end{aligned}$$

For the next case assume that $|\chi| > 2$ then

$$q_i(0, \chi)(v_i \otimes 1) = -\frac{1}{|\chi|} \sum_{\varnothing \neq \psi \in \mathcal{F}(\chi)} \mathcal{M}(\psi)\big(h_i \otimes \pi(\psi)\big) q_i(\chi - \psi)(v_i \otimes 1) = 0$$

by induction. Now assume that $|\varphi| = 1$ (or $\varphi = \chi_b$ for some $b \in A$). Then

$$\begin{aligned} &q_i(\chi_b, \chi)(v_i \otimes 1) \\ &\quad = -\sum_{\psi \in \mathcal{F}(\chi)} \mathcal{M}(\psi)\big(x_{-i} \otimes b\pi(\psi)\big) q_i(0, \chi - \psi)(v_i \otimes 1) \\ &\quad = -\mathcal{M}(\chi)\big(x_{-i} \otimes b\pi(\chi)\big)(v_i \otimes 1) \\ &\qquad\qquad - \sum_{a \in \operatorname{supp} \chi} \mathcal{M}(\chi - \chi_a)\big(x_{-i} \otimes b\pi(\chi - \chi_a)\big) q_i(0, \chi_a)(v_i \otimes 1) \\ &\quad = -\mathcal{M}(\chi)\big(v_{i+1} \otimes b\pi(\chi)\big) \\ &\qquad\qquad - \sum_{a \in \operatorname{supp} \chi} \mathcal{M}(\chi - \chi_a)\big(x_{-i} \otimes b\pi(\chi - \chi_a)\big)(-h_i \otimes a)(v_i \otimes 1) \\ &\quad = -\mathcal{M}(\chi)\big(v_{i+1} \otimes b\pi(\chi)\big) + \sum_{a \in \operatorname{supp} \chi} \mathcal{M}(\chi - \chi_a)\big(x_{-i} \otimes b\pi(\chi - \chi_a)\big)(v_i \otimes a) \end{aligned}$$

$$
\begin{aligned}
&= -\mathcal{M}(\chi)\big(v_{i+1} \otimes b\pi(\chi)\big) + \sum_{a \in \operatorname{supp} \chi} \mathcal{M}(\chi - \chi_a)\big(v_{i+1} \otimes b\pi(\chi)\big) \\
&= -\mathcal{M}(\chi)\big(v_{i+1} \otimes b\pi(\chi)\big) + \sum_{a \in \operatorname{supp} \chi} \frac{(|\chi| - 1)!}{\prod_{c \in \operatorname{supp}(\chi - \chi_a)} (\chi - \chi_a)(c)!} \big(v_{i+1} \otimes b\pi(\chi)\big) \\
&= -\mathcal{M}(\chi)\big(v_{i+1} \otimes b\pi(\chi)\big) + \sum_{a \in \operatorname{supp} \chi} \frac{(|\chi| - 1)!}{\prod_{\substack{c \in \operatorname{supp} \chi \\ c \neq a}} \chi(c)!\big(\chi(a) - 1\big)!} \big(v_{i+1} \otimes b\pi(\chi)\big) \\
&= -\mathcal{M}(\chi)\big(v_{i+1} \otimes b\pi(\chi)\big) + \sum_{a \in \operatorname{supp} \chi} \frac{\chi(a)}{|\chi|} \mathcal{M}(\chi)\big(v_{i+1} \otimes b\pi(\chi)\big) \\
&= -\mathcal{M}(\chi)\big(v_{i+1} \otimes b\pi(\chi)\big) + \mathcal{M}(\chi)\big(v_{i+1} \otimes b\pi(\chi)\big) \\
&= 0.
\end{aligned}
$$

Finally assume that $|\varphi| > 1$. Then

$$
\begin{aligned}
&q_i(\varphi, \chi)(v_i \otimes 1) \\
&\quad = -\frac{1}{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{d \in \operatorname{supp} \varphi} \mathcal{M}(\psi)\big(x_{-i} \otimes d\pi(\psi)\big) q_i(\varphi - \chi_d, \chi - \psi)(v_i \otimes 1) \\
&\quad = -\frac{1}{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{d \in \operatorname{supp} \varphi} \mathcal{M}(\psi) \\
&\qquad \Bigg(-\frac{1}{|\varphi| - 1} \sum_{\psi_1 \in \mathcal{F}(\chi - \psi)} \sum_{d_1 \in \operatorname{supp}(\varphi - \chi_d)} \mathcal{M}(\psi_1) \\
&\qquad\quad \big(x_{-i} \otimes d\pi(\psi)\big)\big(x_{-i} \otimes d_1\pi(\psi_1)\big) q_i(\varphi - \chi_d - \chi_{d_1}, \chi - \psi - \psi_1)\Bigg)(v_i \otimes 1) \\
&\quad = 0,
\end{aligned}
$$

because at least two x_{-i} terms act on a single v_i as 0. □

Lemma 3.1.4. *For all $i \in \{1, \dots, n-1\}$ and $\varphi, \chi \in \mathcal{F}$ with $|\varphi| + |\chi| = k$ we have*

$$q_i(\varphi, \chi)(v_i \otimes 1)^{\otimes k} = (-1)^k v(0, \dots, 0, \chi, \varphi, 0, \dots, 0),$$

where χ is in the i-th position and φ in the $(i+1)$-st.

Proof. We have $q_i(\varphi, \chi)(v_i \otimes 1)^{\otimes k} i = \Delta^{k-1}\big(q_i(\varphi, \chi)\big)(v_i \otimes 1)^{\otimes k}$. By Lemma 3.1.2, this equals

$$\Bigg(\sum_{\substack{\psi \in \circ_k(\chi) \\ \phi \in \circ_k(\varphi)}} q_i\big(\phi(1), \psi(1)\big) \otimes \cdots \otimes q_i\big(\phi(k), \psi(k)\big) \Bigg)(v_i \otimes 1)^{\otimes k},$$

which can be rewritten as

$$\sum_{\substack{\psi\in\circ_k(\chi)\\ \phi\in\circ_k(\varphi)}} \big(q_i(\phi(1),\psi(1))(v_i\otimes 1)\big)\otimes\cdots\otimes\big(q_i(\phi(k),\psi(k))(v_i\otimes 1)\big).$$

By Lemma 3.1.3 we see that the only potentially nonzero terms in the sum are those for which $|\phi(j)|+|\psi(j)|\le 1$ for all $j\in\{1,\dots,k\}$. Since $|\varphi|+|\chi|=k$ if we have $|\psi(j)|+|\phi(j)|=0$ for some $j\in\{1,\dots,n-1\}$, then there is a $r\in\{1,\dots,n-1\}$ such that $|\psi(r)|+|\phi(r)|>1$. So the only potentially nonzero terms in the sum are those for which $|\phi(j)|+|\psi(j)|=1$ for all $j\in\{1,\dots,k\}$. Suppose that $\phi(j)=\chi_a$ and $\psi(j)=0$ for some $j\in\{1,\dots k\}$ and some $a\in A$. Then

$$q_i(\chi_a,0)(v_i\otimes 1)=-(x_{-i}\otimes a)(v_i\otimes 1)=-(v_{i+1}\otimes a).$$

Suppose that $\phi(j)=0$ and $\psi(j)=\chi_a$ for some $j\in\{1,\dots k\}$ and some $a\in A$. Then

$$q_i(0,\chi_a)(v_i\otimes 1)=-(h_i\otimes a)(v_i\otimes 1)=-(v_i\otimes a).$$

So $-(v_{i+1}\otimes a)$ and $-(v_i\otimes a)$ are the only possibilities for factors in the tensor product above. Since we are summing over all possible submultisets of φ and χ, we have the result. □

Lemma 3.1.5. *For all $m\in\mathbb{N}$ and all $\varphi_1,\dots,\varphi_n\in\mathcal{F}$ with $\sum_{i=1}^n|\varphi_i|=m$,*

$$q(\varphi_1,\dots,\varphi_n)(v_1\otimes 1)^{\otimes m}=(-1)^{\sum_{j=1}^n j|\varphi_j|}v(\varphi_1,\dots,\varphi_n).$$

Proof. Since for all $j\in\{1,\dots,n-1\}$ and $k\in\{1,\dots,n\}$,

$$x_{-j}v_k=\delta_{j,k}v_{j+1},\qquad h_jv_k=\delta_{j,k}v_j-\delta_{j+1,k}v_{j+1},$$

so by Lemma 3.1.4 we have

$$\begin{aligned}
&q(\varphi_1,\dots,\varphi_n)(v_1\otimes 1)^{\otimes m}\\
&\quad=q_{n-1}(\varphi_n,\varphi_{n-1})q_{n-2}\big((|\varphi_n|+|\varphi_{n-1}|)\chi_1,\varphi_{n-2}\big)\\
&\qquad\qquad\times\cdots\times q_1\bigg(\Big(\sum_{j=2}^n|\varphi_j|\Big)\chi_1,\varphi_1\bigg)(v_1\otimes 1)^{\otimes m}\\
&\quad=(-1)^m q_{n-1}(\varphi_n,\varphi_{n-1})\dots q_2\bigg(\Big(\sum_{j=3}^n|\varphi_j|\Big)\chi_1,\varphi_2\bigg)v\bigg(\varphi_1,\Big(\sum_{j=2}^n|\varphi_j|\Big)\chi_1,0,\dots,0\bigg)\\
&\quad=(-1)^{|\varphi_1|+2\sum_{j=2}^n|\varphi_j|}q_{n-1}(\varphi_n,\varphi_{n-1})\\
&\qquad\qquad\times\cdots\times q_3\bigg(\Big(\sum_{j=4}^n|\varphi_j|\Big)\chi_1,\varphi_3\bigg)vi\bigg(\varphi_1,\varphi_2,\Big(\sum_{j=3}^n|\varphi_j|\Big)\chi_1,0,\dots,0\bigg)
\end{aligned}$$

$$= (-1)^{\sum_{j=1}^{n-2} j|\varphi_j|} q_{n-1}(\varphi_n, \varphi_{n-1}) v\big(\varphi_1, \dots, \varphi_{n-2}, (|\varphi_{n-1}| + |\varphi_n|)\chi_1, 0\big)$$
$$= (-1)^{\sum_{j=1}^{n} j|\varphi_j|} v(\varphi_1, \dots, \varphi_n). \qquad \square$$

3.2. *The proof of Theorem 3.0.1.*

Proof. By Lemmas 3.1.5 and 2.4.2

$$\left\{ q(\varphi_1, \dots, \varphi_n)(v_1 \otimes 1)^{\otimes m} \;\middle|\; \varphi_1, \dots, \varphi_n \in \mathcal{F}(\boldsymbol{B}),\ \sum_{i=1}^{n} |\varphi_i| = m \right\}$$

is a basis for $S^m(V \otimes A)$. Therefore by Theorem 2.4.1

$$\left\{ q(\varphi_1, \dots, \varphi_n) w_{m\omega_1} \;\middle|\; \varphi_1, \dots, \varphi_n \in \mathcal{F}(\boldsymbol{B}),\ \sum_{i=1}^{n} |\varphi_i| = m \right\}$$

is a basis for $W_A(m\omega_1)$. $\square$

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