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The Hamiltonian problem and t-path traceable graphs

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The problem of characterizing maximal non-Hamiltonian graphs may be naturally extended to characterizing graphs that are maximal with respect to nontraceability and beyond that to t-path traceability. We show how t-path traceability behaves with respect to disjoint union of graphs and the join with a complete graph. Our main result is a decomposition theorem that reduces the problem of characterizing maximal t-path traceable graphs to characterizing those that have no universal vertex. We generalize a construction of maximal nontraceable graphs by Zelinka to t-path traceable graphs.

1. Introduction

The motivating problem for this article is the characterization of maximal non-Hamiltonian (MNH) graphs. The first broad family of MNH graphs was given in [Skupień 1979], and all MNH graphs with ten or fewer vertices were described in [Jamrozik et al. 1982], a paper where Skupień and his coauthors gave three constructions, called types $A1$, $A2$, $A3$, with a similar structure. Zelinka [1998] gave two constructions of graphs that are maximal nontraceable; that is, they have no Hamiltonian path, but the addition of any edge gives a Hamiltonian path. The join of such a graph with a single vertex gives an MNH graph. Zelinka's first family produces, under the join with K_1, the original MNH graphs of Skupień. Zelinka's second family is a broad generalization of the type $A1$, $A2$, and $A3$ graphs of [Jamrozik et al. 1982]. Further examples of infinite families of maximal nontraceable graphs appeared in [Bullock et al. 2008].

In this article, we work with two closely related invariants of a graph G, $\check{\mu}(G)$ and $\mu(G)$. The μ-invariant, introduced by Ore [1961] and also used by Noorvash [1975], is the minimal number of paths in G required to cover the vertex set of G. We define $\check{\mu}(G)$ to be the smallest integer ℓ such that the join of K_ℓ with G is Hamiltonian. We show that $\check{\mu}(G) = \mu(G)$ unless G is Hamiltonian, when $\check{\mu}(G) = 0$. Maximal

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non-Hamiltonian graphs are maximal with respect to $\check{\mu}(G) = 1$, and maximal nontraceable graphs are maximal with respect to $\check{\mu}(G) = 2$. It is useful to broaden the perspective to study, for arbitrary t, graphs that are maximal with respect to $\check{\mu}(G) = t$, which we call t-path traceable graphs.

In Section 2 we show how the $\check{\mu}$ and μ invariants behave with respect to disjoint union of graphs and the join with a complete graph. Section 3 derives the main result, a decomposition theorem that reduces the problem of characterizing maximal t-path traceable graphs to characterizing those that have no universal vertex, which we call *trim*. Section 4 presents a generalization of the Zelinka construction to t-path traceable graphs.

2. Traceability and Hamiltonicity

It will be notationally convenient to say that the complete graphs K_1 and K_2 are Hamiltonian. As justification for this view, consider an undirected graph as a directed graph with each edge having a conjugate edge in the reverse direction. This perspective does not affect the Hamiltonicity of a graph with more than three vertices, but it does give K_2 a Hamiltonian cycle. Similarly, adding loops to any graph with more than two vertices does not alter the Hamiltonicity of the graph, but K_1, with an added loop, has a Hamiltonian cycle.

Let G be a graph. A vertex, $v \in V(G)$, is called a *universal vertex* if $\deg(v) = |V(G)| - 1$. A universal vertex is also known as a dominating vertex. Let $\overline{G}$ denote the *graph complement* of G, having vertex set $V(G)$ and edge set $E(K_n) \setminus E(G)$. We will use the disjoint union of two graphs, $G \sqcup H$ and the join of two graphs $G * H$. The latter is $G \sqcup H$ together with the edges $\{vw \mid v \in V(G) \text{ and } w \in V(H)\}$.

Definition 1. A set of s disjoint paths in a graph G that includes every vertex in G is an *s-path covering* of G. We define the following invariants:

$$\mu(G) := \min\{\, s \in \mathbb{N} \mid \text{there exists an } s\text{-path covering of } G\},$$

$$\check{\mu}(G) := \min\{\, l \in \mathbb{N}_0 \mid K_l * G \text{ is Hamiltonian}\},$$

$$i_H(G) := \begin{cases} 1 & \text{if } G \text{ is Hamiltonian,} \\ 0 & \text{otherwise.} \end{cases}$$

We will say G is *t-path traceable* when $\mu(G) = t$. A set of t disjoint paths that covers a t-path traceable graph G is a *minimal path covering*.

Note that $K_r * (K_s * G) = K_{r+s} * G$. If G is Hamiltonian then so is $K_r * G$ for $r \geq 0$ (in particular, this is true for $G = K_1$ and $G = K_2$).

We now present a series of lemmas that leads to the main result of this section, which is a formula showing how the μ-invariant and $\check{\mu}$-invariant behave with respect to the disjoint union and the join with a complete graph.

Lemma 2. $\check{\mu}(G) = \min\{l \in \mathbb{N}_0 \mid \overline{K}_l * G \textit{ is Hamiltonian}\}.$

Proof. Since $\overline{K}_l * G$ is a subgraph of $K_l * G$, a Hamiltonian cycle in $\overline{K}_l * G$ would also be one in $K_l * G$.

Let $\check{\mu}(G) = a$. Suppose C is a Hamiltonian cycle in $K_a * G$ and write C as $v \sim P_1 \sim Q_1 \sim \cdots \sim P_s \sim Q_s \sim v$, where v is a vertex in G and the paths P_i in G and Q_i in K_a. If any Q_i contains two vertices or more, say u and $w_1, \dots, w_k$ with $k \geq 1$, then we may simply remove all the vertices, except u, and end up with a Hamiltonian graph on K_{a-k}. This contradicts the minimality of $a = \check{\mu}(G)$. Therefore, C must not contain any paths of length greater than two in the subgraph K_a, and any Hamiltonian cycle on $K_a * G$ is also a Hamiltonian cycle on $\overline{K}_a * G$. □

Lemma 3. $\check{\mu}(G) = \mu(G) - i_H(G).$

Proof. If G is Hamiltonian (including K_1 and K_2) then $\check{\mu}(G) = 0$, $\mu(G) = 1$ so the equality holds. Suppose G is non-Hamiltonian with $\mu(G) = t$ and t-path covering $P_1, \dots, P_t$. Let K_t have vertices $u_1, \dots, u_t$. In the graph $K_t * G$, there is a Hamiltonian cycle: $v_1 \sim P_1 \sim v_2 \sim P_2 \sim \cdots \sim v_t \sim P_t \sim v_1$. Thus $\check{\mu}(G) \leq t = \mu(G)$.

Let $\check{\mu}(G) = a$, so there is a Hamiltonian cycle in $K_a * G$. Removing the vertices of K_a breaks the cycle into at most a disjoint paths covering G. Thus $\mu(G) \leq \check{\mu}(G)$. □

Lemma 4.
$$\mu(G \sqcup H) = \mu(G) + \mu(H) \textit{ and}$$
$$\check{\mu}(G \sqcup H) = \check{\mu}(G) + \check{\mu}(H) + i_H(G) + i_H(H).$$

Proof. A path covering of G may be combined with a path covering of H to create one for $G \sqcup H$ so $\mu(G \sqcup H) \leq \mu(G) + \mu(H)$. Conversely, paths in a t-path covering of $G \sqcup H$ can be partitioned into those contained in G and those contained in H, giving a path covering of G and one of H. Consequently, $\mu(G \sqcup H) \geq \mu(G) + \mu(H)$.

Since $G \sqcup H$ is not Hamiltonian we have
$$\begin{aligned}\check{\mu}(G \sqcup H) &= \mu(G \sqcup H) + i_H(G \sqcup H) \\ &= \mu(G) + \mu(H) \\ &= \check{\mu}(G) + i_H(G) + \check{\mu}(H) + i_H(H).\end{aligned}$$ □

Lemma 5. *For any graph G,*
$$\mu(K_s * G) = \max\{1, \mu(G) - s\},$$
$$\check{\mu}(K_s * G) = \max\{0, \check{\mu}(G) - s\}.$$

In particular, if $K_s * G$ is Hamiltonian then $\mu(K_s * G) = 1$ and $\check{\mu}(K_s * G) = 0$; otherwise, $\mu(K_s * G) = \mu(G) - s$ and $\check{\mu}(K_s * G) = \check{\mu}(G) - s$.

Proof. The formula for $\check{\mu}$ is immediate when G is Hamiltonian since we have observed that this forces $K_s * G$ to be Hamiltonian. Otherwise, it follows from

$K_r * (K_s * G) = K_{r+s} * G$: if $\check{\mu}(G) = a$, then $K_r * (K_s * G)$ is Hamiltonian if and only if $r + s \geq a$.

The formula for μ may be derived from the result for $\check{\mu}$ using Lemma 3. $\square$

The main result of this section is the following two formulas for the μ and $\check{\mu}$ invariants of the disjoint union of graphs, and the join with a complete graph.

Proposition 6. *Let $\{G_j\}_{j=1}^m$ be graphs. Then*

$$\mu\Big(\bigsqcup_{j=1}^m G_j\Big) = \sum_{j=1}^m \mu(G_j),$$

$$\check{\mu}\Big(\bigsqcup_{j=1}^m G_j\Big) = \sum_{j=1}^m \check{\mu}(G_j) + \sum_{j=1}^m i_H(G_j).$$

Furthermore,

$$\check{\mu}\bigg(\Big(\bigsqcup_{j=1}^m G_j\Big) \ * \ K_r\bigg) = \max\bigg\{0, \sum_{j=1}^m \check{\mu}(G_j) + \sum_{j=1}^m i_H(G_j) - r\bigg\}.$$

Proof. We proceed by induction. The base case $k = 2$ is exactly Lemma 4. Assume the formula holds for k graphs; we will prove it for $k + 1$ graphs.

$$\begin{aligned}\mu\Big(\bigsqcup_{j=1}^{k+1} G_j\Big) &= \mu\bigg(\Big(\bigsqcup_{j=1}^{k} G_j\Big) \sqcup G_{k+1}\bigg) = \mu\Big(\bigsqcup_{j=1}^{k} G_j\Big) + \mu(G_{k+1}) \\ &= \sum_{j=1}^k \mu(G_j) + \mu(G_{k+1}) = \sum_{j=1}^{k+1} \mu(G_j).\end{aligned}$$

By Lemma 3 and the fact that disjoint graphs are not Hamiltonian, we have

$$\begin{aligned}\check{\mu}\Big(\bigsqcup_{j=1}^m G_j\Big) &= \mu\Big(\bigsqcup_{j=1}^m G_j\Big) + i_H\Big(\bigsqcup_{j=1}^m G_j\Big) \\ &= \sum_{j=1}^m \big(\check{\mu}(G_j) + i_H(G_j)\big) = \sum_{j=1}^m \check{\mu}(G_j) + \sum_{j=1}^m i_H(G_j).\end{aligned}$$

Therefore, we have by Lemma 5,

$$\begin{aligned}\check{\mu}\bigg(\Big(\bigsqcup_{j=1}^m G_j\Big) * K_r\bigg) &= \max\bigg\{0, \check{\mu}\Big(\bigsqcup_{j=1}^m G_j\Big) - r\bigg\} \\ &= \max\bigg\{0, \sum_{j=1}^m \check{\mu}(G_j) + \sum_{j=1}^m i_H(G_j) - r\bigg\}.\end{aligned}$$ $\square$

The following lemma will be useful in the next section. To express it succinctly, we introduce the following Boolean condition. For a graph G and vertex $v \in V(G)$, $T(v, G)$ is true if and only if v is a terminal vertex in some minimal path covering of G.

Lemma 7. *Let $v \in V(G)$ and $w \in V(H)$. Then we have*

$$\mu((G \sqcup H) + vw) = \begin{cases} \mu(G \sqcup H) - 1 & \text{if } T(v, G) \text{ and } T(w, H), \\ \mu(G \sqcup H) & \text{otherwise.} \end{cases}$$

Proof. Let $\mu(G) = c$, $\mu(H) = d$ and $\mu((G \sqcup H) + vw) = t$. Clearly, $t \le c + d$.

Let $R_1, \ldots, R_t$ be a minimal path cover of $(G \sqcup H) + vw$. If no R_i contains vw then this is also a minimal path cover of $(G \sqcup H)$ so $t = c + d$. Suppose R_1 contains vw and note that R_1 is the only path with vertices in both G and H. Removing vw gives two paths $P \subseteq G$ and $Q \subseteq H$. Paths P and Q along with $R_2, \ldots, R_t$ cover $G \sqcup H$, so $t + 1 \ge c + d$. Thus, t can either be $c + d$ or $c + d - 1$.

If $t = c + d - 1$, then we have the minimal $(t+1)$-path covering $P, Q, R_2, \ldots, R_t$ of $G \sqcup H$, as above. We note that v must be a terminal point of P and w must be a terminal point of Q, by construction. This path covering may be partitioned into a c-path covering of G containing P and a d-path covering of H containing Q. Thus, $T(v, G)$ and $T(w, G)$ hold.

Conversely, suppose $T(u, G)$ and $T(w, H)$ both hold. Let $P_1, \ldots, P_c$ be a minimal path of G with v a terminal vertex of P_1 and let $Q_1, \ldots, Q_d$ be a minimal path cover of H with w a terminal vertex of Q_1. The edge vw knits P_1 and Q_1 into a single path and $P_1 \sim Q_1, P_1, \ldots, P_c, Q_1, \ldots, Q_d$ is a $c + d - 1$ cover of $(G \sqcup H) + vw$. Consequently, $t \le c + d - 1$.

Thus, $T(u, G)$ and $T(w, H)$ both hold if and only if $t = c + d - 1$. Otherwise, $t = c + d$. □

Corollary 8. *Let $v \in V(G)$ and $w \in V(H)$. Then we have*

$$\check{\mu}((G \sqcup H) + vw) = \begin{cases} \check{\mu}(G \sqcup H) - 2 & \text{if } G = H = K_1, \\ \check{\mu}(G \sqcup H) - 1 & \text{if } T(v, G) \text{ and } T(w, H), \\ \check{\mu}(G \sqcup H) & \text{otherwise.} \end{cases}$$

Proof. Let $\delta = 1$ if $T(v, G)$ and $T(w, H)$ are both true and $\delta = 0$ otherwise. Then

$$\begin{aligned} \check{\mu}((G \sqcup H) + vw) &= \mu((G \sqcup H) + vw) - i_H((G \sqcup H) + vw) \\ &= \mu((G \sqcup H) - \delta - i_H((G \sqcup H) + vw). \end{aligned}$$

The final term is -1 if and only if $G = H = K_1$. □

3. Decomposing maximal t-path traceable graphs

In this section we prove our main result, a maximal t-path traceable graph may be uniquely written as the join of a complete graph and a disjoint union of graphs that are also maximal with respect to traceability, but which are also either complete or have no universal vertex. We work with the families of graphs $\mathcal{M}_t$ for $t \geq 0$ and $\mathcal{N}_t$ for $t \geq 1$:

$$\mathcal{M}_t := \{G \mid \check{\mu}(G) = t \text{ and } \check{\mu}(G+e) < t,\ \forall e \in E(\overline{G})\},$$
$$\mathcal{N}_t := \{G \in \mathcal{M}_t \mid G \text{ is connected and has no universal vertex}\}.$$

The set $\mathcal{M}_0$ is the set of complete graphs. The set $\mathcal{M}_1$ is the set of graphs with a Hamiltonian path but no Hamiltonian cycle, that is, maximal non-Hamiltonian graphs. For $t > 1$, $\mathcal{M}_t$ is also the set of graphs G such $\mu(G) = t$ and $\mu(G+e) = t-1$ for any $e \in E(\overline{G})$. We will call these *maximal t-path traceable graphs*. A graph in $\mathcal{N}_t$ will be called *trim*.

Proposition 9. *For* $0 \leq r < t$, $G \in \mathcal{M}_t$ *if and only if* $K_r * G \in \mathcal{M}_{t-r}$.

Proof. We have $\check{\mu}(K_r * G) = \check{\mu}(G) - r$, by Lemma 5, so we just need to show that $K_r * G$ is maximal if and only if G is maximal. The only edges that can be added to $K_r * G$ are those between vertices of G, that is, $E(\overline{K_r * G}) = E(\overline{G})$. For such an edge e,

$$\check{\mu}((K_r * G) + e) = \check{\mu}(K_r * (G+e)) = \check{\mu}(G+e) - r. \tag{1}$$

Thus, $\check{\mu}(G+e) = \check{\mu}(G) - 1$ if and only if $\check{\mu}((K_r * G) + e) = \check{\mu}(K_r * G) - 1$. ☐

Note that the proposition is false for $r = t > 0$ since $K_r * G$ will not be a complete graph and $\mathcal{M}_0$ is the set of complete graphs. The proof breaks down in (1).

As a key step before the main theorem, the next lemma shows that in a maximal graph, each vertex is either universal or it is a terminal vertex in a minimal path covering (but not both).

Lemma 10. *Let* $c \geq 1$ *and* $G \in \mathcal{M}_c$. *For any two nonadjacent vertices* v, w *in* G, *there is a c-path covering of G in which both v and w are terminal points of paths. Moreover, a vertex* $v \in V(G)$ *is a terminal point in some c-path covering if and only if v is not universal.*

Proof. Suppose $c > 1$ and let v, w be nonadjacent in G. Since G is maximal, $G + vw$ has a $(c-1)$-path covering, $P_1, \ldots, P_{c-1}$. The edge vw must be contained in some P_i because G has no $(c-1)$-path covering. Removing that edge gives a c-path covering of G with v and w as terminal vertices. The special case $c = 1$ is well known, adding the edge vw gives a Hamiltonian cycle, and removing it leaves a path with endpoints v and w. A consequence is that any nonuniversal vertex is the terminal point of some path in a c-path covering.

Suppose $P_1, \dots, P_c$ is a c-path covering of $G \in \mathcal{M}_c$ with v a terminal point of P_i. Then v is not adjacent to any of the terminal points of P_j for $j \neq i$, for otherwise two paths could be combined into a single one. In the case $c = 1$, v cannot be adjacent to the other terminal point of P_1, otherwise G would have a Hamiltonian cycle. Consequently, a universal vertex is not a terminal point in a c-path covering of G. □

Proposition 11. *Let $G \in \mathcal{M}_c$ and $H \in \mathcal{M}_d$. The following are equivalent*:

(1) $G \sqcup H \in \mathcal{M}_{c+d+i_H(G)+i_H(H)}$.

(2) *Each of G and H is either complete or has no universal vertex.*

Proof. We have already shown that $\check{\mu}(G \sqcup H) = c + d + i_H(G) + i_H(H)$. We have to consider whether adding an edge to $G \sqcup H$ reduces the $\check{\mu}$-invariant. There are three cases to consider: the extra edge may be in $E(\overline{G})$ or $E(\overline{H})$ or it may join a vertex in G to one in H. Since G is maximal, adding an edge to G is either impossible, when G is complete, or it reduces the $\check{\mu}$-invariant of G. This edge would also reduce the $\check{\mu}$-invariant of $G \sqcup H$ by Lemma 4. The case for adding an edge of H is the same. Consider the edge vw for $v \in V(G)$ and $w \in V(H)$. By Corollary 8 the $\check{\mu}$-invariant will drop if and only if v is the terminal point of a path in a minimal path covering of G and similarly for w in H, that is, $T(v, G)$ and $T(w, H)$. Clearly this holds for all vertices in a complete graph. Lemma 10 shows that $T(v, G)$ holds for $G \in \mathcal{M}_c$ with $c > 0$ if and only if v is not a universal vertex in G. Thus, in order for $G \sqcup H$ to be maximal, G must either be complete or be maximal itself and have no universal vertex, and similarly for H. □

Theorem 12. *For any $G \in \mathcal{M}_t$, $t > 0$, G may be uniquely decomposed as*

$$K_r * (G_1 \sqcup \dots \sqcup G_m),$$

where r is the number of universal vertices of G, and each G_j is either complete or $G_j \in \mathcal{N}_{t_j}$ for some $t_j > 0$. Furthermore $t = \sum_{j=1}^m t_j + \sum_{j=1}^m i_H(G_j) - r$.

Proof. Suppose $G \in \mathcal{M}_t$ and let r be the number of universal vertices of G. Let m be the number of components in the graph obtained by removing the universal vertices from G, let $G_1, \dots G_m$ be the components and let $\check{\mu}(G_j) = t_j$. Then $G = K_r * (G_1 \sqcup \dots \sqcup G_m)$.

Proposition 6 shows that $t = \sum_{j=1}^m t_j + \sum_{j=1}^m i_H(G_j) - r$. By Proposition 9, we have that $G \in \mathcal{M}_t$ if and only if $G_1 \sqcup \dots \sqcup G_m \in \mathcal{M}_{t+r}$. Each G_i must be maximal, otherwise the disjoint union would not be maximal (add an appropriate edge to a G_i in Proposition 6). Inductively applying Proposition 11 to $G_1 \sqcup \dots \sqcup G_m \in \mathcal{M}_{t+r}$, where $t + r = \sum_{j=1}^m t_j + \sum_{j=1}^m i_H(G_j)$, we have that each G_j is complete or is trim ($G_j \in \mathcal{N}_{t_j}$ for $t_j > 0$). □

4. Trim maximal t-path traceable graphs

Skupień [1979] discovered the first family of maximal non-Hamiltonian graphs, that is, graphs in $\mathcal{M}_1$. These graphs are formed by taking the join of K_r with the disjoint union of $r+1$ complete graphs [Marczyk and Skupień 1991]. The smallest graph in $\mathcal{N}_2$ is shown in Figure 1. Chvátal [1973] identified its join with K_1 as the smallest maximal non-Hamiltonian graph that is not 1-tough, that is, not one of the Skupień family. Jamrozik, Kalinowski and Skupień [1982] generalized this example to three different families. Family $A1$ replaces each edge u_iv_i in Figure 1 with an arbitrary complete graph containing u_i and replaces the K_3 formed by the u_i with an arbitrary complete graph. The result — a type $A1$ graph — has four cliques, the first three disjoint from each other but each intersecting the fourth clique in a single vertex. An $A1$ graph is in $\mathcal{N}_2$ and its join with K_1 gives a maximal non-Hamiltonian graph. Family $A2$ is formed by taking the join with K_2 of the disjoint union of a complete graph and an $A1$ graph. Theorem 12 shows that the resulting graph is in $\mathcal{M}_1$. Family $A3$ is a modification of the $A1$ family based on the graph in Figure 2, which is in $\mathcal{N}_2$.

More than two decades later, Bullock, Frick, Singleton and van Aardt [2008] recognized that two constructions of Zelinka [1998] give maximal nontraceable graphs, that is, elements of $\mathcal{M}_2$. Zelinka's first construction is like the Skupień family: formed from $r+1$ complete graphs followed by the join with K_{r-1}. The Zelinka type II family contains graphs in $\mathcal{N}_2$ that are a significant generalization of the graphs in Figures 1 and 2. In this section we generalize this family further to get graphs in $\mathcal{N}_t$ for arbitrary t. Our starting point is the graph in Figure 3, which is in $\mathcal{N}_3$.

Example 13. Consider K_m with $m = 2t-1$ and vertices $u_1, \dots, u_m$. Let G be the graph containing K_m along with vertices $v_1, \dots, v_{2t-1}$ and edges u_iv_i. The case with $t = 3$ and $m = 5 = 2t-1$ is Figure 3. We claim $G \in \mathcal{N}_t$.

One can readily check that this graph is t-path covered using $v_{2i-1} \sim u_{2i-1} \sim u_{2i} \sim v_{2i}$ for $i = 1, \dots, t-1$ and $v_{2t-1} \sim u_{2t-1} \sim u_{2t} \sim \cdots \sim u_m$. We check that G is maximal. By the symmetry of the graph, we need only consider the addition

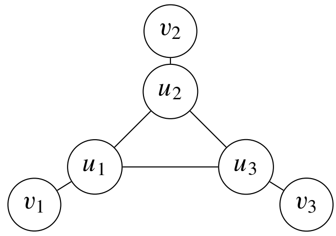

Figure 1. Smallest graph in $\mathcal{N}_2$.

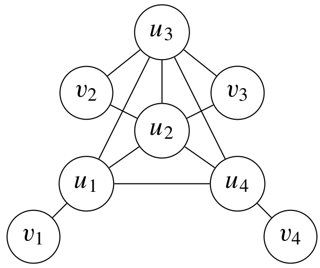

Figure 2. The join of this graph with K_1 is the smallest graph in the $A3$ family.

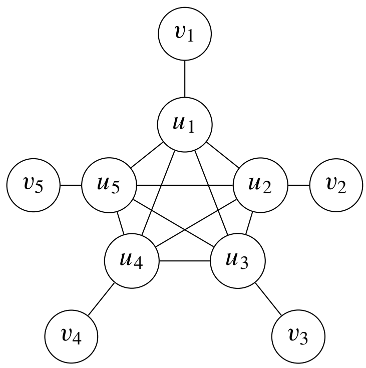

Figure 3. Whirligig in $\mathcal{N}_3$.

of the edge v_1u_m or v_1u_2 or v_1v_2. In each case, the last and the first paths listed above may be combined into one, either

$$v_{2t-1} \sim u_{2t-1} \sim \cdots \sim u_m \sim v_1 \sim u_1 \sim u_2 \sim v_2, \text{ or}$$

$$v_{2t-1} \sim u_{2t-1} \sim \cdots \sim u_m \sim u_1 \sim v_1 \sim u_2 \sim v_2, \text{ or}$$

$$v_{2t-1} \sim u_{2t-1} \sim \cdots \sim u_m \sim u_1 \sim v_1 \sim v_2 \sim u_2.$$

Thus, adding an edge creates a $(t-1)$-path covered graph, proving maximality.

The next proposition shows that the previous example is the only way to have a trim maximal t-path covered graph with $2t-1$ degree-one vertices. We start with a technical lemma.

Lemma 14. *Let G be a connected graph and u_1, v_1, v_2, $v_3 \in V(G)$ with $\deg(v_i) = 1$, and u adjacent to v_1 and v_2 but not v_3. Then $\mu(G) = \mu(G + uv_3)$.*

Proof. Let $P_1, \ldots, P_r$ be a minimal path covering of $G + uv_3$; it is enough to show that there are r-paths covering G. If the covering doesn't include uv_3, then $P_1, \ldots, P_r$ also give a minimal path covering of G, establishing the claim of the lemma. Otherwise, suppose uv_3 is an edge of P_1. We consider two cases.

Suppose P_1 contains the edge uv_1 (or similarly uv_2). Then P_1 has v_1 as a terminal point and one of the other paths, say P_2, must be a length-0 path containing simply v_2. Let Q be obtained by removing uv_1 and uv_3 from P_1. Then $v_1 \sim u \sim v_2, Q, P_3, \ldots, P_r$, gives an r-path covering of G.

Suppose P_1 contains neither uv_1 nor uv_2. Then each of v_1 and v_2 must be on a length-0 path in the covering, say P_2 and P_3 are these paths. Furthermore u must not be a terminal point of P_1; if it were, the path could be extended to include v_1 or v_2, reducing the number of paths required to cover G. Removing u from P_1 yields two paths, Q_1, Q_2. Then $v_1 \sim u \sim v_2, Q_1, Q_2, P_4, \ldots, P_r$ gives an r-path cover of G. This proves the lemma. □

Proposition 15. *Let $G \in \mathcal{N}_t$. The number of degree-one vertices in G is at most $2t - 1$. This occurs if and only if the $2t - 1$ vertices of degree-one have distinct neighbors and removing the degree-one vertices leaves a complete graph.*

Proof. Each degree-one vertex must be a terminal point in a path covering. So any graph G covered by t paths can have at most $2t$ degree-one vertices. Aside from the case $t = 1$ and $G = K_2$, we can see that a graph with $2t$ degree-one vertices cannot be maximal t-path traceable as follows. It is easy to check that a $2t$ star is not t-path traceable (it is also not trim). A t-path traceable graph with $2t$ degree-one vertices must therefore have an interior vertex w that is not connected to at least one of the degree-one vertices v. Such a graph is not maximal because the edge vw can be added leaving $2t - 1$ degree-one vertices. This resulting graph cannot be $(t - 1)$-path covered.

Suppose that $G \in \mathcal{N}_t$ with $2t - 1$ degree-one vertices, $v_1, \ldots, v_{2t-1}$. Lemma 14 shows that no two of the v_i can be adjacent to the same vertex, for that would violate maximality of G. So, the v_i have distinct neighbors. Furthermore, all the vertices except the v_i can be connected to each other and a path covering will still require at least t paths since there remain $2t - 1$ degree-one vertices. This proves the necessity of the structure claimed in the proposition. The previous example showed that the graph is indeed in $\mathcal{N}_t$. □

We can now generalize the Zelinka family.

Construction 16. Let $U_0, U_1, \ldots, U_{2t-1}$ be disjoint sets of vertices and

$$U = \bigsqcup_{i=0}^{2t-1} U_i.$$

Let $m_i = |U_i|$ and assume that for $i > 0$ the U_i are nonempty, so $m_i > 0$. For $i = 1, \ldots, 2t - 1$ (but not $i = 0$) and $j = 1, \ldots, m_i$, let V_{ij} be nonempty sets of vertices disjoint from each other and from U. Form the graph W with vertex set $U \sqcup \big(\bigsqcup_{i=1}^{2t-1} \big(\bigsqcup_{j=1}^{m_i} V_{ij}\big)\big)$ and edges uu' for $u, u' \in U$ and uv for any $u \in U_i$

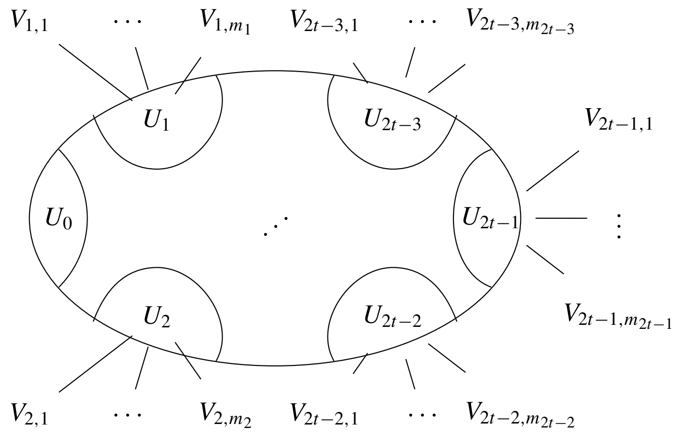

Figure 4. Generalization of the whirligig, W.

and $v \in V_{ij}$ with $i = 1, \dots, 2t-1$ and $j = 1, \dots, m_i$ and all edges within each set V_{ij}. The cliques of this graph are K_U and $K_{U_i \sqcup V_{ij}}$ for each $i = 1, \dots, 2t-1$ and $j = 1, \dots, m_i$.

The graph in Figure 2 has $m_0 = 0$, $m_1 = m_2 = 1$ and $m_3 = 2$, and the graph in Figure 4 indicates the general construction.

Theorem 17. *The graph W in Construction 16 is a trim, maximal t-path traceable graph.*

Proof. We must show that W is t-path covered and not $(t-1)$-path covered, and that the addition of any edge yields a $(t-1)$-path covered graph. The argument is analogous to the one in Example 13.

Let R be a Hamiltonian path in U_0. For each $i = 1, \dots, 2t-1$ and $j = 1, \dots, m_i$, let Q_{ij} be a Hamiltonian path in $K_{V_{ij}}$. Let P_i be the path

$$P_i : Q_{i1} \sim u_{i1} \sim Q_{i2} \sim u_{i2} \sim \cdots \sim Q_{im_i} \sim u_{im_i},$$

and let $\overleftarrow{P_i}$ be the reversal of P_i.

Since there is an edge $u_{im_i} u_{jm_j}$ there is a path $P_i \sim \overleftarrow{P}_j$ for any $i \neq j \in \{1, \dots, 2t-1\}$. Therefore the graph W has a t-path covering $P_{2i-1} \sim \overleftarrow{P}_{2i}$ for $i = 1, \dots, (t-1)$, along with $P_{2t-1} \sim R$. We leave to the reader the argument that there is no $(t-1)$-path cover.

To show W is maximal we show that after adding an edge e, we can join two paths in the t-path cover above, with a bit of rearrangement. There are three types of edges to consider, the edge e might join V_{ij} to $U_{i'}$ for $i \neq i'$; or V_{ij} to $V_{ij'}$ for $j \neq j'$; or V_{ij} to $V_{i'j'}$ for $i \neq i'$. Because of the symmetry of W, we may assume

$i = 1$ and $j = 1$ and that the vertex chosen from $V_{ij} = V_{1,1}$ is the initial vertex of $Q_{1,1}$. Other simplifications due to symmetry will be evident in what follows.

In the first case there are two subcases — determined by $i' \geq 2t$ or not — and after permutation, we may consider the edge e from the initial vertex of $Q_{1,1}$ to the terminal vertex of R, or to the terminal vertex of P_{2t-1}. We can then join two paths in the t-path cover: either $P_{2t-1} \sim R \overset{e}{\sim} P_1 \sim \overleftarrow{P}_2$ or $P_{2t-1} \overset{e}{\sim} P_1 \sim R \sim \overleftarrow{P}_2$.

Suppose next that we join the initial vertex of Q_{11} with the terminal vertex of Q_{12}. We then rearrange P_1 and join two paths in the t-path cover to get

$$P_{2t-1} \sim R \sim u_{1,1} \sim Q_{1,1} \overset{e}{\sim} Q_{1,2} \sim u_{1,2} \sim \cdots \sim Q_{1m_1} \sim u_{1m_1} \sim \overleftarrow{P}_2.$$

Finally, suppose that we join the initial vertex of $Q_{1,1}$ with the initial vertex of $Q_{2t-1,1}$. Then we rearrange to

$$\overleftarrow{R} \sim \overleftarrow{P}_{2t-1} \overset{e}{\sim} P_1 \sim \overleftarrow{P}_2. \qquad \square$$

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