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We utilize a theorem of B. Feigin and S. Loktev to give explicit bases for the global Weyl modules for the map algebras of the form $\mathfrak{sl}_n \otimes A$ of highest weight $m\omega_1$. These bases are given in terms of specific elements of the universal enveloping algebra, $U(\mathfrak{sl}_n \otimes A)$, acting on the highest weight vector.

1. Introduction

Let \mathfrak{g} be a simple finite-dimensional complex Lie algebra. For the loop algebras $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, the global Weyl modules were introduced by Chari and Pressley [2001]. Feigin and Loktev [2004] extended these global Weyl modules to the case where the Laurent polynomials above were replaced by the coordinate ring of a complex affine variety. Chari, Fourier and Khandai [Chari et al. 2010] then generalized this definition to the map algebras, $\mathfrak{g} \otimes A$, where A is a commutative, associative, complex unital algebra. Feigin and Loktev [2004] also gave an isomorphism which explicitly determines the structure of the global Weyl modules for the map algebras of \mathfrak{sl}_n of highest weight $m\omega_1$.

The goal of this work is to use the structure isomorphism given by Feigin and Loktev to give nice bases for the global Weyl modules for the map algebras $\mathfrak{sl}_n \otimes A$ of \mathfrak{sl}_n of highest weight $m\omega_1$. These bases will be given in terms of specific elements of $U(\mathfrak{sl}_n \otimes A)$ acting on the highest weight vector. This was done in [Chamberlin 2011] in the case $n = 2$, but the case $n > 2$ has not previously appeared in the literature.

2. Preliminaries

2.1. The structure of \mathfrak{sl}_n . Recall that \mathfrak{sl}_n is the Lie algebra of all complex traceless matrices. The Lie bracket is the commutator bracket given by $[A, B] = AB - BA$.

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Given any matrix $[b_{i,j}]$, define $\varepsilon_k([b_{i,j}]) := b_{k,k}$. For $i \in \{1, \dots, n-1\}$, define $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. Define

$$R^\pm := \{\pm(\alpha_i + \dots + \alpha_j) \mid 1 \leq i < j \leq n-1\}$$

to be the positive and negative roots respectively, and define $R = R^+ \cup R^-$ to be the set of roots. Let $e_{i,j}$ be the $n \times n$ matrix with a one in the i -th row and j -th column and zeros in every other position. Define $h_i := h_{\alpha_i} = e_{i,i} - e_{i+1,i+1}$, for $i \in \{1, \dots, n-1\}$. Then $\mathfrak{h} := \text{span}\{h_i \mid 1 \leq i \leq n\}$ is a Cartan subalgebra of \mathfrak{sl}_n . Given $\alpha = \alpha_i + \dots + \alpha_j \in R^+$, define $x_\alpha := e_{i,j}$ and $x_{-\alpha} := e_{j,i}$. Then $\{h_i, x_{\pm\alpha} \mid 1 \leq i \leq n-1, \alpha \in R\}$ is a Chevalley basis for \mathfrak{sl}_n . Given $i \in \{1, \dots, n-1\}$, define $x_i := x_{\alpha_i} = e_{i,i+1}$ and $x_{-i} := x_{-\alpha_i} = e_{i+1,i}$. Note that, for all $1 \leq i \leq n-1$, $\text{span}\{x_{-i}, h_i, x_i\} \cong \mathfrak{sl}_2$.

Define nilpotent subsuperalgebras $\mathfrak{n}^\pm := \text{span}\{x_\alpha \mid \alpha \in R^\pm\}$ and note that $\mathfrak{sl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Define the set of fundamental weights $\{\omega_1, \dots, \omega_{n-1}\} \subset \mathfrak{h}^*$ by $\omega_i(h_j) = \delta_{i,j}$ for all $i, j \in \{1, \dots, n-1\}$. Define $P^+ := \text{span}_{\mathbb{Z}_{\geq 0}}\{\omega_1, \dots, \omega_{n-1}\}$ to be the set of dominant integral weights.

2.2. Map algebras and Weyl modules. For the remainder of this work fix a commutative, associative, complex unital algebra A . Define the map algebra of \mathfrak{sl}_n to be $\mathfrak{sl}_n \otimes A$ with Lie bracket given by linearly extending the bracket

$$[z \otimes a, w \otimes b] = [z, w] \otimes ab$$

for all $z, w \in \mathfrak{sl}_n$ and $a, b \in A$.

Define $U(\mathfrak{sl}_n \otimes A)$ to be the universal enveloping algebra of $\mathfrak{sl}_n \otimes A$.

As in [Chari et al. 2010] we define the global Weyl model for $\mathfrak{sl}_n \otimes A$ of highest weight $\lambda \in P^+$ to be the module generated by a vector w_λ , called the highest weight vector, with relations

$$(x \otimes a)w_\lambda = 0, \quad (h \otimes 1)w_\lambda = \lambda(h)w_\lambda, \quad (x_{-i} \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0,$$

for all $a \in A$, $x \in \mathfrak{n}^+$, $h \in \mathfrak{h}$, and $1 \leq i \leq n-1$.

2.3. Multisets. Given any set S , define a multiset of elements of S to be a multiplicity function $\chi : S \rightarrow \mathbb{Z}_{\geq 0}$. Define $\mathcal{F}(S) := \{\chi : S \rightarrow \mathbb{Z}_{\geq 0} \mid \text{supp } \chi < \infty\}$. For $\chi \in \mathcal{F}(S)$, define $|\chi| := \sum_{s \in S} \chi(s)$. Notice that $\mathcal{F}(S)$ is an abelian monoid under function addition. For $\psi, \chi \in \mathcal{F}(S)$, we write $\psi \subseteq \chi$ if $\psi(s) \leq \chi(s)$ for all $s \in S$. Define $\mathcal{F}(\chi)(S) := \{\psi \in \mathcal{F}(S) \mid \psi \subseteq \chi\}$. In the case $S = A$, the S will be omitted from the notation, so that $\mathcal{F} := \mathcal{F}(A)$ and $\mathcal{F}(\chi) = \mathcal{F}(\chi)(A)$.

If $\psi, \chi \in \mathcal{F}$ with $\psi \in \mathcal{F}(\chi)$ we define $\chi - \psi$ by standard function subtraction.

Also define $\pi : \mathcal{F} - \{0\} \rightarrow A$ by

$$\pi(\psi) := \prod_{a \in A} a^{\psi(a)},$$

and extend π to \mathcal{F} by setting $\pi(0) = 1$. Define $\mathcal{M} : \mathcal{F} \rightarrow \mathbb{Z}$ by

$$\mathcal{M}(\psi) := \frac{|\psi|!}{\prod_{a \in A} \psi(a)!}.$$

$\mathcal{M}(\psi) \in \mathbb{Z}$ for all $\psi \in \mathcal{F}$ because if $\text{supp } \psi = \{a_1, \dots, a_k\}$ then $\mathcal{M}(\psi)$ is the multinomial coefficient

$$\binom{|\psi|}{\psi(a_1), \dots, \psi(a_k)}.$$

For $s \in S$, define χ_s to be the characteristic function of the set $\{s\}$. Then, for all $\chi \in \mathcal{F}(S)$,

$$\chi = \sum_{s \in S} \chi(s) \chi_s.$$

2.4. The symmetric tensor space. Given any vector space W , there is an action of the symmetric group S_k on $W^{\otimes k} = W \otimes W \otimes \dots \otimes W$ (k times) given by

$$\sigma(w_1 \otimes w_2 \otimes \dots \otimes w_k) = w_{\sigma^{-1}(1)} \otimes w_{\sigma^{-1}(2)} \otimes \dots \otimes w_{\sigma^{-1}(k)}, \quad \text{where } w_1, \dots, w_k \in W.$$

For any vector space W , define its k -th symmetric tensor space

$$S^k(W) = \text{span} \left\{ \sum_{\sigma \in S_k} \sigma(w_1 \otimes \dots \otimes w_k) \mid w_1, \dots, w_k \in W \right\}.$$

Define $V \cong \mathbb{C}^n$ to be an \mathfrak{sl}_n -module via left matrix multiplication and write the basis as $v_1 := (1, 0, \dots, 0)$, and for $i \in \{1, \dots, n+m-1\}$, set $v_{i+1} := x_{-i} v_i$. Then $V \otimes A$ is an $\mathfrak{sl}_n \otimes A$ -module under the action $(z \otimes a)(w \otimes b) = zw \otimes ab$.

Given $\varphi_1, \dots, \varphi_n \in \mathcal{F}$ with $k := \sum_{i=1}^n |\varphi_i|$, define

$$w(\varphi_1, \dots, \varphi_n) := \bigotimes_{a_1 \in \text{supp } \varphi_1} (v_1 \otimes a_1)^{\otimes \varphi_1(a_1)} \otimes \dots \otimes \bigotimes_{a_n \in \text{supp } \varphi_n} (v_n \otimes a_n)^{\otimes \varphi_n(a_n)} \in (V \otimes A)^{\otimes k}$$

and

$$v(\varphi_1, \dots, \varphi_n) := \sum_{\sigma \in S_k} \sigma(w(\varphi_1, \dots, \varphi_n)) \in S^k(V \otimes A).$$

We will need the following theorem:

Theorem 2.4.1 [Feigin and Loktev 2004, Theorem 6]. *For all $m \in \mathbb{N}$, $W_A(m\omega_1) \cong S^m(V \otimes A)$ via the map given by*

$$w_{m\omega_1} \mapsto (v_1 \otimes 1)^{\otimes m}.$$

We will also need the following lemma:

Lemma 2.4.2. *Let \mathbf{B} be a basis for A . Then the set*

$$\mathfrak{B} := \left\{ v(\varphi_1, \dots, \varphi_n) \mid \varphi_1, \dots, \varphi_n \in \mathcal{F}(\mathbf{B}), \sum_{i=1}^n |\varphi_i| = m \right\}$$

is a basis for $S^m(V \otimes A)$.

Proof. \mathfrak{B} spans $S^m(V \otimes A)$ because \mathbf{B} spans A and v_1, \dots, v_n span V . \mathfrak{B} is linearly independent because the set

$$\{(v_{j_1} \otimes b_1) \otimes \dots \otimes (v_{j_m} \otimes b_m) \mid j_1, \dots, j_m \in \{1, \dots, n\}, b_1, \dots, b_m \in \mathbf{B}\}$$

is a basis for $(V \otimes A)^{\otimes m}$ and hence is linearly independent. □

Given $k \in \mathbb{N}$, define $\Delta^{k-1} : U(\mathfrak{sl}_n \otimes A) \rightarrow U(\mathfrak{sl}_n \otimes A)^{\otimes k}$ by extending the map $\mathfrak{sl}_n \otimes A \rightarrow U(\mathfrak{sl}_n \otimes A)^{\otimes k}$ given by

$$\Delta^{k-1}(z \otimes a) = \sum_{j=0}^{k-1} 1^{\otimes j} \otimes (z \otimes a) \otimes 1^{\otimes k-1-j}.$$

Note that $\Delta^{k-1}(1) = 1^{\otimes k}$, not $k1^{\otimes k}$.

Since $V \otimes A$ is a $U(\mathfrak{sl}_n \otimes A)$ -module, $(V \otimes A)^{\otimes m}$ is a left $U(\mathfrak{sl}_n \otimes A)$ -module with u acting as $\Delta^{m-1}(u)$ followed by coordinatewise module actions. Moreover $S^m(V \otimes A)$ is a submodule under this action. Thus $S^m(V \otimes A)$ is a left $U(\mathfrak{sl}_n \otimes A)$ -module under this Δ^{m-1} action.

2.5. For all $i = 1, \dots, n-1$ and $\chi, \varphi \in \mathcal{F}$, recursively define $q_i(\varphi, \chi) \in U(\mathfrak{sl}_n \otimes A)$ as follows:

$$\begin{aligned} q_i(0, 0) &:= 1, \\ q_i(0, \chi) &:= -\frac{1}{|\chi|} \sum_{0 \neq \psi \in \mathcal{F}(\chi)} \mathcal{M}(\psi)(h_i \otimes \pi(\psi))q_i(0, \chi - \psi), \\ q_i(\varphi, \chi) &:= -\frac{1}{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{d \in \text{supp } \varphi} \mathcal{M}(\psi)(x_{-i} \otimes d\pi(\psi))q_i(\varphi - \chi_d, \chi - \psi). \end{aligned}$$

Given $\varphi_n, \dots, \varphi_1 \in \mathcal{F}$, define

$$\begin{aligned} q(\varphi_1, \dots, \varphi_n) &:= q_{n-1}(\varphi_n, \varphi_{n-1})q_{n-2}((|\varphi_n| + |\varphi_{n-1}|)\chi_1, \varphi_{n-2}) \\ &\quad \times \dots \times q_2\left(\left(\sum_{j=3}^n |\varphi_j|\right)\chi_1, \varphi_2\right)q_1\left(\left(\sum_{k=2}^n |\varphi_k|\right)\chi_1, \varphi_1\right). \end{aligned}$$

Remark. The $q_i(0, \chi)$ coincide with the $p_i(\chi)$ defined in [Bagci and Chamberlin 2014].

3. Main theorem

The main result of this work is the theorem stated below.

Theorem 3.0.1. *Given a basis \mathbf{B} for A and $m \in \mathbb{Z}_{>0}$, the set*

$$\left\{ q(\varphi_1, \dots, \varphi_n)w_{m\omega_1} \mid \varphi_1, \dots, \varphi_n \in \mathcal{F}(\mathbf{B}), \sum_{i=1}^n |\varphi_i| = m \right\}$$

is a basis for $W_A(m\omega_1)$.

The proof of this theorem will be given after several lemmas and propositions.

3.1. Necessary lemmas and propositions.

Proposition 3.1.1. *For all $k \in \mathbb{N}$ $\Delta^k = (1^{\otimes k-1} \otimes \Delta^1) \circ \Delta^{k-1}$.*

Proof. The case $k = 1$ is trivial. For $k \geq 2$ and $u \in U(\mathfrak{sl}_n \otimes A)$ we have

$$\begin{aligned} (1^{\otimes k-1} \otimes \Delta^1)(\Delta^{k-1}(u)) &= (1^{\otimes k-1} \otimes \Delta^1) \left(\sum_{j=0}^{k-1} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-1-j} \right) \\ &= (1^{\otimes k-1} \otimes \Delta^1) \left(\sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-1-j} + 1^{\otimes k-1} \otimes u \right) \\ &= \sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-2-j} \otimes \Delta^1(1) + 1^{\otimes k-1} \otimes \Delta^1(u) \\ &= \sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-2-j} \otimes 1 \otimes 1 + 1^{\otimes k-1} \otimes (u \otimes 1 + 1 \otimes u) \\ &= \sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-j} + 1^{\otimes k-1} \otimes u \otimes 1 + 1^{\otimes k-1} \otimes 1 \otimes u \\ &= \sum_{j=0}^k 1^{\otimes j} \otimes u \otimes 1^{\otimes k-j} = \Delta^k(u). \end{aligned}$$

□

Given $\chi \in \mathcal{F}$ and $k \in \mathbb{N}$, define

$$\circ_k(\chi) = \left\{ \psi : \{1, \dots, k\} \rightarrow \mathcal{F}(\chi) \mid \sum_{j=1}^k \psi(j) = \chi \right\}.$$

Lemma 3.1.2. *For all $i \in \{1, \dots, n-1\}$,*

$$\Delta^{k-1}(q_i(\varphi, \chi)) = \sum_{\substack{\psi \in \circ_k(\chi) \\ \phi \in \circ_k(\varphi)}} q_i(\phi(1), \psi(1)) \otimes \dots \otimes q_i(\phi(k), \psi(k))$$

Proof. This can be proven by induction on k . The case $k = 1$ is trivial. In the case $k = 2$ the lemma becomes

$$\Delta^1(q_i(\varphi, \chi)) = \sum_{\substack{\psi \in \mathcal{F}(\chi) \\ \phi \in \mathcal{F}(\varphi)}} q_i(\phi, \psi) \otimes q_i(\varphi - \phi, \chi - \psi).$$

This can be proven by induction on $|\varphi|$. For $k > 2$ use Proposition 3.1.1. The details in the \mathfrak{sl}_2 case can be found in [Chamberlin 2011]. This can be extended to the \mathfrak{sl}_n case via the injection $\Omega_i : \mathfrak{sl}_2 \otimes A \rightarrow \mathfrak{sl}_n \otimes A$ given by

$$\Omega_i(x^- \otimes a) = x_{-i} \otimes a, \quad \Omega_i(h \otimes a) = h_i \otimes a, \quad \Omega_i(x^+ \otimes a) = x_i \otimes a,$$

for all $i \in \{1, \dots, n - 1\}$ and $a \in A$. □

Lemma 3.1.3. *For all $\varphi, \chi \in \mathcal{F}$ with $|\varphi| + |\chi| > 1$ and all $i \in \{1, \dots, n - 1\}$, $q_i(\varphi, \chi)(v_i \otimes 1) = 0$.*

Proof. Assume that $\varphi = 0$. This case will proceed by induction on $|\chi| > 1$. If $|\chi| = 2$ (so that $\chi = \{a, b\}$ for some $a, b \in A$) we have

$$\begin{aligned} q_i(0, \{a, b\})(v_i \otimes 1) &= [(h_i \otimes a) \otimes (h_i \otimes b) - (h_i \otimes ab)](v_i \otimes 1) \\ &= (h_i \otimes a) \otimes (v_i \otimes b) - (v_i \otimes ab) \\ &= (v_i \otimes ab) - (v_i \otimes ab) \\ &= 0. \end{aligned}$$

For the next case assume that $|\chi| > 2$ then

$$q_i(0, \chi)(v_i \otimes 1) = -\frac{1}{|\chi|} \sum_{\emptyset \neq \psi \in \mathcal{F}(\chi)} \mathcal{M}(\psi)(h_i \otimes \pi(\psi))q_i(\chi - \psi)(v_i \otimes 1) = 0$$

by induction. Now assume that $|\varphi| = 1$ (or $\varphi = \chi_b$ for some $b \in A$). Then

$$\begin{aligned} q_i(\chi_b, \chi)(v_i \otimes 1) &= - \sum_{\psi \in \mathcal{F}(\chi)} \mathcal{M}(\psi)(x_{-i} \otimes b\pi(\psi))q_i(0, \chi - \psi)(v_i \otimes 1) \\ &= -\mathcal{M}(\chi)(x_{-i} \otimes b\pi(\chi))(v_i \otimes 1) \\ &\quad - \sum_{a \in \text{supp } \chi} \mathcal{M}(\chi - \chi_a)(x_{-i} \otimes b\pi(\chi - \chi_a))q_i(0, \chi_a)(v_i \otimes 1) \\ &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) \\ &\quad - \sum_{a \in \text{supp } \chi} \mathcal{M}(\chi - \chi_a)(x_{-i} \otimes b\pi(\chi - \chi_a))(-h_i \otimes a)(v_i \otimes 1) \\ &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \sum_{a \in \text{supp } \chi} \mathcal{M}(\chi - \chi_a)(x_{-i} \otimes b\pi(\chi - \chi_a))(v_i \otimes a) \end{aligned}$$

$$\begin{aligned}
 &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \sum_{a \in \text{supp } \chi} \mathcal{M}(\chi - \chi_a)(v_{i+1} \otimes b\pi(\chi)) \\
 &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \sum_{a \in \text{supp } \chi} \frac{(|\chi| - 1)!}{\prod_{c \in \text{supp}(\chi - \chi_a)} (\chi - \chi_a)(c)!} (v_{i+1} \otimes b\pi(\chi)) \\
 &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \sum_{a \in \text{supp } \chi} \frac{(|\chi| - 1)!}{\prod_{\substack{c \in \text{supp } \chi \\ c \neq a}} \chi(c)! (\chi(a) - 1)!} (v_{i+1} \otimes b\pi(\chi)) \\
 &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \sum_{a \in \text{supp } \chi} \frac{\chi(a)}{|\chi|} \mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) \\
 &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) \\
 &= 0.
 \end{aligned}$$

Finally assume that $|\varphi| > 1$. Then

$$\begin{aligned}
 &q_i(\varphi, \chi)(v_i \otimes 1) \\
 &= -\frac{1}{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{d \in \text{supp } \varphi} \mathcal{M}(\psi)(x_{-i} \otimes d\pi(\psi)) q_i(\varphi - \chi_d, \chi - \psi)(v_i \otimes 1) \\
 &= -\frac{1}{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{d \in \text{supp } \varphi} \mathcal{M}(\psi) \\
 &\quad \left(-\frac{1}{|\varphi| - 1} \sum_{\psi_1 \in \mathcal{F}(\chi - \psi)} \sum_{d_1 \in \text{supp}(\varphi - \chi_d)} \mathcal{M}(\psi_1) \right. \\
 &\quad \left. (x_{-i} \otimes d\pi(\psi))(x_{-i} \otimes d_1\pi(\psi_1)) q_i(\varphi - \chi_d - \chi_{d_1}, \chi - \psi - \psi_1) \right) (v_i \otimes 1) \\
 &= 0,
 \end{aligned}$$

because at least two x_{-i} terms act on a single v_i as 0. □

Lemma 3.1.4. For all $i \in \{1, \dots, n - 1\}$ and $\varphi, \chi \in \mathcal{F}$ with $|\varphi| + |\chi| = k$ we have

$$q_i(\varphi, \chi)(v_i \otimes 1)^{\otimes k} = (-1)^k v(0, \dots, 0, \chi, \varphi, 0, \dots, 0),$$

where χ is in the i -th position and φ in the $(i+1)$ -st.

Proof. We have $q_i(\varphi, \chi)(v_i \otimes 1)^{\otimes k} = \Delta^{k-1}(q_i(\varphi, \chi))(v_i \otimes 1)^{\otimes k}$. By Lemma 3.1.2, this equals

$$\left(\sum_{\substack{\psi \in \circ_k(\chi) \\ \phi \in \circ_k(\varphi)}} q_i(\phi(1), \psi(1)) \otimes \dots \otimes q_i(\phi(k), \psi(k)) \right) (v_i \otimes 1)^{\otimes k},$$

which can be rewritten as

$$\sum_{\substack{\psi \in \circ_k(\chi) \\ \phi \in \circ_k(\varphi)}} (q_i(\phi(1), \psi(1))(v_i \otimes 1)) \otimes \cdots \otimes (q_i(\phi(k), \psi(k))(v_i \otimes 1)).$$

By Lemma 3.1.3 we see that the only potentially nonzero terms in the sum are those for which $|\phi(j)| + |\psi(j)| \leq 1$ for all $j \in \{1, \dots, k\}$. Since $|\varphi| + |\chi| = k$ if we have $|\psi(j)| + |\phi(j)| = 0$ for some $j \in \{1, \dots, n - 1\}$, then there is a $r \in \{1, \dots, n - 1\}$ such that $|\psi(r)| + |\phi(r)| > 1$. So the only potentially nonzero terms in the sum are those for which $|\phi(j)| + |\psi(j)| = 1$ for all $j \in \{1, \dots, k\}$. Suppose that $\phi(j) = \chi_a$ and $\psi(j) = 0$ for some $j \in \{1, \dots, k\}$ and some $a \in A$. Then

$$q_i(\chi_a, 0)(v_i \otimes 1) = -(x_{-i} \otimes a)(v_i \otimes 1) = -(v_{i+1} \otimes a).$$

Suppose that $\phi(j) = 0$ and $\psi(j) = \chi_a$ for some $j \in \{1, \dots, k\}$ and some $a \in A$. Then

$$q_i(0, \chi_a)(v_i \otimes 1) = -(h_i \otimes a)(v_i \otimes 1) = -(v_i \otimes a).$$

So $-(v_{i+1} \otimes a)$ and $-(v_i \otimes a)$ are the only possibilities for factors in the tensor product above. Since we are summing over all possible submultisets of φ and χ , we have the result. □

Lemma 3.1.5. *For all $m \in \mathbb{N}$ and all $\varphi_1, \dots, \varphi_n \in \mathcal{F}$ with $\sum_{i=1}^n |\varphi_i| = m$,*

$$q(\varphi_1, \dots, \varphi_n)(v_1 \otimes 1)^{\otimes m} = (-1)^{\sum_{j=1}^n j|\varphi_j|} v(\varphi_1, \dots, \varphi_n).$$

Proof. Since for all $j \in \{1, \dots, n - 1\}$ and $k \in \{1, \dots, n\}$,

$$x_{-j}v_k = \delta_{j,k}v_{j+1}, \quad h_jv_k = \delta_{j,k}v_j - \delta_{j+1,k}v_{j+1},$$

so by Lemma 3.1.4 we have

$$\begin{aligned} & q(\varphi_1, \dots, \varphi_n)(v_1 \otimes 1)^{\otimes m} \\ &= q_{n-1}(\varphi_n, \varphi_{n-1})q_{n-2}((|\varphi_n| + |\varphi_{n-1}|)\chi_1, \varphi_{n-2}) \\ & \quad \times \cdots \times q_1\left(\left(\sum_{j=2}^n |\varphi_j|\right)\chi_1, \varphi_1\right)(v_1 \otimes 1)^{\otimes m} \\ &= (-1)^m q_{n-1}(\varphi_n, \varphi_{n-1}) \cdots q_2\left(\left(\sum_{j=3}^n |\varphi_j|\right)\chi_1, \varphi_2\right)v\left(\varphi_1, \left(\sum_{j=2}^n |\varphi_j|\right)\chi_1, 0, \dots, 0\right) \\ &= (-1)^{|\varphi_1|+2\sum_{j=2}^n |\varphi_j|} q_{n-1}(\varphi_n, \varphi_{n-1}) \\ & \quad \times \cdots \times q_3\left(\left(\sum_{j=4}^n |\varphi_j|\right)\chi_1, \varphi_3\right)v_i\left(\varphi_1, \varphi_2, \left(\sum_{j=3}^n |\varphi_j|\right)\chi_1, 0, \dots, 0\right) \end{aligned}$$

$$\begin{aligned} &= (-1)^{\sum_{j=1}^{n-2} j|\varphi_j|} q_{n-1}(\varphi_n, \varphi_{n-1})v(\varphi_1, \dots, \varphi_{n-2}, (|\varphi_{n-1}| + |\varphi_n|)\chi_1, 0) \\ &= (-1)^{\sum_{j=1}^n j|\varphi_j|} v(\varphi_1, \dots, \varphi_n). \end{aligned} \quad \square$$

3.2. The proof of Theorem 3.0.1.

Proof. By Lemmas 3.1.5 and 2.4.2

$$\left\{ q(\varphi_1, \dots, \varphi_n)(v_1 \otimes 1)^{\otimes m} \mid \varphi_1, \dots, \varphi_n \in \mathcal{F}(\mathbf{B}), \sum_{i=1}^n |\varphi_i| = m \right\}$$

is a basis for $S^m(V \otimes A)$. Therefore by Theorem 2.4.1

$$\left\{ q(\varphi_1, \dots, \varphi_n)w_{m\omega_1} \mid \varphi_1, \dots, \varphi_n \in \mathcal{F}(\mathbf{B}), \sum_{i=1}^n |\varphi_i| = m \right\}$$

is a basis for $W_A(m\omega_1)$. □

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
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