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Matrix completions for linear matrix equations

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A matrix completion problem asks whether a partial matrix composed of specified and unspecified entries can be completed to satisfy a given property. This work focuses on determining which patterns of specified and unspecified entries correspond to partial matrices that can be completed to solve three different matrix equations. We approach this problem with two techniques: converting the matrix equations into linear equations and examining bases for the solution spaces of the matrix equations. We determine whether a particular pattern can be written as a linear combination of the basis elements. This work classifies patterns as admissible or inadmissible based on the ability of their corresponding partial matrices to be completed to satisfy the matrix equation. Our results present a partial or complete characterization of the admissibility of patterns for three homogeneous linear matrix equations.

1. Introduction

A matrix completion problem asks whether a partial matrix, one with some entries given and others freely chosen, can be completed to satisfy a desired property. In this work, we classify patterns for entries in a partial matrix so that the partial matrix can almost always be completed to satisfy certain linear matrix equations. We establish limits on the number of specified entries in patterns and on the locations of specified and unspecified entries.

Examples of matrix completion problems include determining completions for M-matrices and inverse M-matrices where the desired property is that a nonnegative partial matrix pattern of any order has an inverse M-matrix [Johnson and Smith 1996], where M-matrices are Z-matrices such that each eigenvalue of the matrix has positive real parts. A Z-matrix is one whose off-diagonal entries are less than or equal to zero. The inverse M-matrix completion problem can also be evaluated using a graph theoretic approach [Hogben 1998; 2000]. Other classical matrix

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completion problems involve completing partial Hermitian matrices and positive definite matrices to determine which partial positive definite matrices have a positive definite completion [Grone et al. 1984], while others look at completing TP or TN matrices with the goal of preserving low-rank [Johnson and Wei 2013]. A TP, or totally positive, matrix is a square matrix such that the determinant of each square submatrix (including minors) is positive. Equivalently, each of the eigenvalues of such a matrix is nonnegative. TN matrices are totally nonnegative matrices.

Another matrix completion problem is the titled completion problem, which asks if, given a conventional partial matrix, there exist values for the unspecified entries resulting in a conventional matrix that is either doubly nonnegative (DN) or completely positive (CP) [Drew et al. 2000]. Additionally, for partial matrices that are symmetric and have specified entries along the diagonal, it is known there is a P-matrix completion if and only if every given principal submatrix has a positive determinant [Johnson and Kroschel 1996]. Any 4×4 pattern also has a P-completion if it contains eight or fewer off-diagonal positions [DeAlba and Hogben 2000]. A graph theoretic approach can also be used to evaluate the P-completion problem [Hogben 2001]. There are also results for matrix completions involving the Euclidean distance. For example, for every partial distance matrix in $\mathbb{R}^k$ such that the graph of specified entries is chordal, there exists a completion to a distance matrix in $\mathbb{R}^k$ [Bakonyi and Johnson 1995]. These classic matrix completion problems determine the condition under which a partial matrix can be completed, so that the resulting matrix has a certain property. Only one matrix is involved in these problems, the partial matrix itself.

In this work, we determine if a partial matrix can be completed to satisfy certain matrix equations. In this case the admissibility of a pattern is relative to other matrices in the matrix equation. We focus on determining which patterns of specified and unspecified entries for partial matrices can almost always be completed to satisfy the following matrix equations: the skew-symmetric equation $AX - A^TX = 0$, the commutativity equation $AX - XA = 0$, and the skew-Lyapunov equation $AX - XA^T = 0$. It is not possible to, in general, solve these matrix completion problems for all matrices A. So, we look to solve the completion for almost all matrices A. That is, we assume A has a certain property that almost all matrices satisfy, and we show that any partial matrix can be completed for almost all of these "generic" A. In this work, we assume either A has distinct eigenvalues or is nonderogatory.

We use two approaches to classify patterns. The column space approach converts the matrix equations to linear equations and uses linearly independent columns to determine unspecified entry locations. The nullspace approach uses a basis of the solution space of a homogeneous matrix equation to determine specified entry locations. We classify patterns as admissible or inadmissible based on the ability or inability of corresponding partial matrices to be completed to satisfy the matrix equation for a "generic" matrix A.

We discuss the important ideas and definitions relevant to completions of matrix equations in Section 2. Sections 3 and 4 explain the two principle methods used for classifying partial matrix patterns: the column space and nullspace approaches. We apply the column space and nullspace approaches to the skew-symmetric, commutativity, and skew-Lyapunov equations in Section 5 to classify patterns for these equations.

2. Preliminaries

In this section, we define a partial matrix pattern, a partial matrix, a partial matrix completion, and the admissibility or inadmissibility of matrix patterns. We include relevant definitions and theorems from linear algebra, including the Kronecker product and the vec function.

Definition 2.1. An $n \times n$ *partial matrix pattern*

$$\alpha = \{(i_t, j_t) \mid 1 \le i_t, j_t \le n,\ t = 1, \dots, n\}$$

is a set of specified entry locations in an $n \times n$ matrix. For a partial matrix pattern α, the $n \times n$ rectangular array $\mathcal{X} = [x_{ij}]$ is an *α-partial matrix* if the only specified entries correspond to the locations in α.

A pattern describes locations in a matrix as specified or unspecified. A pattern becomes a partial matrix when the specified entry locations have values assigned.

Definition 2.2. A *completion* of an α-partial matrix $\mathcal{X} = [x_{ij}]$ is a matrix $\widehat{\mathcal{X}} = [\hat{x}_{ij}] \in M_n(\mathbb{R})$ in which $\hat{x}_{ij} = x_{ij}$ whenever $(i, j) \in \alpha$.

Throughout this paper, $\mathcal{X}$ will represent a partial matrix, and $\widehat{\mathcal{X}}$ will represent a completion of $\mathcal{X}$. For example, consider a 3×3 pattern $\alpha = \{(1, 1), (1, 3), (2, 2), (3, 2), (3, 3)\}$. The following are the pattern α, an α-partial matrix $\mathcal{X}$, and a completion $\widehat{\mathcal{X}}$:

$$\alpha = \begin{bmatrix} \# & \square & \# \\ \square & \# & \square \\ \square & \# & \# \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} 1 & x_{12} & 4 \\ x_{21} & 5 & x_{23} \\ x_{31} & 9 & 11 \end{bmatrix}, \quad \widehat{\mathcal{X}} = \begin{bmatrix} 1 & 15 & 4 \\ 13 & 5 & 19 \\ 2 & 9 & 11 \end{bmatrix}.$$

Definition 2.3. An $n \times n$ partial matrix pattern α is *admissible* for the matrix equation

$$A_1 X B_1 + A_2 X B_2 + \cdots + A_k X B_k = C$$

if for all α-partial matrices $\mathcal{X}$ there exists a completion $\widehat{\mathcal{X}}$ such that

$$A_1 \widehat{\mathcal{X}} B_1 + A_2 \widehat{\mathcal{X}} B_2 + \cdots + A_k \widehat{\mathcal{X}} B_k = C,$$

where $A_1, A_2, \dots, A_k, B_1, B_2, \dots, B_k, C \in M_n(\mathbb{R})$.

Because the admissibility of a pattern, in this work, depends on the fully specified matrices in the matrix equation, the problem of classifying admissible patterns becomes unwieldy without some restrictions on these matrices. In this paper, we restrict our attention to two large categories of matrices: nonderogatory matrices and matrices with distinct eigenvalues. These restrictions are necessary in order to calculate the maximum number of specified entry locations for the matrix equations we examine. Both nonderogatory and distinct eigenvalues are "generic" matrix properties in the sense that almost all matrices satisfy these properties.

There may be some versions of matrix equations for which a given partial matrix may not be completed to satisfy the particular instance of the matrix equation. For example with the 2×2 pattern $\alpha = \{(1, 2), (2, 2)\}$, not all α-partial matrices can be completed to commute with a diagonal matrix with distinct eigenvalues. However, the only matrices A for which not all of these α-partial matrices can be completed to commute with A are those matrices A with a 0 in the $(1, 2)$ position. The set of such matrices is a set of measure zero. So we say that α is admissible for the commutativity equation in general, which is to say that α is admissible for the matrix equation $AX - XA = 0$ for almost all "generic" A, which we show in Section 5.

In Sections 3 and 4, we construct conditions for the admissibility of patterns given matrices $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k, C$. For the matrix equations in Section 5, there is only one matrix A that is fully specified, so admissibility of a pattern for the general form of a matrix equation means any partial matrix can be completed for almost all "generic" A. Admissibility depends on the matrix equation as well; a pattern may be admissible for $AX - A^T X = 0$ but not admissible for $AX - XA^T = 0$. The matrix equation for which a pattern is admissible or inadmissible should be clear from context.

Definition 2.4. An admissible pattern α is *maximally admissible* if and only if $|\beta| \leq |\alpha|$ for every admissible pattern β.

In Section 4 we show the dimension of the solution space of the matrix equations gives the size of the maximally admissible patterns

Definition 2.5. The *Kronecker product* of $A = [a_{ij}] \in M_{m,n}(\mathbb{R})$ and $B = [b_{ij}] \in M_{p,q}(\mathbb{R})$ is denoted by $A \otimes B$ and is defined to be the block matrix

$$A \otimes B \equiv \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in M_{mp,nq}(\mathbb{R}).$$

Definition 2.6. Given $A = [a_{ij}] \in M_{m,n}(\mathbb{R})$, the function $\operatorname{vec} : M_{m,n}(\mathbb{R}) \to \mathbb{R}^{mn}$ is defined as

$$\operatorname{vec}(A) = [a_{11} \cdots a_{m1}\, a_{12} \cdots a_{m2} \cdots a_{1n} \cdots a_{mn}]^T.$$

The following theorem describes how to use the vec function and Kronecker product to transform linear matrix equations into linear equations.

Theorem 2.7 [Neudecker 1969]. *If A, B, $I \in M_n(\mathbb{R})$, where I is the identity matrix, then*

$$\operatorname{vec}(AB) = (I \otimes A)\operatorname{vec}(B) = (B^T \otimes I)\operatorname{vec}(A).$$

The following notation describes the submatrices corresponding to certain rows or columns.

Definition 2.8. If $A \in M_{m,n}(\mathbb{R})$ and $\varepsilon \subseteq \{1, \dots, m\}$, then $A[\varepsilon]$ is defined as the submatrix of A lying in the rows ε. The notation $A[s]$ may also be used to indicate the s-th row in A.

Definition 2.9. If $A \in M_{m,n}(\mathbb{R})$ and $\varepsilon \subseteq \{1, \dots, n\}$, then $A(\varepsilon)$ is defined as the submatrix of A lying in the columns ε. The notation $A(s)$ may also be used to indicate the s-th column in A.

For example, let $A \in M_3(\mathbb{R})$ and let $\varepsilon = \{1, 3\}$. If we have

$$A = \begin{bmatrix} 35 & 24 & 19 \\ 39 & 76 & 14 \\ 12 & 7 & 20 \end{bmatrix}, \text{ then } A[\varepsilon] = \begin{bmatrix} 35 & 24 & 19 \\ 12 & 7 & 20 \end{bmatrix} \text{ and } A(\varepsilon) = \begin{bmatrix} 35 & 19 \\ 39 & 14 \\ 12 & 20 \end{bmatrix}.$$

3. The column space approach

The vec function is a vector space isomorphism which is used to convert linear matrix equations into linear equations. In this section, we show that unspecified entry locations in maximally admissible patterns correspond to full rank submatrices of a certain matrix.

Let $A_1, \dots, A_k, B_1, \dots, B_k, C$ be $n \times n$ real matrices. Applying Theorem 2.7 to the matrix equation $A_1 X B_1 + \cdots + A_k X B_k = C$ yields the linear equation

$$(B_1^T \otimes A_1 + \cdots + B_k^T \otimes A_k)\operatorname{vec}(X) = \operatorname{vec}(C).$$

The solution space of $A_1 X B_1 + A_2 X B_2 + \cdots + A_k X B_k = \mathbb{0}$ is isomorphic to the nullspace of ${B_1}^T \otimes A_1 + {B_2}^T \otimes A_2 + \cdots + {B_k}^T \otimes A_k$. Throughout this section, we denote this $n^2 \times n^2$ matrix $B_1^T \otimes A_1 + \cdots + B_k^T \otimes A_k$ as K.

Lemma 3.1. *Let $A_1, \dots, A_k, B_1, \dots, B_k, C \in M_n(\mathbb{R})$ and α be an $n \times n$ partial matrix pattern. There exists a completion $\widehat{\mathcal{X}}$ of the α-partial matrix $\mathcal{X}$ satisfying $A_1 \widehat{\mathcal{X}} B_1 + \cdots + A_k \widehat{\mathcal{X}} B_k = C$ if and only if*

$$\operatorname{vec}(C) - \sum_{(i,j)\in\alpha} x_{ij} K(i + (j-1)n) \in \operatorname{span}\{K(i + (j-1)n) \mid (i, j) \notin \alpha\}.$$

Proof. The matrix equation $A_1\mathcal{X}B_1+\cdots+A_k\mathcal{X}B_k=C$ is equivalent to the equation $(B_1^T\otimes A_1+\cdots+B_k^T\otimes A_k)\operatorname{vec}(\mathcal{X})=\operatorname{vec}(C)$ where $\mathcal{X}$ has specified and unspecified entries. As above, let $K=B_1^T\otimes A_1+\cdots+B_k^T\otimes A_k$. Separating the specified and unspecified entries of $\mathcal{X}$ we rewrite this equation as

$$\sum_{(i,j)\notin\alpha} x_{ij}K(i+(j-1)n)+\sum_{(i,j)\in\alpha} x_{ij}K(i+(j-1)n)=\operatorname{vec}(C),$$

where x_{ij} are the entries in the partial matrix $\mathcal{X}$. In the first sum, the entries are unspecified while in the second sum, the entries x_{ij} are specified. Moving the specified entries to the right-hand side yields the linear equation

$$\sum_{(i,j)\notin\alpha} x_{ij}K(i+(j-1)n)=\operatorname{vec}(C)-\sum_{(i,j)\in\alpha} x_{ij}K(i+(j-1)n).$$

This is solvable if and only if the vector on the right-hand side lies in $\operatorname{span}\{K(i+(j-1)n \mid (i,j)\notin\alpha\}$. □

This lemma tells us precisely when a partial matrix can be completed to satisfy a linear matrix equation and describes the linear system that must be solvable in order to complete a partial matrix. If C is the zero matrix, then the condition for the existence of a completion simplifies to

$$\sum_{(i,j)\in\alpha} x_{ij}K(i+(j-1)n)\in\operatorname{span}\{K(i+(j-1)n)\mid (i,j)\notin\alpha\},$$

which can be answered by determining which sets of columns of K have rank equal to the rank of K. With some abuse of notation, let $K(\alpha)$ denote the submatrix of columns of K corresponding to specified entries and $K(\bar{\alpha})$ denote the submatrix of columns of K corresponding to unspecified entries.

Theorem 3.2. *Let $A_1,\ldots,A_k,B_1,\ldots,B_k,C\in M_n(\mathbb{R})$, α be an $n\times n$ partial matrix pattern, and $K=B_1^T\otimes A_1+\cdots+B_k^T\otimes A_k$. Then, the following statements are equivalent*:

(1) *For a given α-partial matrix $\mathcal{X}$ and any $C\in M_n(\mathbb{R})$ such that* $\operatorname{vec}(C)\in\operatorname{span}\{K(1),\ldots,K(n^2)\}$, *there exists a completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ such that $A_1\widehat{\mathcal{X}}B_1+\cdots+A_k\widehat{\mathcal{X}}B_k=C$.*

(2) $\operatorname{rank}(K)=\operatorname{rank}(K(\bar{\alpha}))$.

Proof. Assuming (1), by Lemma 3.1 $\operatorname{vec}(C)-\sum_{(i,j)\in\alpha}x_{ij}K(i+(j-1)n)$ is in the span of $\{K(i+(j-1)n)\mid(i,j)\notin\alpha\}$ for all $\operatorname{vec}(C)$ in the span of the columns of K. Since it is possible to choose C so that it is any vector in the column space of K, it follows that the column space of K is contained in the column space of $K(\bar{\alpha})$, and $\operatorname{rank}(K)=\operatorname{rank}(K(\bar{\alpha}))$, proving the second statement.

Assuming (2), for any $\mathrm{vec}(C)$ in the column space of $K(\bar{\alpha})$ and any α-partial matrix $\mathcal{X}$, the column space $K(\bar{\alpha})$ is the column space of K, since $K(\bar{\alpha})$ is contained in the column space of K and both matrices have the same rank. In particular, the column space of $K(\alpha)$ is contained in the column space of $K(\bar{\alpha})$. Then for any $\mathrm{vec}(C)$ in the column space of K, $\mathrm{vec}(C)$ lies in the column space of $K(\bar{\alpha})$, and by Lemma 3.1 there exists a completion $\widehat{\mathcal{X}}$ of the α-partial matrix $\mathcal{X}$ such that $A_1\widehat{\mathcal{X}}B_1 + \cdots + A_k\widehat{\mathcal{X}}B_k = C$, establishing the first statement. □

In this paper, the specific matrix equations of interest are homogeneous. The following corollary gives the condition that we use to classify patterns for this column space approach: the rank of the columns of K corresponding to unspecified entries must equal the rank of K. That is, the sets of columns of K with full rank correspond to unspecified entry locations in admissible patterns.

Corollary 3.3. *Let $A_1, \ldots, A_k, B_1, \ldots, B_k \in M_n(\mathbb{R})$, α be an $n \times n$ matrix pattern, and $K = B_1^T \otimes A_1 + \cdots + B_k^T \otimes A_k$. Then, the following statements are equivalent*:

(1) *The matrix pattern α is admissible for the matrix equation $A_1XB_1 + \cdots + A_kXB_k = 0$.*

(2) $\mathrm{rank}(K) = \mathrm{rank}(K(\bar{\alpha}))$.

Proof. This follows from the definition of admissibility, Theorem 3.2, and the fact that $\mathbb{0}$ is in the span of the columns of K. □

Corollary 3.3 gives the size of a maximally admissible pattern, namely $n^2 - \mathrm{rank}(K)$.

Corollary 3.4. *Let $A_1, \ldots, A_k, B_1, \ldots, B_k \in M_n(\mathbb{R})$ and $K = B_1^T \otimes A_1 + \cdots + B_k^T \otimes A_k$. If α is an admissible $n \times n$ partial matrix pattern for the matrix equation $A_1XB_1 + \cdots + A_kXB_k = 0$,*

$$|\alpha| \leq n^2 - \mathrm{rank}(K).$$

Proof. If $|\alpha| > n^2 - \mathrm{rank}(K)$, then the number of columns corresponding to unspecified entries is strictly less than the $\mathrm{rank}(K)$ and condition (2) of Corollary 3.3 can never be satisfied. □

Given a linear matrix equation, the patterns α that are admissible are exactly the patterns that set unspecified entries against a set of columns of K whose span is equal to the span of all the columns of K. With the column space approach we think of the specified entries of X as removing certain columns from K. We then look at the submatrix formed by the remaining columns of K and determine its rank. An α-partial pattern is admissible if the rank of the columns of K corresponding to unspecified entries is equal to the rank of K.

The following lemmas establish two basic properties of matrix patterns: subpatterns of admissible patterns are admissible and patterns that contain inadmissible patterns are inadmissible.

Lemma 3.5. *Let α and β be partial matrix patterns such that α is admissible for the matrix equation $A_1XB_1 + \cdots + A_kXB_k = 0$, where $A_1, \ldots, A_k, B_1, \ldots, B_k \in M_n(\mathbb{R})$, and let $K = B_1^T \otimes A_1 + \cdots + B_k^T \otimes A_k$. If $\beta \subseteq \alpha$, then β is admissible for the matrix equation $A_1XB_1 + \cdots + A_kXB_k = 0$.*

Proof. By Corollary 3.3, α is admissible if and only if $\operatorname{rank}(K(\bar{\alpha})) = \operatorname{rank}(K)$. Since $\beta \subseteq \alpha$, $\operatorname{rank}(K(\bar{\alpha})) \leq \operatorname{rank}(K(\bar{\beta}))$. This forces $\operatorname{rank}(K(\bar{\beta})) = \operatorname{rank}(K)$, and β is admissible for the matrix equation $A_1XB_1 + \cdots + A_kXB_k = 0$. □

Lemma 3.6. *Let α and β be partial matrix patterns such that α is inadmissible for the matrix equation $A_1XB_1 + \cdots + A_kXB_k = 0$, where $A_1, \ldots, A_k, B_1, \ldots, B_k \in M_n(\mathbb{R})$, and let $K = B_1^T \otimes A_1 + \cdots + B_k^T \otimes A_k$. If $\alpha \subseteq \beta$, then β is also inadmissible for the matrix equation $A_1XB_1 + \cdots + A_kXB_k = 0$.*

Proof. By Corollary 3.3, α is admissible if and only if $\operatorname{rank}(K(\bar{\alpha})) = \operatorname{rank}(K)$. Since α is inadmissible, $\operatorname{rank}(K(\bar{\alpha})) < \operatorname{rank}(K)$. Since $\alpha \subseteq \beta$, $\operatorname{rank}(K(\bar{\beta})) \leq \operatorname{rank}(K(\bar{\alpha}))$. This forces $\operatorname{rank}(K(\bar{\beta})) < \operatorname{rank}(K)$, and β is also inadmissible for the matrix equation $A_1XB_1 + \cdots + A_kXB_k = 0$. □

4. The nullspace approach

In this section we develop a second criterion for admissible patterns for the homogeneous matrix equation $A_1XB_1 + \cdots + A_kXB_k = 0$. We show that if the specified entry locations of a pattern correspond to full rank submatrices of a matrix constructed from a basis of the solution space of the homogeneous matrix equation, the pattern is admissible. We also construct a basis for the solution space of two special cases of this matrix equation

Nullspace criterion. Given a partial matrix, we need to determine if the specified entries of the partial matrix can be written as a linear combination of basis elements for the solution space of $A_1XB_1 + \cdots + A_kXB_k = 0$. Let $\{V_1, V_2, \ldots, V_n\}$ be a basis for the solution space, then $\{\operatorname{vec}(V_1), \operatorname{vec}(V_2), \ldots, \operatorname{vec}(V_n)\}$ is a basis for the nullspace of $B_1^T \otimes A_1 + \cdots + B_k^T \otimes A_k$. Throughout this paper we denote this matrix $[\operatorname{vec}(V_1)\ \operatorname{vec}(V_2)\ \cdots\ \operatorname{vec}(V_n)]$ as N.

The partial matrix has a completion if there exist scalars $c_1, \ldots, c_n$ such that the specified entries of $\mathcal{X}$ satisfy

$$\mathcal{X} = c_1V_1 + c_2V_2 + \cdots + c_nV_n.$$

Applying the vec function to this equation yields

$$\operatorname{vec}(\mathcal{X}) = [\operatorname{vec}(V_1)\ \operatorname{vec}(V_2)\ \cdots\ \operatorname{vec}(V_n)]\boldsymbol{c} = N\boldsymbol{c},$$

where $\boldsymbol{c} = [c_1\, c_2\, \cdots\, c_n]^T$. Only the rows in $\mathrm{vec}(\mathcal{X})$ which are specified are of interest because the unspecified entries can be freely chosen. Let $\varepsilon = \{i + (j-1)n \mid (i, j) \in \alpha\}$, the set of integer values corresponding to the rows of $\mathrm{vec}(\mathcal{X})$ which contain specified entries. Solving the equation

$$\mathrm{vec}(\mathcal{X})[\varepsilon] = \big[\mathrm{vec}(V_1)[\varepsilon]\ \mathrm{vec}(V_2)[\varepsilon]\ \cdots\ \mathrm{vec}(V_n)[\varepsilon]\big] = N[\varepsilon]\boldsymbol{c}$$

is equivalent to determining if the specified entries of $\mathcal{X}$ can be written as a linear combination of basis elements to the solution space of our linear equation.

The following theorem describes the nullspace condition for admissibility: the submatrix of rows of N corresponding to specified entries must have rank at least equal to the number of specified entries in $\mathcal{X}$

Theorem 4.1. *Let α be an $n \times n$ partial matrix pattern and $\{V_1, V_2, \ldots, V_\ell\}$ be a basis for the solution space of the matrix equation $A_1 X B_1 + \cdots + A_k X B_k = 0$. The matrix pattern α is admissible for this matrix equation if and only if* $\mathrm{rank}(N[\varepsilon]) \geq |\alpha|$, *where $\varepsilon = \{i + (j-1)n \mid (i, j) \in \alpha\}$ and*

$$N[\varepsilon] = \big[\mathrm{vec}(V_1)[\varepsilon]\ \mathrm{vec}(V_2)[\varepsilon]\ \cdots\ \mathrm{vec}(V_\ell)[\varepsilon]\big].$$

Proof. The matrix completion problem is equivalent to determining if there exists a solution to the linear equation $\mathrm{vec}(X)[\varepsilon] = N[\varepsilon]\boldsymbol{c}$. $N[\varepsilon]$ is an $n \times |\alpha|$ matrix, so this equation is solvable for all $\mathrm{vec}(X)[\varepsilon]$ if and only if $\mathrm{rank}(N[\varepsilon]) \geq |\alpha|$. If so, there exists a completion $\widehat{\mathcal{X}}$ for any $\mathcal{X}$ satisfying $A_1 \widehat{\mathcal{X}} B_1 + \cdots + A_k \widehat{\mathcal{X}} B_k = 0$.

If $\mathrm{rank}(N[\varepsilon]) < |\alpha|$, then $\mathrm{vec}(X)[\varepsilon] = N[\varepsilon]\boldsymbol{c}$ has a solution if $\mathrm{vec}(X)[\varepsilon]$ lies in the span of the columns of $N[\varepsilon]$. Since $\mathrm{rank}(N[\varepsilon]) < |\alpha|$ and $\mathrm{vec}(X)[\varepsilon]$ is an $|\alpha|$-dimensional vector, there exists an α-partial matrix $\mathcal{X}$ such that $\mathrm{vec}(X)[\varepsilon]$ does not lie in the span of the columns of $N[\varepsilon]$. Hence for this α-partial matrix $\mathcal{X}$ there does not exist a completion of $A_1 X B_1 + \cdots + A_k X B_k = 0$. Since this α does not have a completion for all α-partial matrices, α is inadmissible. □

For maximal patterns, the condition for admissibility is that the number of specified entries in $\mathcal{X}$ must equal the rank of $N[\varepsilon]$.

Corollary 4.2. *Let α be an $n \times n$ partial matrix pattern for the matrix equation $A_1 X B_1 + \cdots + A_k X B_k = 0$ and let $\{V_1, V_2, \ldots, V_\ell\}$ be a basis for the solution space of the given matrix equation. An admissible pattern α is maximal if and only if $|\alpha| = \ell$.*

Proof. First assume that the admissible pattern is maximally admissible to show that the number of specified entries equals the dimension of the solution space. For the pattern to be admissible, the rank of $N[\varepsilon]$ must be greater than $|\alpha|$, but also must not exceed the number of columns in $N[\varepsilon]$. Then, the greatest possible value for the rank of $N[\varepsilon]$ is ℓ, namely the dimension of the solution space.

We next assume that the number of specified entries equals the dimension of the solution space to show that the admissible pattern is maximal. Then, since the dimension of the solution space is ℓ, $|\alpha| = \ell$. Since $N[\varepsilon]$ has ℓ columns and by Theorem 4.1, $|\alpha| \leq \operatorname{rank}(N[\varepsilon]) \leq \ell$, the rank of $N[\varepsilon]$ must equal ℓ. Therefore, α is maximally admissible because the dimension of α is as large as possible while maintaining admissibility. □

Construction of bases for the nullspace. We construct a basis for the solution space of the matrix equation $AX + XB = 0$ using eigenvectors of the matrices A and B. This basis is used to classify patterns for the commutativity equation $AX - XA = 0$ and the skew-Lyapunov equation $AX - XA^T = 0$ in Section 5.

Theorem 4.3 [Horn and Johnson 1991]. *Let $A \in M_n(\mathbb{R})$ and $B \in M_m(\mathbb{R})$ be given. If λ is an eigenvalue of A and $\mathbf{x} \in \mathbb{C}^n$ is a corresponding eigenvector of A, and if μ is an eigenvalue of B and $\mathbf{y} \in \mathbb{C}^m$ is a corresponding eigenvector of B, then $\lambda + \mu$ is an eigenvalue of $(I_m \otimes A) + (B \otimes I_n)$, and $\mathbf{y} \otimes \mathbf{x} \in \mathbb{C}^{nm}$ is a corresponding eigenvector. Every eigenvalue of $(I_m \otimes A) + (B \otimes I_n)$ arises as such a sum of eigenvalues of A and B, and $I_m \otimes A$ commutes with $B \otimes I_n$. If the set of eigenvalues of A equals $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and the set of eigenvalues of B equals $\{\mu_1, \mu_2, \ldots, \mu_n\}$, then the set of eigenvalues of $(I_m \otimes A) + (B \otimes I_n)$ equals $\{\lambda_i + \mu_j \mid i = 1, \ldots, n,\ j = 1, \ldots, m\}$ (including algebraic multiplicities in all three cases).*

We use the lemma below to construct bases for $I \otimes A - A^T \otimes I$ and $I \otimes A - A \otimes I$.

Lemma 4.4. *If $\{\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n\}$ and $\{\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^n\}$ are each linearly independent sets of nonzero vectors, then $\{\boldsymbol{y}^1 \otimes \boldsymbol{x}^1, \boldsymbol{y}^2 \otimes \boldsymbol{x}^2, \ldots, \boldsymbol{y}^n \otimes \boldsymbol{x}^n\}$ is linearly independent.*

Proof. Let

$$\boldsymbol{x}^i = [x_1^i\ x_2^i\ \cdots\ x_n^i]^T \quad \text{and} \quad \boldsymbol{y}^i = [y_1^i\ y_2^i\ \cdots\ y_n^i]^T.$$

By the definition of the Kronecker product,

$$\boldsymbol{y}^i \otimes \boldsymbol{x}^i = [y_1^i \boldsymbol{x}^i\ y_2^i \boldsymbol{x}^i\ \cdots\ y_n^i \boldsymbol{x}^i]^T.$$

We want to show that

$$a_1(\boldsymbol{y}^1 \otimes \boldsymbol{x}^1) + a_2(\boldsymbol{y}^2 \otimes \boldsymbol{x}^2) + \cdots + a_n(\boldsymbol{y}^n \otimes \boldsymbol{x}^n) = \mathbb{0} \quad \text{only when } a_1 = a_2 = \cdots = a_n = 0.$$

Using the Kronecker product definition, this can be rewritten as

$$\begin{gathered}
(a_1 y_1^1)\boldsymbol{x}^1 + (a_2 y_1^2)\boldsymbol{x}^2 + \cdots + (a_n y_1^n)\boldsymbol{x}^n = \mathbb{0}, \\
(a_1 y_2^1)\boldsymbol{x}^1 + (a_2 y_2^2)\boldsymbol{x}^2 + \cdots + (a_n y_2^n)\boldsymbol{x}^n = \mathbb{0}, \\
\vdots \\
(a_1 y_n^1)\boldsymbol{x}^1 + (a_2 y_n^2)\boldsymbol{x}^2 + \cdots + (a_n y_n^n)\boldsymbol{x}^n = \mathbb{0}.
\end{gathered}$$

Since $\boldsymbol{x}^1, \boldsymbol{x}^2, \dots, \boldsymbol{x}^n$ are linearly independent,

$$a_1\boldsymbol{y}^1 = \mathbb{0},\ a_2\boldsymbol{y}^2 = \mathbb{0},\ \dots,\ a_n\boldsymbol{y}^n = \mathbb{0}.$$

Since $\boldsymbol{y}^1, \boldsymbol{y}^2, \dots, \boldsymbol{y}^n$ are nonzero vectors, there exists at least one nonzero entry in each vector. This implies that $a_1 = a_2 = \cdots = a_n = 0$. Therefore $\{\boldsymbol{y}^1 \otimes \boldsymbol{x}^1, \boldsymbol{y}^2 \otimes \boldsymbol{x}^2, \dots, \boldsymbol{y}^n \otimes \boldsymbol{x}^n\}$ is linearly independent. □

Remark 4.5. If we further assume that A has distinct eigenvalues, then the nullities of $(I \otimes A) - (A^T \otimes I)$ and $(I \otimes A) - (A \otimes I)$ are both n (see Section 5). This and Lemma 4.4 imply that $\{\boldsymbol{y}^1 \otimes \boldsymbol{x}^1, \boldsymbol{y}^2 \otimes \boldsymbol{x}^2, \dots, \boldsymbol{y}^n \otimes \boldsymbol{x}^n\}$ is a basis for the nullspace of $(I \otimes A) - (A^T \otimes I)$, where $\{\boldsymbol{x}_1, \dots \boldsymbol{x}_n\}$ is a basis of eigenvectors for A corresponding to eigenvalues $\lambda_1, \dots, \lambda_n$ and $\{\boldsymbol{y}_1, \dots \boldsymbol{y}_n\}$ is a basis of eigenvectors for $-A^T$ corresponding to eigenvalues $-\lambda_1, \dots, -\lambda_n$. Similarly $\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^1, \boldsymbol{x}^2 \otimes \boldsymbol{x}^2, \dots, \boldsymbol{x}^n \otimes \boldsymbol{x}^n\}$ is a basis for the nullspace of $(I \otimes A) - (A \otimes I)$.

5. Admissible patterns for certain matrix equations

In this section, we apply the column space and nullspace approaches to three matrix equations: the skew-symmetric equation, the commutativity equation, and the skew-Lyapunov equation. For the skew-symmetric equation, we completely characterize admissible patterns. For the other two matrix equations we classify certain patterns as admissible or inadmissible.

For the skew-symmetric equation, $AX - A^T X = 0$, Theorem 5.2 states that a maximal pattern is admissible if and only if it contains one specified entry in each column of an α-partial matrix $\mathcal{X}$. We also show all admissible patterns are subpatterns of maximal patterns.

For the commutativity equation, $AX - XA = 0$, Theorem 5.8 states that maximal patterns with no diagonal entries specified are inadmissible. Theorem 5.9 states that patterns in which all of the specified entries are in the same row or in the same column are admissible.

For the skew-Lyapunov equation, $AX - XA^T = 0$, Theorem 5.12 states that a pattern is admissible if all of the specified entries reside in the i-th row or column without (i, j) and (j, i) both being in the pattern for any j. Corollary 5.15 states that if any pattern contains two specified entries which are located across the main diagonal from each other, then the pattern is inadmissible.

Patterns for the skew-symmetric equation. Applying the vec function to $AX - A^T X = 0$ yields the linear equation

$$(I \otimes (A - A^T))\,\mathrm{vec}(X) = \mathbb{0}.$$

The matrix $A - A^T$ is skew-symmetric, so $(I \otimes (A - A^T))$ is a block diagonal matrix and is skew-symmetric. We denote $I \otimes (A - A^T)$ as S_A.

Since $A - A^T$ is skew-symmetric, it is also diagonalizable and its eigenvalues are purely imaginary or zero [Rukmangadachari 2010]. The rank of $A - A^T$ is dependent upon whether n is odd or even.

In this section, we assume that $A - A^T$ has maximum rank. So $\operatorname{rank}(A - A^T) = n$ if n is even, and $\operatorname{rank}(A - A^T) = n - 1$ if n is odd. The set of matrices A with which $\operatorname{rank}(A - A^T)$ is strictly less that the maximum possible rank is a set of measure zero. So in this section our "generic" property of A is that $\operatorname{rank}(A - A^T)$ is maximal.

Since S_A is a block-diagonal matrix consisting of the matrix $A - A^T$ down the main diagonal, $\operatorname{rank}(S_A) = n \cdot \operatorname{rank}(A - A^T)$. By Corollary 4.2 maximally admissible patterns for S_A contain n specified entries for n odd. Since the nullity of S_A is zero when n is even, only the empty pattern, the pattern with no specified entries is admissible.

From this point forward, we only consider the case when n is odd. We first construct a basis for the nullspace of S_A in order to apply the nullspace approach.

Lemma 5.1. *Let $A \in M_n(\mathbb{R})$ with n odd and* $\operatorname{rank}(A - A^T) = n - 1$, *and let $\{\boldsymbol{v}\}$ be a basis for the nullspace of $A - A^T$. If n is odd, then*

$$\mathcal{B} = \big\{[\boldsymbol{v}\ \mathbb{0}\ \cdots\ \mathbb{0}], [\mathbb{0}\ \boldsymbol{v}\ \mathbb{0}\ \cdots\ \mathbb{0}], \ldots, [\mathbb{0}\ \cdots\ \mathbb{0}\ \boldsymbol{v}]\big\}$$

is a basis for the solution space of $AX - A^T X = 0$.

Proof. Each element of $\mathcal{B}$ is a solution to $AX - A^T X = 0$. The matrices in $\mathcal{B}$ are clearly linearly independent. The dimension of the solution space of $AX - A^T X = 0$ is n, and $\mathcal{B}$ contains n elements. So $\mathcal{B}$ is a basis for the solution space of $AX - A^T X = 0$. □

We now consider maximally admissible patterns for the skew-symmetric equation, and determine whether they are admissible or inadmissible.

Theorem 5.2. *Let α be an $n \times n$ partial matrix pattern with $|\alpha| = n$, and let n be odd. The matrix pattern α is maximally admissible for the matrix equation $AX - A^T X = 0$ for almost all A with* $\operatorname{rank}(A - A^T) = n - 1$ *if and only if $\alpha = \{(i_1, 1), (i_2, 2), \ldots, (i_n, n)\}$, where $1 \le i_k \le n$.*

Proof. We first show that if α is admissible, then $\alpha = \{(i_1, 1), (i_2, 2), \ldots, (i_n, n)\}$, where $1 \le i_k \le n$. We proceed by contraposition, assuming that

$$\alpha \neq \{(i_1, 1), (i_2, 2), \ldots, (i_n, n)\}$$

to show that α is inadmissible. By Lemma 5.1, a basis for the solution space of $(A - A^T)X = 0$ is $\{V_1, \ldots, V_n\}$ where the i-th column of V_i is $\boldsymbol{v}$ and all other columns only contain zeros. Following the nullspace approach, the matrix completion problem is equivalent to solving

$$\mathrm{vec}(\mathcal{X})[\varepsilon] = \big[\mathrm{vec}(V_1)[\varepsilon]\ \mathrm{vec}(V_2)[\varepsilon]\ \cdots\ \mathrm{vec}(V_n)[\varepsilon]\big]\boldsymbol{c},$$

where $\boldsymbol{c} = [c_1\, c_2\ \cdots\ c_n]^T$ and $\varepsilon = \{i + (j-1)n \mid (i, j) \in \alpha\}$. Let N be the matrix containing the column vectors of the basis elements, so

$$N[\varepsilon] = \big[\mathrm{vec}(V_1)[\varepsilon]\ \mathrm{vec}(V_2)[\varepsilon]\ \cdots\ \mathrm{vec}(V_n)[\varepsilon]\big].$$

From our assumption, there exists at least one column in $\mathcal{X}$ that does not have a specified entry. Without loss of generality, assume that the k-th column in $\mathcal{X}$ does not have a specified entry. Any row in $\mathrm{vec}(V_k)$ that contains an element of $\boldsymbol{v}$ will be excluded when $\mathrm{vec}(V_k)$ is restricted to $\mathrm{vec}(V_i)[\varepsilon]$. We have, then, that $\mathrm{vec}(V_k)[\varepsilon] = \mathbb{0}$. The rank of $N[\varepsilon]$ is therefore strictly less than $|\alpha|$, and therefore α is inadmissible.

We next show that if $\alpha = \{(i_1, 1), (i_2, 2), \ldots, (i_n, n)\}$, where $1 \le i_k \le n$, then α is admissible. Following the nullspace approach as above, this completion problem is equivalent to

$$\begin{aligned}\mathrm{vec}(X)[\varepsilon] &= \big[\mathrm{vec}(V_1)[\varepsilon]\ \mathrm{vec}(V_2)[\varepsilon]\ \cdots\ \mathrm{vec}(V_n)[\varepsilon]\big]\boldsymbol{c}\\ &= \begin{bmatrix} v_{i_1} & 0 & \ldots & 0\\ 0 & v_{i_2} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \ldots & 0 & v_{i_n}\end{bmatrix}\begin{bmatrix} c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix},\end{aligned}$$

where v_{i_ℓ} are entries in $\boldsymbol{v}$. For almost all A, $v_{i_\ell} \neq 0$ for all $1 \le \ell \le n$, and the rank of $N[\varepsilon]$ is n. This means that the columns of $N[\varepsilon]$ spans $\mathbb{R}^n$, and therefore any values that can be specified for $\mathcal{X}$ are in the span of the columns of $N[\varepsilon]$. So any α-partial matrix for the α pattern can be completed to satisfy the skew-symmetric equation, and $\alpha = \{(i_1, 1), (i_2, 2), \ldots, (i_n, n)\}$ is admissible. □

This tells us that α is maximally admissible if and only if α contains exactly one specified entry in each column. Again "almost all" is used to say that these patterns are admissible for the given matrix equation, with A satisfying the given conditions, except for a set of matrices A of measure zero. In this case, we can be more specific. The set of matrices that these patterns are not admissible for are those matrices A for which the vector $\boldsymbol{v}$ has zero entries, where $\boldsymbol{v}$ is the basis for the nullspace of $A - A^T$. The following theorem shows that admissible patterns appear as subpatterns of maximal patterns.

Theorem 5.3. *Let $A \in M_n(\mathbb{R})$ be nonderogatory with n odd. A pattern β is admissible for the matrix equation $AX - A^T X = 0$ for almost all A with* $\mathrm{rank}(A - A^T) = n - 1$ *if and only if $\beta \subseteq \{(i_1, 1), (i_2, 2), \ldots, (i_n, n)\}$ with $1 \le i_k \le n$.*

Proof. By Theorem 5.2, $\alpha = \{(i_1, 1), (i_2, 2), \dots, (i_n, n)\}$ is admissible. If $\beta \subseteq \alpha$ then by Lemma 3.5 β is also admissible.

If $\beta \nsubseteq \{(i_1, 1), (i_2, 2), \dots, (i_n, n)\}$ then $\{(i, k), (j, k)\} \subseteq \beta$ for some $i \neq j$. Then $\varepsilon = \{i + (k-1)n,\ j + (k-1)n\}$ and

$$\begin{bmatrix}\operatorname{vec}(V_1)[\varepsilon] & \operatorname{vec}(V_2)[\varepsilon] & \cdots & \operatorname{vec}(V_n)[\varepsilon]\end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & v_i & 0 & \cdots & 0 \\ 0 & \cdots & 0 & v_j & 0 & \cdots & 0 \end{bmatrix}.$$

This matrix does not have full rank, so the pattern $\{(i, k), (j, k)\}$ is inadmissible by the nullspace criterion. Since $\{(i, k), (j, k)\} \subseteq \beta$, β is inadmissible by Lemma 3.6. □

Finally we give formulas for the number of maximally admissible and admissible patterns.

Corollary 5.4. *For $A \in M_n(\mathbb{R})$ where n is odd and* $\operatorname{rank}(A - A^T) = n - 1$, *the number of maximally admissible patterns for the skew-symmetric equation is n^n.*

Proof. From Theorem 5.2, if α is admissible for the skew-symmetric equation, each column in $\mathcal{X}$ has one specified entry. Each of the n columns has n possible locations where an entry can be specified, so the total number of admissible patterns is n^n. □

Corollary 5.5. *For $A \in M_n(\mathbb{R})$ where n is odd and* $\operatorname{rank}(A - A^T) = n - 1$, *the number of admissible patterns for the skew-symmetric equation is* $(1 + n)^n$.

Proof. We have by Theorem 5.3 that if $\beta \subseteq \alpha$, where $\alpha = \{(i_1, 1), (i_2, 2), \dots, (i_n, n)\}$ and $1 \leq i_k \leq n$, then β is admissible for the skew-symmetric equation.

Suppose β has i specified entries, there are $\binom{n}{i}$ choices for columns and n choices within each column. Summing over i and using the binomial theorem, the total number of admissible patterns is

$$\sum_{i=0}^{n} \binom{n}{i} n^i = (1 + n)^n. \qquad \square$$

Patterns for the commutativity equation. We next classify patterns for the commutativity equation, $AX - XA = 0$. The conditions under which two matrices commute are well known, but there still are interesting questions that can be asked about matrix commutativity with regard to partial matrix completions [Horn and Johnson 1991]. We are interested in finding answers to the following: if given a partial matrix pattern α and a matrix A, what are the conditions on the specific entries in an α-partial matrix $\mathcal{X}$ so that $\mathcal{X}$ has a completion that commutes with A? Which patterns α allow any α-partial matrix $\mathcal{X}$ to be completed to commute with almost all $A \in M_n(\mathbb{R})$?

We use the column space approach to convert the matrix equation into a linear equation. The vec function applied to the commutativity equation yields $[(I \otimes A) - (A^T \otimes I)] \operatorname{vec}(X) = \mathbb{0}$. We denote $(I \otimes A) - (A^T \otimes I)$ as Ω_A.

Lemma 5.6 [Horn and Johnson 1991]. *If $A \in M_n(\mathbb{R})$ has k eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, then the dimension of the nullspace of Ω_A is*

$$\sum_{i=1}^{k} m_a(\lambda_i) m_g(\lambda_i),$$

where $m_a(\lambda)$, $m_g(\lambda)$ are the algebraic and geometric multiplicities of λ respectively.

Lemma 5.7 [Horn and Johnson 1991]. *For $A \in M_n(\mathbb{R})$, the dimension of the commutant of A is at least n, and the dimension of the commutant is equal to n if and only if A is nonderogatory.*

Because the solutions to the commutativity equation are exactly the elements of the commutant, the rank of Ω_A is $n^2 - n$ if and only if A is nonderogatory. Maximal patterns for the commutativity equation contain at most n specified entries for A nonderogatory.

We use two different bases for the nullspace of Ω_A to classify admissible and inadmissible patterns. If A is nonderogatory, then only polynomials in A commute with A [Horn and Johnson 1985]. So one basis for the null space of Ω_A is

$$\{\text{vec}(I), \text{vec}(A), \text{vec}(A^2), \ldots, \text{vec}(A^{n-1})\}.$$

By Remark 4.5 if A has distinct eigenvalues then $\{\boldsymbol{y}^1 \otimes \boldsymbol{x}^1, \boldsymbol{y}^2 \otimes \boldsymbol{x}^2, \ldots, \boldsymbol{y}^n \otimes \boldsymbol{x}^n\}$ is also a second basis for the nullspace where $\{\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n\}$ is a set of eigenvectors for A and $\{\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^n\}$ is a set of eigenvectors for $-A^T$ corresponding to eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_n\}$ respectively.

We first show maximally admissible patterns must have a diagonal entry specified.

Theorem 5.8. *Let α be an $n \times n$ partial matrix pattern with $|\alpha| = n$ and $A \in M_n(\mathbb{R})$ be nonderogatory. If $(i, i) \notin \alpha$ for all $1 \le i \le n$, then any α-partial matrix $\mathcal{X}$ is inadmissible for the matrix equation $AX - XA = 0$.*

Proof. Using the nullspace approach and the basis $\{\text{vec}(I), \text{vec}(A), \ldots, \text{vec}(A^{n-1})\}$, the partial matrix completion problem for the commutativity equation is equivalent to solving

$$\text{vec}(\mathcal{X})[\varepsilon] = [\text{vec}(I)[\varepsilon] \;\; \text{vec}(A)[\varepsilon] \;\; \text{vec}(A^2)[\varepsilon] \; \ldots \; \text{vec}(A^{n-1})[\varepsilon]]\boldsymbol{c},$$

where $\varepsilon = \{i + (j-1)n \mid (i, j) \in \alpha\}$.

From our assumption, we have that $(i, i) \notin \alpha$ for all $1 \le i \le n$. That is, no entries along the main diagonal are specified. Then, any row in $\text{vec}(I)$ that contains a 1 will be excluded in $\text{vec}(I)[\varepsilon]$, so $\text{vec}(I)[\varepsilon] = \mathbb{0}$.

This means that $\text{rank}(N[\varepsilon]) < n = |\alpha|$. By Theorem 4.1, α is inadmissible. □

We now partially classify maximally admissible patterns for the commutativity equation.

Theorem 5.9. *Let α be an $n \times n$ partial matrix pattern with $|\alpha| = n$. If $\alpha = \{(i, 1), (i, 2), \dots, (i, n)\}$ or $\alpha = \{(1, j), (2, j), \dots, (n, j)\}$ where $1 \le i, j \le n$, then α is maximally admissible for the commutativity equation $AX - XA = 0$ for almost all A, where all A have distinct eigenvalues.*

Proof. By Remark 4.5, $\{\boldsymbol{y}^1 \otimes \boldsymbol{x}^1, \boldsymbol{y}^2 \otimes \boldsymbol{x}^2, \dots, \boldsymbol{y}^n \otimes \boldsymbol{x}^n\}$ is a basis for the nullspace of Ω_A where $\{\boldsymbol{x}^1, \boldsymbol{x}^2, \dots, \boldsymbol{x}^n\}$ is a set of eigenvectors for A and $\{\boldsymbol{y}^1, \boldsymbol{y}^2, \dots, \boldsymbol{y}^n\}$ is a set of eigenvectors for $-A^T$.

Following the nullspace approach, the commutativity matrix completion problem is equivalent to solving

$$\begin{aligned}\operatorname{vec}(\mathcal{X})[\varepsilon] &= [\operatorname{vec}(\boldsymbol{y}^1 \otimes \boldsymbol{x}^1)[\varepsilon]\ \operatorname{vec}(\boldsymbol{y}^2 \otimes \boldsymbol{x}^2)[\varepsilon]\ \dots\ \operatorname{vec}(\boldsymbol{y}^n \otimes \boldsymbol{x}^n)[\varepsilon]]\boldsymbol{c} \\ &= [x_i^1 \boldsymbol{y}^1\ x_i^2 \boldsymbol{y}^2\ \dots\ x_i^n \boldsymbol{y}^n]\boldsymbol{c},\end{aligned}$$

where $\boldsymbol{c} = [c_1\, c_2\, \dots\, c_n]^T$ and $\varepsilon = \{i + (j-1)n \mid (i, j) \in \alpha\}$.

Since $\{\boldsymbol{y}^1, \boldsymbol{y}^2, \dots, \boldsymbol{y}^n\}$ is linearly independent, $\{x_i^1 \boldsymbol{y}^1, x_i^2 \boldsymbol{y}^2, \dots, x_i^n \boldsymbol{y}^n\}$ is linearly independent because its elements are scalar multiples of the elements in the linearly independent set $\{\boldsymbol{y}^1, \boldsymbol{y}^2, \dots, \boldsymbol{y}^n\}$ and for almost all A, we have $x_j^i \ne 0$, because for almost all A, it follows that $x_j^i \ne 0$. As a result, the columns of $N[\varepsilon]$ span $\mathbb{R}^n$. As such, any $\operatorname{vec}(\mathcal{X})[\varepsilon]$ lies in the span of the columns of $N[\varepsilon]$. Therefore α is admissible.

The proof that $\alpha = \{(1, j), (2, j), \dots, (n, j)\}$, where $1 \le j \le n$, is admissible is similar. □

This shows that patterns including an entire row or entire column of specified entries is maximally admissible. For specific n, we can show that there exist other admissible patterns, and we conjecture that a pattern with n specified entries is admissible if and only if it has at least one diagonal entry specified. The following corollary describes a subset of admissible patterns.

Corollary 5.10. *If*

$$\beta \subseteq \{(i, 1), (i, 2), \dots, (i, n)\} \quad \textit{or} \quad \beta \subseteq \{(1, j), (2, j), \dots, (n, j)\},$$

where $1 \le i, j \le n$, then β is admissible.

Proof. This follows by Theorem 5.9 and Lemma 3.5. □

Patterns for the skew-Lyapunov equation. Lastly we classify patterns for the skew-Lyapunov equation, $AX - XA^T = 0$. Applying the vec function to $AX - XA^T = 0$ yields the linear equation $[(I \otimes A) - (A \otimes I)] \operatorname{vec}(X) = \mathbb{0}$. We denote $(I \otimes A) - (A \otimes I)$ as Ψ_A. The rank of Ψ_A determines the maximum number of specified entries in an admissible pattern. In this section, we assume A has distinct eigenvalues, and consider the rank of Ψ_A under this condition. The following result gives us an upper bound for the nullity of Ψ_A.

Lemma 5.11 [Morris 2015]. *Let $A \in M_n(\mathbb{R})$ and $B \in M_n(\mathbb{R})$ be similar matrices with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, then*

$$\operatorname{nullity}(I_n \otimes A + (-B^T) \otimes I_n) \leq \sum_{i=1}^{k} {a_i}^2$$

and

$$n^2 - \sum_{i=1}^{k} m_a(\lambda_i)^2 \leq \operatorname{rank}(I_n \otimes A + (-B^T) \otimes I_n) \leq n^2.$$

For $A \in M_n(\mathbb{R})$ with distinct eigenvalues, the maximum nullity of Ψ_A is n, and we can construct n linearly independent vectors in the nullspace.

Since the nullity of Ψ_A is n, maximally admissible patterns for $AX - XA^T = 0$ will have n specified entries. We proceed by determining a basis for the solution space of the skew-Lyapunov equation. This is equivalent to finding a basis for the nullspace of Ψ_A.

The following theorem partially classifies maximally admissible patterns for the skew-Lyapunov equation. Maximally admissible patterns contain n specified entries by Corollary 3.4. We first show that if the same numbered column and row have a total of n specified entries, then the pattern is admissible.

Theorem 5.12. *Let $A \in M_n(\mathbb{R})$ with distinct eigenvalues and α be an $n \times n$ partial matrix pattern. Given $k \in \{1, \ldots, n\}$, if exactly one of (k, i) or (i, k) is in α for all $1 \leq i \leq n$, then α is maximally admissible for the matrix equation $AX - XA^T = 0$ for almost all A, where all A have distinct eigenvalues.*

Proof. Noting that the rows of N corresponding to the (i, j) and (j, i) entries are equal, this theorem is a special case of Theorem 5.9 with $\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^1, \boldsymbol{x}^2 \otimes \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n \otimes \boldsymbol{x}^n\}$ as a basis for the solution space. □

Corollary 5.13. *For $A \in M_n(\mathbb{R})$ with distinct eigenvalues, if $\beta \subseteq \{(1, k), \ldots, (n, k)\}$ or $\beta \subseteq \{(k, 1), \ldots, (k, n)\}$ then β is admissible for the matrix equation $AX - XA^T = 0$ for almost all A, where all A have distinct eigenvalues.*

Proof. This follows by Theorem 5.12 and Lemma 3.5. □

We next classify patterns as inadmissible. If α is admissible, then there are no pairs of specified entries which reside opposite the main diagonal from each other. Equivalently, if there exists a pair of specified entries such that they are across the main diagonal from each other, then the pattern will be inadmissible.

Theorem 5.14. *For $A \in M_n(\mathbb{R})$, if $\alpha = \{(i, j), (j, i)\}$ such that $i \neq j$ and $1 \leq i, j \leq n$, then α is inadmissible for the skew-Lyapunov equation $AX - XA^T = 0$.*

Proof. By Remark 4.5, $\{\boldsymbol{x}^1 \otimes \boldsymbol{x}^1, \boldsymbol{x}^2 \otimes \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n \otimes \boldsymbol{x}^n\}$ is a basis for the nullspace of Ψ_A where $\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n\}$ is a basis of eigenvectors for A. Following the nullspace approach, we form $N[\varepsilon]$ where $\varepsilon = \{i + j(n-1) \mid (i, j) \in \alpha\}$. So,

$$N[\varepsilon] = \begin{bmatrix} x_{1j}x_{1i} & x_{2j}x_{2i} & \dots & x_{nj}x_{ni} \\ x_{1j}x_{1i} & x_{2j}x_{2i} & \dots & x_{nj}x_{ni} \end{bmatrix}$$

and we have that $\operatorname{rank}(N[\varepsilon]) = 1$ which is strictly less than the size of this pattern, 2. So by Theorem 4.1 the pattern $(i, j), (j, i)$ with $i \neq j$ is inadmissible for the matrix equation $AX - XA^T = 0$. □

Corollary 5.15. *If $\alpha = \{(i, j), (j, i)\} \subseteq \beta$ where $i \neq j$ then β is inadmissible for the matrix equation $AX - XA^T = 0$.*

Proof. This follows by Theorem 5.14 and Lemma 3.6. □

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