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Domination with decay in triangular matchstick arrangement graphs

Jill Cochran, Terry Henderson, Aaron Ostrander and Ron Taylor

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We provide results for the exponential dominating numbers and total exponential dominating numbers of a family of triangular grid graphs. We then prove inequalities for these numbers and compare them with inequalities that hold more generally for exponential dominating numbers of graphs.

1. Introduction

A *dominating set* of a graph G is a set $S \subseteq V(G)$ such that every $v \in V(G)$ is either in S or is adjacent to a member of S. A *total dominating set* of a graph G is a set $S \subseteq V(G)$ such that every $v \in V(G)$ is adjacent to a member of S. The vertices in S are called *dominating vertices* or *dominators*, and a vertex adjacent to a dominator is said to be *dominated* by that dominator. In most kinds of domination a dominator is considered to dominate itself, but this is not the case for *total domination* where each dominator must be dominated by another dominator.

When considering domination at a distance, a *k-dominating set* of a graph G is a set $S \subseteq V(G)$ such that every $v \in V(G)$ is either in S or is a distance of k or less from any member of S. More examples of domination at a distance have been investigated in [Erwin 2004; Slater 1976].

In [Dankelmann et al. 2009] the authors introduce *exponential domination*, a variety of distance domination where the dominating power of a vertex decreases exponentially with the distance from that vertex. In this paper, we consider exponential domination and introduce a variation of exponential domination which we call *total exponential domination*. In the rest of the paper we sometimes talk about exponential domination or total exponential domination just in terms of domination when the context is clear.

For a connected graph G and $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of G induced by S. For $u \in S$ and $v \in V(G) \setminus S$ we define $d_S(u, v)$ to be the distance

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between u and v in $G[V(G) \setminus (S \setminus \{u\})]$; i.e., minimum length paths do not include other dominators.

For exponential domination we use the same weight function as in [Dankelmann et al. 2009], given by

$$w_S(v) = \begin{cases} \sum_{u \in S} 2^{-d_S(u,v)+1}, & v \notin S, \\ 2, & v \in S. \end{cases}$$

For total exponential domination we use a similar weight function given by

$$w_S^t(v) = \begin{cases} \sum_{u \in S} 2^{-d_S(u,v)+1}, & v \notin S, \\ \sum_{u \in S, u \neq v} 2^{-d_S(u,v)+1}, & v \in S. \end{cases}$$

Note that the only difference between these two weight functions is that $w_S(u) = 2$ but $w_S^t(u)$ depends on the distribution of the other dominators for $u \in S$.

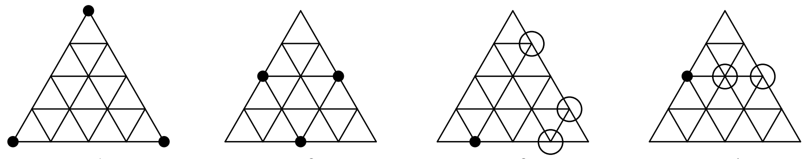
As in [Dankelmann et al. 2009], if for each $v \in V(G)$ (or equivalently $v \in V(G) \setminus S$) we have that $w_S(v) \geq 1$, then S is an *exponential dominating set* of G. The *exponential dominating number* of a graph G, denoted by $\gamma_e(G)$, is the smallest cardinality of an exponential dominating set of G. Similarly, if for each $v \in V(G)$ we have that $w_S^t(v) \geq 1$, then S is a *total exponential dominating set* of G. The *total exponential dominating number* of a graph G, denoted by $\gamma_{te}(G)$, is the smallest cardinality of a total exponential dominating set of G. For an arbitrary S and arbitrary $v \in V(G) \setminus S$, if $w_S(v) \geq 1$ or $w_S^t(v) \geq 1$ then v is *exponentially dominated* or *totally exponentially dominated* by S.

We restrict ourselves to a particular family of *triangular grid graphs*. A triangular grid graph is a graph G such that $V(G)$ can be put in a correspondence with points $(x, y) = \left(\frac{1}{2}a - b, \frac{\sqrt{3}}{2}a\right)$, where $a, b \in \mathbb{Z}$; additionally, we require that in this correspondence two vertices can be adjacent only if their corresponding points are separated by unit distance (this is the same definition that is found in [Gordon et al. 2008]). We denote by G_n the graph whose vertices correspond with the points in

$$\left\{\left(\tfrac{1}{2}a - b, \tfrac{\sqrt{3}}{2}a\right) \;\middle|\; a, b \in \mathbb{Z},\ 0 \leq b \leq a \leq n\right\}$$

and which has as many edges as possible; G_n is called the *triangular matchstick arrangement graph of side n*. This is the family of graphs which we consider in this paper. The *corners* of G_n are those vertices corresponding to $a = b = 0$, $a = b = n$, and $a = n$, $b = 0$. The *perimeter* of G_n is the set of vertices and edges that lie on the minimal length paths between the corners. Any one of these minimal length paths is a *perimeter edge*; note that each perimeter edge of G_n contains n edges.

In Section 2 we determine the exponential dominating numbers for G_n up to $n = 7$. In Section 3 we provide upper bounds for exponential dominating numbers for arbitrary G_n. In Section 4 we determine the total exponential dominating numbers for G_n up to $n = 5$. In Section 5 we use arguments similar to those from Section 3 to provide upper bounds for total exponential dominating numbers for arbitrary G_n.



G_4^1 G_4^2 G_4^3 G_4^4

Figure 1. Graphs for Lemma 5.

2. Base cases for exponential domination

We use the following lemmas in proving Theorem 8.

Lemma 1. $$\gamma_e(G_n) \leq \gamma_e(G_{n+1}).$$

Lemma 2. *If there exists an arrangement of dominators that dominates G_n where a dominator is placed at a corner vertex, then the graph is also dominated by the arrangement of dominators produced by moving the corner dominator to a vertex adjacent to it and leaving the rest of the dominators in their original positions.*

Lemma 3. $$\gamma_e(G_1) = 1.$$

Lemma 4. $$\gamma_e(G_2) = 2.$$

Proof. To see that $\gamma_e(G_2) \leq 2$, note that picking any two vertices of G_2 to be dominators suffices to dominate the graph.

Suppose $\gamma_e(G_2) = 1$. For every vertex in $V(G_2)$ there is a second vertex that is a distance of 2 away. Thus no matter where the dominator is placed there always is one vertex with only a weight of $\frac{1}{2}$, so G_2 is not dominated, which is a contradiction. □

Lemma 5. $$\gamma_e(G_4) = 3.$$

Proof. The graphs G_4^i referred to in this proof are contained in Figure 1. To see that $\gamma_e(G_4) \leq 3$, consider G_4^1 or G_4^2 (from now on all vertices appearing as bullet points are dominators). If $\gamma_e(G_4) < 3$ then we can dominate G_4 with two dominators. We obviously must dominate the corners of G_4, and by Lemma 2 we can assume that no dominator is in a corner.

Supposing that the two dominators are at a distance of 1 from two corners (to ensure that at the least those corners are dominated), we produce graph G_4^3, where one of the circled vertices is also a dominator. The graph is not dominated in any of these cases.

Supposing that the two dominators are each at a distance of 2 from a single corner (to ensure that one corner is dominated), we produce G_4^4, where one of the circled vertices is also a dominator. G_4 is not dominated in either case. This suffices to prove the lemma. □

Lemma 6. $$\gamma_e(G_6) = 4.$$

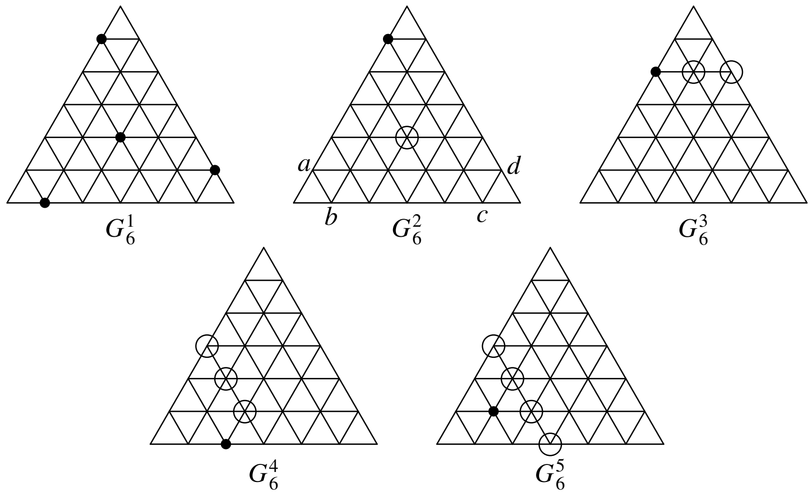


Figure 2. Graphs for Lemma 6.

Proof. The graphs G_6^i referred to in this proof are contained in Figure 2. To see that $\gamma_e(G_6) \leq 4$, consider G_6^1.

If $\gamma_e(G_6) < 4$ then we can dominate G_6 with three dominators. We first consider the case where each dominator is a distance of 1 from each corner. Doing so we produce G_6^2, where one of a or b and one of c or d is a dominator. In each of these cases the circled vertex is not dominated.

Considering next the case where one of the corners has two dominators at a distance of 2, we must place the third dominator on G_6^3, where one of the circled vertices is a dominator. We cannot place a third dominator in either of these cases so that all of the corners are dominated.

We now consider placing two dominators at a distance of 3 from a corner and the third dominator at a distance of 2 from the same corner. Doing so, we generate G_6^4 or G_6^5, where two of the circled vertices are dominators. In any such graph only one of the corners is dominated. This suffices to prove the lemma. □

Lemma 7. $$\gamma_e(G_7) = 5.$$

Proof. The graphs G_7^i referred to in this proof are contained in Figure 3. To see that $\gamma_e(G_7) \leq 5$, consider G_7^1. If $\gamma_e(G_7) < 5$, then four dominators suffice to dominate the graph. We first try to dominate G_7 by placing three dominators so that each lies at a distance of 1 from each corner. Doing so, we produce G_7^2 or G_7^3.

Notice that the vertices a, b and c in G_7^2 have domination of $\frac{17}{32}$ due to the first three dominators. So the fourth dominator must be placed at either of the circled vertices so that a and b will both have domination greater than 1. However, doing so, the domination of c is either $\frac{21}{32}$ or $\frac{25}{32}$, so we cannot dominate G_7 with four dominators by starting with G_7^2.

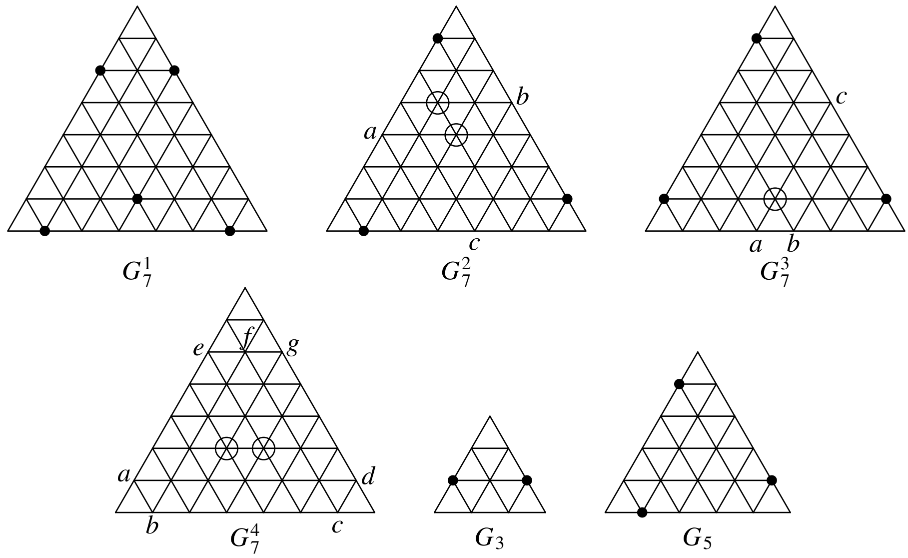


Figure 3. Graphs for Lemma 7 and Theorem 8.

Note that the vertices a and b in G_7^3 have domination $\frac{13}{32}$ due to the first three dominators. So the fourth dominator must be placed at the circled vertex in order for both a and b to have domination greater than 1. However, doing so, the domination of c is $\frac{25}{32}$, so we cannot dominate G_7 with four dominators by starting with G_7^3.

We next try to dominate G_7 by placing two dominators a distance of 1 from two corners and two other dominators a distance of 2 from the third corner. Doing so, we produce G_7^4, where one of a and b, one of c and d, and two of e, f, and g are dominators. In each of these graphs one of the circled vertices fails to be dominated. This exhausts all of the ways that we can ensure that all of the corners are dominated, which suffices to prove the lemma. □

Theorem 8. *The exponential domination numbers for G_1 through G_7 are*

n	1	2	3	4	5	6	7
$\gamma_e(G_n)$	1	2	2	3	3	4	5

Proof. Lemmas 3–7 provide values for $\gamma_e(G_n)$ for $n \in \{1, 2, 4, 6, 7\}$. To see that $\gamma_e(G_3) \le 2$, consider G_3 in Figure 3; by Lemmas 1 and 4, $\gamma_e(G_3) = 2$. To see that $\gamma_e(G_5) \le 3$, consider G_5 in Figure 3; by Lemmas 1 and 5 we see that $\gamma_e(G_5) = 3$. □

Theorem 9. *The exponential domination numbers for G_{10} through G_{15} are bounded as follows*:

n	8	9	10	11	12	13	14	15
$\gamma_e(G_n) \le$	6	6	7	9	10	11	12	13

Proof. Consult Figure 4 for graphs that satisfy these bounds. □

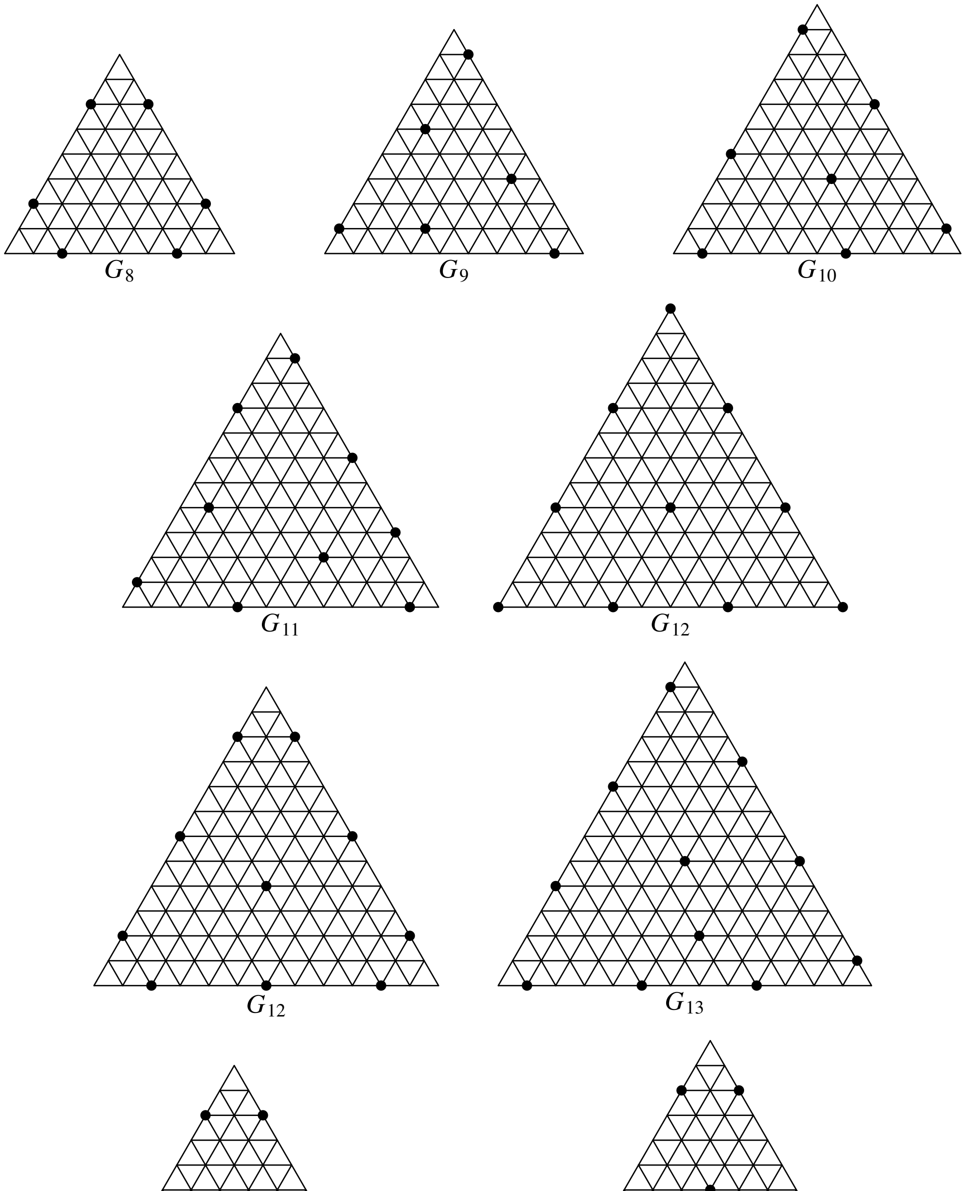


Figure 4. Graphs for Theorem 9.

3. Inequalities for exponential domination

We will now determine the total exponential dominating numbers for G_n up to $n = 5$.

Theorem 10 [Dankelmann et al. 2009]. *If G is a connected graph of size n then*

$$\gamma_e(G) \le \tfrac{2}{5}(n+2).$$

Applying this inequality to triangular grid graphs, we have the bound

$$\gamma_e(G_n) \le \frac{2}{5}\left(\binom{n+2}{2} + 2\right) = \tfrac{1}{5}(n^2 + 3n + 6).$$

The theorem in [Dankelmann et al. 2009] that bounds $\gamma_e(G)$ is established by considering an exponentially dominated spanning tree of G and not G itself. In establishing our bounds, we make use of the fact that G_{n+1} can be constructed from G_n by a set of elementary operations that are dependent on n. Our strongest bounds arise from considering how we can construct a distribution of dominators that dominates G_{n+r} based on a distribution of dominators that dominates G_n.

Lemma 11. *Suppose G_{4n} can be dominated by a set of m dominators where each of the corners are dominated by two dominators placed on the perimeter at a distance of 2 from each corner*:

(1) *If dominators are placed along the rest of the perimeter edge between those two corners with a distance of 4 between each dominator*, *then G_{4n+4} can be dominated in a similar manner with $m + n + 3$ dominators.*

(2) *If dominators are placed along two of the perimeter edges with a distance of 4 between each dominator*, *then G_{4n+8} can be dominated in a similar manner by $m + 2n + 6$ dominators.*

(3) *If dominators are placed along the rest of the perimeter with a distance of 4 between each dominator*, *then G_{4n+12} can be dominated in a similar manner by $m + 3n + 9$ dominators.*

Proof. (1) Consider the labeled G_{4n+4} implied by Figure 5. A labeled G_{4n} can be seen by removing the lower four rows of vertices from the G_{4n+4} with the lower perimeter of G_{4n}, including the row of vertices labeled $\{1, 2, \dots, n\}$. Both G_{4n} and G_{4n+4} have dominators placed as described in the hypotheses of the lemma. It can be confirmed that all of the vertices in the additional four rows of G_{4n+4} are dominated by this arrangement of dominators. If the G_{4n} is dominated by m dominators, then we have dominated G_{4n+4} by adding $n + 1$ dominators along the lower perimeter and two dominators on the other perimeters; thus we have dominated G_{4n+4} with $m + n + 3$ dominators.

(2) Similarly, consider the labeled G_{4n+8} implied by Figure 5. If G_{4n} is dominated by m dominators then we have dominated G_{4n+8} by adding $n + 2$ dominators along

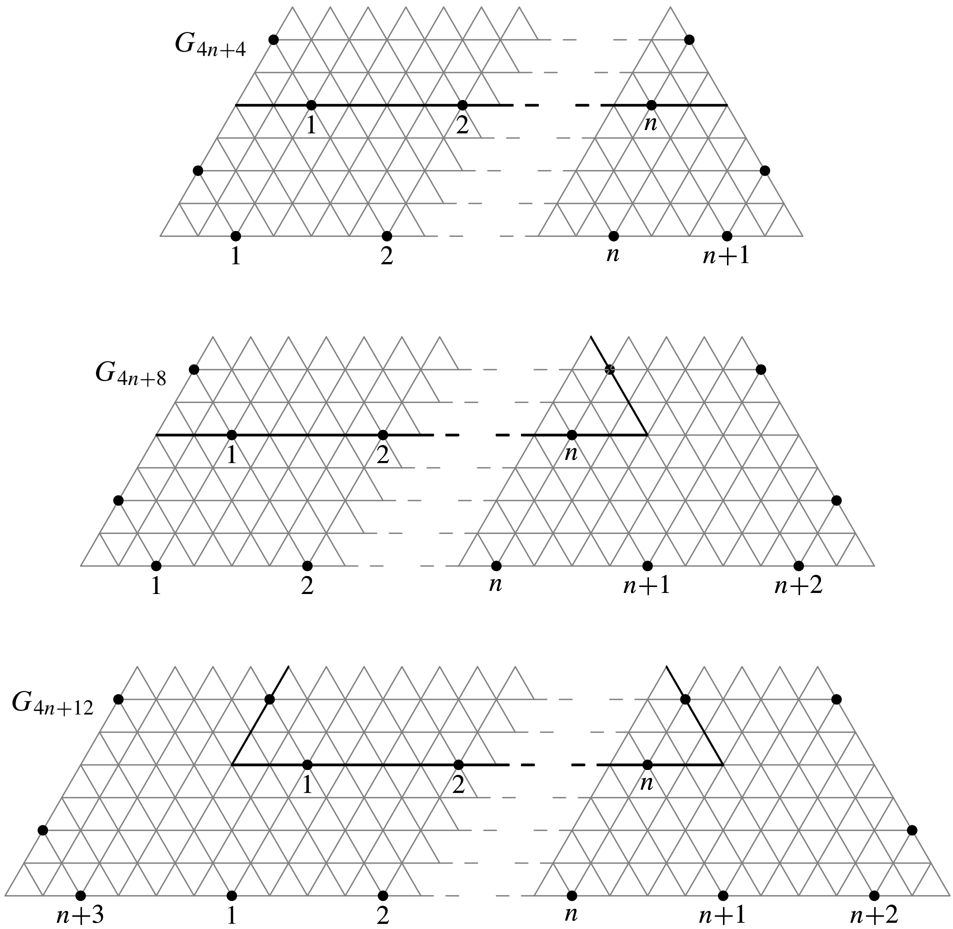


Figure 5. Graphs for Lemma 11.

two perimeter edges and two dominators on the other perimeter edge; thus we have dominated G_{4n+8} with $m+2n+6$ dominators.

(3) Similarly, consider the labeled G_{4n+12} implied by Figure 5. If G_{4n} is dominated by m dominators then we have dominated G_{4n+12} by adding $n+3$ dominators along each perimeter edge; thus we have dominated G_{4n+12} with $m+3n+9$ dominators. □

Lemma 12. *If* G_{15n} *can be dominated by m dominators where*

(1) *dominators are placed at two corners*, *and*

(2) *along the perimeter edge between those corners dominators are placed with a distance of* 3 *between them*,

then G_{15n+15} *can be dominated in a similar manner by* $m+13+15n$ *dominators.*

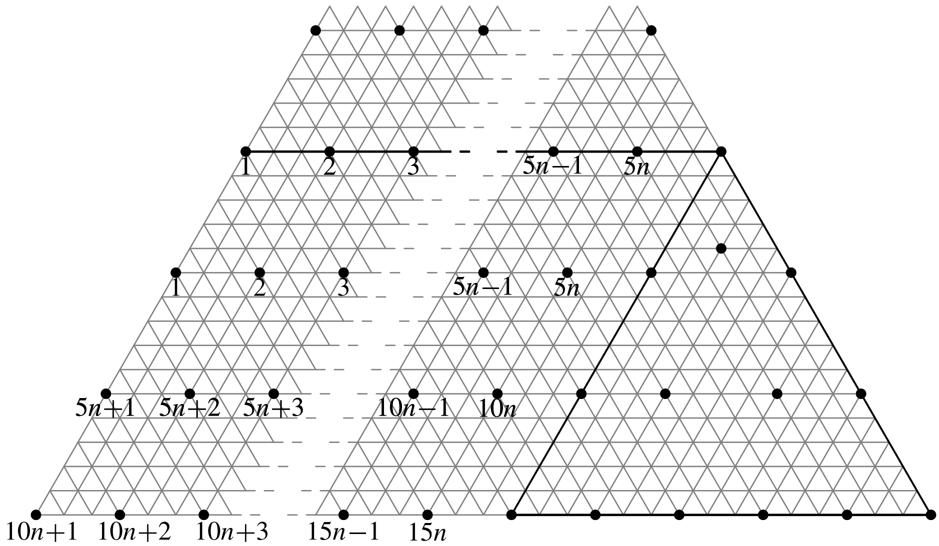


Figure 6. Graph for Lemma 12.

Proof. Consider the labeled G_{15n+15} implied by Figure 6. The lower perimeter edge of a G_{15n} is emboldened, as is part of the perimeter of a G_{15} dominated by 14 dominators. If dominators are placed in the lower 15 rows of G_{15n+15}, outside of the bold G_{15}, as suggested by the placement of the labeled dominators, then both G_{15n} and G_{15n+15} have dominator distributions as described in the hypotheses of the lemma. It can be confirmed that G_{15n+15} is dominated provided that G_{15n} is dominated. If G_{15n} is dominated by m dominators then we have dominated G_{15n+15} by placing $15n$ dominators in a regular pattern in the lower 15 rows and 13 dominators in the remaining space; thus we have dominated G_{15n+15} with $m+13+15n$ dominators. □

Note that this lemma makes use of a G_{15} that is dominated using 14 dominators; however, from Theorem 9 we see that the exponential dominating number is at most 13. We use more dominators than necessary here in order to produce a consistent pattern of dominators along the lower perimeter edge of each subsequently constructed graph.

Theorem 13. *The following inequalities hold for $n \geq 0$:*

(1) $\gamma_e(G_{4n}) \leq \frac{1}{2}(n(n+5))$.

(2) $\gamma_e(G_{4+8n}) \leq 2n^2+6n+3$.

(3) $\gamma_e(G_{4+12n}) \leq \frac{1}{2}(9n^2+15n+6)$.

(4) $\gamma_e(G_{8+4n}) \leq \frac{1}{2}(n^2+9n+12)$.

(5) $\gamma_e(G_{8n}) \leq 2n(n+2)$.

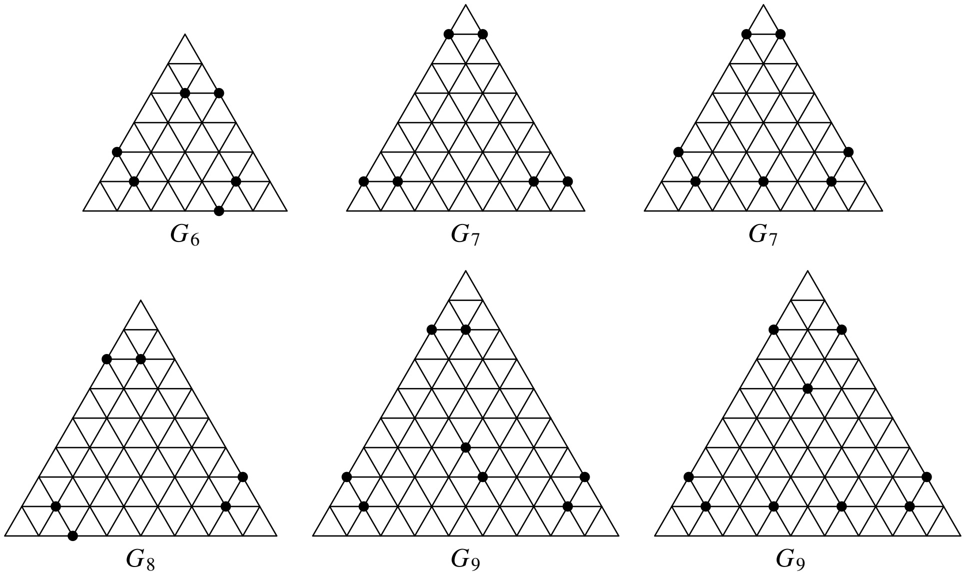


Figure 7. Graphs for Theorem 13.

(6) $\gamma_e(G_{8+12n}) \leq \frac{1}{2}(9n^2+21n+12)$.

(7) $\gamma_e(G_{12+4n}) \leq \frac{1}{2}(n^2+11n+20)$.

(8) $\gamma_e(G_{12+8n}) \leq 2n^2+10n+10$.

(9) $\gamma_e(G_{12n}) \leq \frac{1}{2}(9n^2+9n+2)$.

(10) $\gamma_e(G_{15n}) \leq \frac{1}{2}(15n^2+11n+2)$.

Proof. We will prove inequalities (1)–(3) and (10); inequalities (4)–(9) can be proven by means similar to those used to prove (1)–(3) using the same lemmas. The dominated G_8 used for inequalities (4)–(6) and the dominated G_{12} used for inequalities (7)–(9) can be found in Figure 4; the second G_{12} that appears in Figure 4 is the one we use because it is the only one that satisfies the hypotheses of Lemma 11.

From Figure 1 we see that G_4^2 satisfies the hypotheses of Lemma 11. Lemma 5 also implies that $\gamma_e(G_4) \leq 3$, so we see that inequality (1) holds for the case where $n=1$, and inequalities (2) and (3) hold for the case where $n=0$.

Suppose that inequality (1) holds for all n up to some m; also suppose that G_{4m} can be dominated in agreement with the hypotheses of Lemma 11 by a number of dominators less than or equal to the bound provided by inequality (1). Then $\gamma_e(G_{4m}) \leq \frac{1}{2}(m^2+5m)$, and by Lemma 11 we see that G_{4m+4} can be dominated by

$$\tfrac{1}{2}(m^2+5m)+m+3 = \tfrac{1}{2}(m^2+7m+6) = \tfrac{1}{2}(m+1)(m+6)$$

dominators. Thus $\gamma_e(G_{4(m+1)}) \leq \frac{1}{2}(m+1)((m+1)+5)$, which proves inequality (1).

Suppose that inequality (2) holds for all n up to some m; also suppose that G_{4+8m} can be dominated in agreement with the hypotheses of Lemma 11 by a number of dominators less than or equal to the bound provided by inequality (2). Then $\gamma_e(G_{4+8m}) = \gamma_e(G_{4(1+2m)}) \leq 2m^2 + 6m + 3$, and by Lemma 11 we see that $G_{4(1+2m)+8}$ can be dominated by

$$(2m^2 + 6m + 3) + 2(1 + 2m) + 6 = 2m^2 + 10m + 11 = 2(m + 1)^2 + 6(m + 1) + 3$$

dominators. Thus

$$\gamma_e(G_{4(1+2m)+8}) = \gamma_e(G_{4+8(m+1)}) \leq 2(m + 1)^2 + 6(m + 1) + 3,$$

which proves inequality (2).

Suppose that inequality (3) holds for all n up to some m; also suppose that G_{4+12m} can be dominated in agreement with the hypotheses of Lemma 11 by a number of dominators less than or equal to the bound provided by inequality (3). Then $\gamma_e(G_{4+12m}) = \gamma_e(G_{4(1+3m)}) \leq \frac{1}{2}(9m^2 + 15m + 6)$, and by Lemma 11 we see that $G_{4(1+3m)+12}$ can be dominated by

$$\tfrac{1}{2}(9m^2+15m+6)+3(1+3m)+9=\tfrac{1}{2}(9m^2+33m+30)=\tfrac{1}{2}\big(9(m+1)^2+15(m+1)+6\big)$$

dominators. Thus

$$\gamma_e(G_{4(1+3m)+12}) = \gamma_e(G_{4+12(m+1)}) \leq \tfrac{1}{2}\big(9(m + 1)^2 + 15(m + 1) + 6\big),$$

which proves inequality (3).

From Figure 6 we see that G_{15} can be dominated by 14 dominators in a way that satisfies the hypotheses of Lemma 12. This implies that $\gamma_e(G_{15}) \leq 14$, so we see inequality 10 holds for the case where $n = 1$. Suppose that inequality (10) holds for all n up to some m; also suppose that G_{15m} can be dominated in agreement with the hypotheses of Lemma 12 by a number of dominators less than or equal to the bound provided by inequality (10). Then $\gamma_e(G_{15m}) \leq \frac{1}{2}(15m^2 + 11m + 2)$, and by Lemma 12 we see that $G_{15(m+1)}$ can be dominated by

$$\tfrac{1}{2}(15m^2+11m+2)+13+15m=\tfrac{1}{2}(15m^2+41m+28)=\tfrac{1}{2}\big(15(m+1)^2+11(m+1)+2\big)$$

dominators. Thus,

$$\gamma_e(G_{15(m+1)}) \leq \tfrac{1}{2}\big(15(m + 1)^2 + 11(m + 1) + 2\big),$$

which proves inequality (10). □

This provides us with the following corollary.

Corollary 14. *The following inequalities hold for* $n \in \mathbb{Z}^+$ *as specified*:

(1) $\gamma_e(G_n) \leq \frac{1}{32}(n^2 + 12n + 32)$, *where* $n \bmod 4 = 0$.

(2) $\gamma_e(G_n) \leq \frac{1}{30}(n^2 + 11n + 30)$, *where* $n \bmod 15 = 0$.

The first inequality here is implied by inequalities (3), (6), and (9) in Theorem 13, and the second inequality is implied by inequality (10). The other inequalities from Theorem 13 do not provide bounds that are as good as these.

4. Base cases for total exponential domination

The following lemma is the analogue of Lemma 1 for total exponential domination.

Lemma 15. $$\gamma_{te}(G_n) \leq \gamma_{te}(G_{n+1}).$$

Lemma 16. $$\gamma_{te}(G_1) = 2.$$

Proof. To see that $\gamma_{te}(G_1) \leq 2$, note that picking any two vertices as dominators suffices to dominate G_1. To see that $\gamma_{te}(G_1) \neq 1$, note that a single dominator can never dominate an entire graph. □

Lemma 17. $$\gamma_{te}(G_3) = 3.$$

Proof. To see that $\gamma_{te}(G_3) \leq 3$, consider the first graph in Figure 8. Suppose that $\gamma_{te}(G_3) < 3$. Then the graph is dominated with two dominators. In a graph with only two dominators, the dominators must be adjacent since otherwise both of them will not have weight greater than 1. Any G_3 with two adjacent dominators will be one of the graphs shown in Figure 8, none of which is dominated. □

Lemma 18. $$\gamma_{te}(G_5) = 5.$$

Proof. The graphs referred to in this proof appear in Figure 9. To see that $\gamma_{te}(G_5) \leq 5$, consider the graph G_5. Supposing that $\gamma_{te}(G_5) < 5$, we can dominate the graph with four dominators. Since each corner must have a weight greater than or equal to 1, we organize this proof according to the ways that corners can be dominated by the fewest number of dominators. Note that 1 can be written as a sum of four or fewer powers of $\frac{1}{2}$ (not necessarily unique) with numerators of 1 in the following five ways: 1, $2\left(\frac{1}{2}\right)$, $\frac{1}{2}+2\left(\frac{1}{4}\right)$, $\frac{1}{2}+\frac{1}{4}+2\left(\frac{1}{8}\right)$, $4\left(\frac{1}{4}\right)$. We consider each of these possible combinations of weights separately.

1: One way for all of the corners to have weight at least 1 is to place dominators at a distance of 1 from each of the corners. Doing so, we produce either G_5^1 or G_5^2. In these graphs each dominator has weight less than $\frac{1}{2}$, so in order for the dominators to be dominated we must place another dominator no more than a distance of 1 away from each. It is not possible to do this with a single dominator, so there is

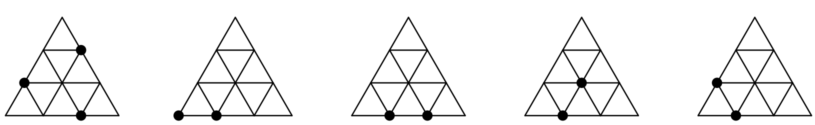


Figure 8. Graphs for Lemma 17.

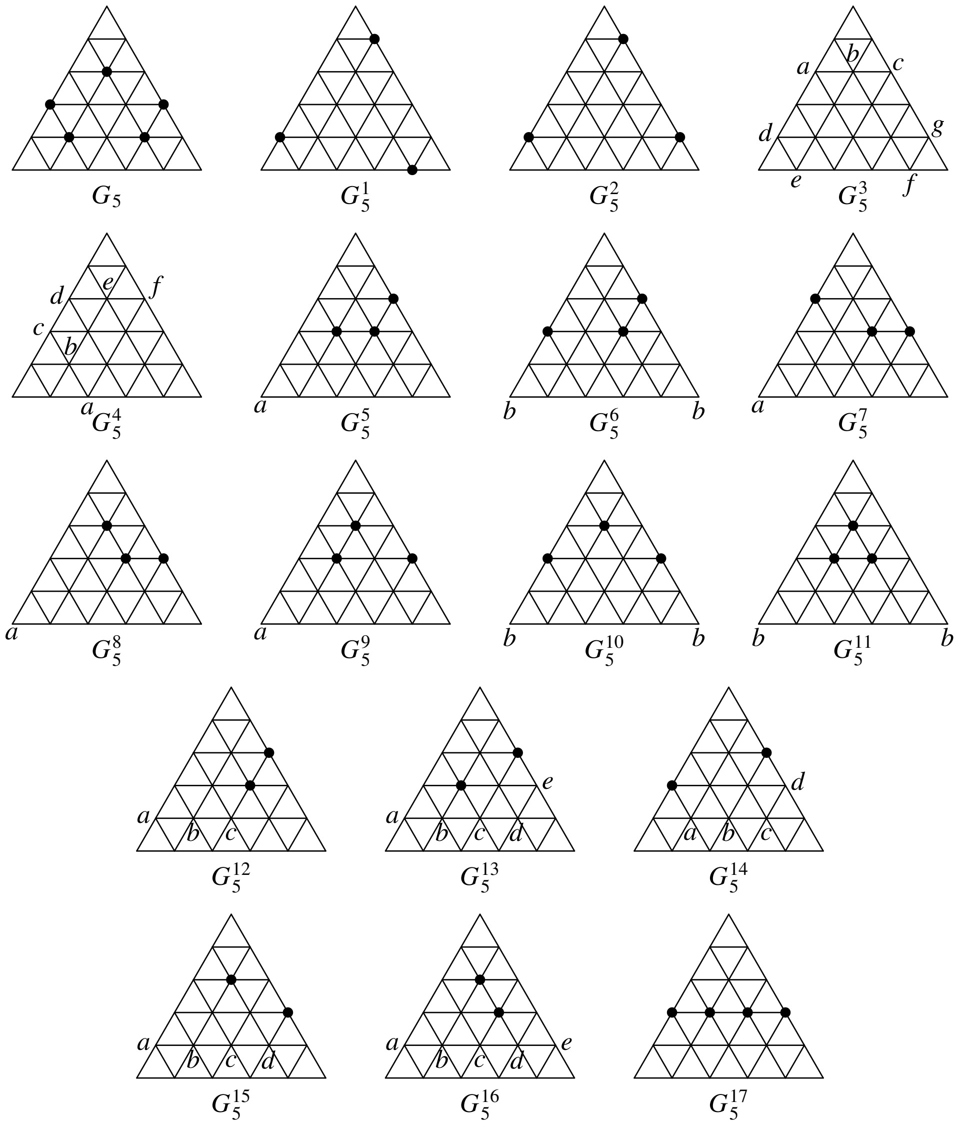


Figure 9. Graphs for Lemma 18.

not a configuration of dominators that dominates the graph where each corner is adjacent to a dominator.

$2(\frac{1}{2})$: Another method to dominate all of the corners is to place two dominators a distance of 2 away from one corner and to place the other dominators adjacent to the other corners. Doing this we produce G_5^3, where two of a, b, and c are dominators, one of d and e is a dominator, and one of f and g is a dominator. It can be confirmed that in each of these cases the graph fails to be dominated.

The next way to consider having corners dominated is to place two dominators at a distance of 2 from one corner, and to do the same for a second corner. Doing so we produce G_5^4, where two of a, b, and c are dominators and two of d, e, and f are dominators. It can be confirmed that in each of these cases the graph fails to be dominated.

$\mathbf{\frac{1}{2} + 2(\frac{1}{4})}$**:** The third case involves dominating a single corner by placing one dominator at a distance of 2 and two dominators at a distance of 3 in such a way that the dominators don't interfere with one another. If we begin by doing this for the top corner, we make one of the graphs from G_5^5 to G_5^{11}. In those graphs with vertices labeled a, each labeled vertex has domination less than $\frac{1}{2}$, so a dominator must be placed adjacent to it. It is easy to confirm that doing so will never suffice to dominate the graph by considering the lower corner vertex opposite to the labeled vertex. In those graphs with vertices labeled b, the weight of each labeled vertex is at least $\frac{1}{2}$ but less than $\frac{3}{4}$, so a dominator must be placed at distance of 2 or closer. Since the labeled vertices are a distance of 5 apart this is not possible.

$\mathbf{\frac{1}{2} + \frac{1}{4} + 2(\frac{1}{8})}$**:** The only other cases that need to be considered are those in which all four dominators are used to dominate a single corner. This can be achieved by placing dominators in a configuration with one dominator at a distance of 2, one dominator at a distance of 3, and two dominators at a distance of 4 (using $d_S(u, v)$). Doing so, we produce one graph from G_5^{12} to G_5^{16}, where two of the vertices labeled by letters are dominators. It can be confirmed that in each case the graph fails to be dominated (this can be easily done by considering domination of the other two corners).

$\mathbf{4(\frac{1}{4})}$**:** If we try to dominate G_5 using four dominators all at a distance of 3 from one of the corners then we produce G_5^{17}, which is not dominated. □

Theorem 19. *The total exponential domination numbers for G_1 through G_5 are*

n	1	2	3	4	5
$\gamma_{te}(G_n)$	2	2	3	3	5

Proof. Lemmas 16–18 provide $\gamma_{te}(G_n)$ for $n \in \{1, 3, 5\}$. To see that $\gamma_{te}(G_2) \leq 2$, consider the graph in Figure 10; by Lemmas 15 and 16 we see that $\gamma_{te}(G_2) = 2$. To see that $\gamma_{te}(G_4) \leq 3$, consider the graph in Figure 10; by Lemmas 15 and 17 we see that $\gamma_{te}(G_4) = 3$. □

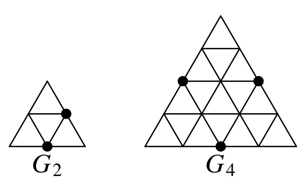


Figure 10. Graphs for Theorem 19.

Theorem 20. *The total exponential domination numbers for G_6 through G_9 are bounded as follows*:

n	6	7	8	9
$\gamma_{te}(G_n) \leq$	6	6	6	8

Proof. Consult Figure 7 for graphs that satisfy these bounds. □

5. Inequalities for total exponential domination

Lemma 21. *If G_{2n+1} can be dominated by a set of m dominators so that there exists a subgraph of G_{2n+1} isomorphic to G_{2n} that contains all of the dominators and such that*

(1) *two of the corners of the G_{2n} subgraph are adjacent to two dominators*, *and*

(2) *dominators are placed along the rest of perimeter edge between those two corners with a distance of 2 between each dominator*,

then G_{2n+7} can be dominated in a similar way by $m+n+5$ dominators.

Proof. Consider the labeled G_{2n+7} implied by Figure 11. The lower perimeter edge of a G_{2n+1} has been emboldened. The lower perimeter edge of a G_{2n+6} subgraph corresponds with the second-lowest set of edges and vertices, including the vertices labeled by $\{1, 2, \ldots, n+2, n+3\}$; this graph has dominators placed as described above. A labeled G_{2n} can be produced by removing the lower seven rows of vertices from the G_{2n+7}; this G_{2n} has dominators placed as described above and is a subgraph of the G_{2n+1} whose lower perimeter edge is bold. It can be confirmed that all of the vertices in the lower six rows are dominated by an arrangement of dominators like the one depicted in Figure 11. If G_{2n+1} is dominated by m dominators then we have dominated G_{2n+7} by adding a total of $n+5$ dominators, thereby dominating G_{2n+7} with $m+n+5$ dominators. □

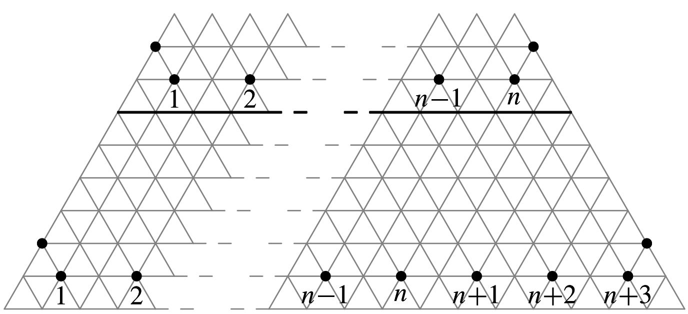


Figure 11. Graph for Lemma 21.

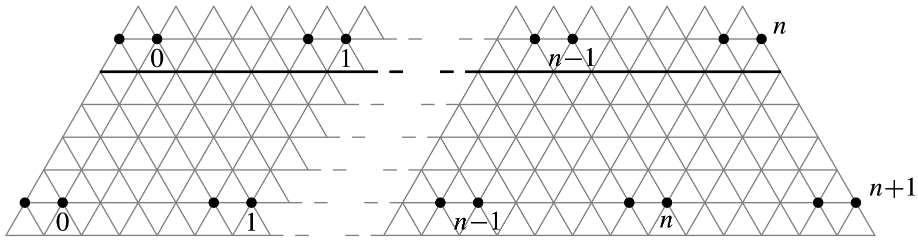


Figure 12. Graph for Lemma 22.

Lemma 22. *If G_{5n+2} can be dominated by a set of m dominators so that there exists a subgraph of G_{5n+2} isomorphic to G_{5n+1} that contains all of the dominators and such that*

(1) *two of the corners of the subgraph are dominators*,

(2) *along the perimeter edge of the subgraph between those dominators there are two dominators adjacent to each of the above-mentioned dominators*, *and*

(3) *along the rest of the perimeter there occur pairs of adjacent dominators with a distance of* 4 *between each pair*,

then G_{5n+7} can be dominated in a similar way by $m+4+2n$ dominators.

Proof. Consider the labeled G_{5n+7} implied by Figure 12. The lower perimeter edge of a G_{5n+2} subgraph has been emboldened. The lower perimeter edge of a G_{5n+6} subgraph corresponds with the second-lowest set of edges and vertices, including the vertices labeled by $\{1, 2, \dots, n, n+1\}$; this graph has dominators placed as described above. A labeled G_{5n+1} can be produced by removing the lower six rows of vertices from the G_{5n+7}; this graph has dominators placed as described above and is a subgraph of the G_{5n+2} whose lower perimeter edge has been emboldened. It can be confirmed that all of the vertices in the lower five rows are dominated by an arrangement of dominators like the one depicted in Figure 12. If the G_{5n+2} is dominated by m dominators then we have dominated G_{5n+7} by adding a total of $2n+4$ dominators, thereby dominating G_{5n+7} with $m+4+2n$ dominators. □

Theorem 23. *The following inequalities hold for $n \geq 0$*:

(1) $\gamma_{te}(G_{5+6n}) \leq \frac{1}{2}(3n^2+11n+10)$.

(2) $\gamma_{te}(G_{7+6n}) \leq \frac{1}{2}(3n^2+13n+14)$.

(3) $\gamma_{te}(G_{9+6n}) \leq \frac{1}{2}(3n^2+15n+18)$.

(4) $\gamma_{te}(G_{2+5n}) \leq (n+1)(n+2)$.

Proof. We will prove inequalities (1) and (4). The proofs for inequalities (2) and (3) are similar to the proof of inequality (1). The dominated G_7 and G_9 used to prove inequalities (2) and (3) can be found in Figure 7; we use the second G_7 and second G_9 that appear because they are the only ones that satisfy the hypotheses of Lemma 21.

From Figure 9 we see that G_5 can be dominated by five dominators in a way that satisfies the hypotheses of Lemma 21. This implies that $\gamma_{te}(G_5) \leq 5$, so we see that inequality (1) is satisfied for the case where $n = 0$. Suppose that inequality (1) holds for all n up to some m; also suppose that G_{5+6m} can be dominated in agreement with the hypotheses of Lemma 21 by a number of dominators less than or equal to the bound provided by inequality (1). Then $\gamma_{te}(G_{5+6m}) = \gamma_{te}(G_{2(3m+2)+1}) \leq \frac{1}{2}(3m^2 + 11m + 10)$, and by Lemma 21 we see that $G_{5+6(m+1)}$ can be dominated by

$$\begin{aligned}\tfrac{1}{2}(3m^2 + 11m + 10) + (3m + 2) + 5 &= \tfrac{1}{2}(3m^2 + 17m + 24)\\ &= \tfrac{1}{2}\big(3(m+1)^2 + 11(m+1) + 10\big)\end{aligned}$$

dominators. Thus

$$\gamma_{te}(G_{5+6(m+1)}) \leq \tfrac{1}{2}\big(3(m+1)^2 + 11(m+1) + 10\big),$$

which proves inequality (1).

From Figure 10 we see that G_2 can be dominated in a way that satisfies the hypotheses of Lemma 22. This implies that $\gamma_{te}(G_2) \leq 2$, so we see that inequality (4) holds in the case where $n = 0$. For some $m > 0$ suppose that G_{2+5m} can be dominated in agreement with the hypotheses of Lemma 22 by a number of dominators less than or equal to the bound provided by inequality (4). Then $\gamma_{te}(G_{2+5m}) \leq m^2 + 3m + 2$, and by Lemma 22, G_{7+5m} can be dominated by

$$(m^2 + 3m + 2) + 4 + 2m = m^2 + 5m + 6 = (m+1)^2 + 3(m+1) + 2$$

dominators. Thus

$$\gamma_{te}(G_{7+5m}) = \gamma_{te}(G_{2+5(m+1)}) \leq (m+1)^2 + 3(m+1) + 2,$$

which proves inequality (4). □

Corollary 24. *The following inequalities hold for n as specified*:

(1) $\gamma_{te}(G_n) \leq \frac{1}{24}(n^2 + 12n + 35)$, *where n is odd and* $n \geq 5$.

(2) $\gamma_{te}(G_n) \leq \frac{1}{25}(n^2 + 11n + 24)$, *where* $n \bmod 5 = 2$.

6. Conclusion

In this paper we have proven the values of $\gamma_e(G_n)$ for $n \leq 7$ and $\gamma_{te}(G_n)$ for $n \leq 5$. We also provided bounds on $\gamma_e(G_n)$ for $n \leq 15$ and $\gamma_{te}(G_n)$ for $n \leq 9$. We made use of the regular structure of triangular matchstick arrangement graphs to establish bounds on $\gamma_e(G_n)$ and $\gamma_{te}(G_n)$ for arbitrary n. The constructive methods we used produced inequalities that are significantly tighter than those found in [Dankelmann et al. 2009]. These techniques are particularly promising since the family of triangular grid graphs is just one family of graphs where G_{n+1} can be constructed from G_n by adding edges and vertices in a regularly defined manner. Similar methods could be used with recursively constructible families of graphs (studied in [Noy and Ribó 2004]) and regular n-gon grid graphs, such as square grid graphs, as in [Gonçalves et al. 2011].

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