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Madeleine Burkhart and Joel Foisy

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Enumerating spherical n-links

Madeleine Burkhart and Joel Foisy

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We investigate spherical links: that is, disjoint embeddings of 1-spheres and 0-spheres in the 2-sphere, where the notion of a split link is analogous to the usual concept. In the quest to enumerate distinct nonsplit n-links for arbitrary n, we must consider when it is possible for an embedding of circles and an even number of points to form a nonsplit link. The main result is a set of necessary and sufficient conditions for such an embedding. The final section includes tables of the distinct embeddings that yield nonsplit n-links for $4 \leq n \leq 8$.

1. Introduction

The enumeration of links in 3-space is well-studied [Hoste 2005]. However, there has not been much study of a planar/spherical analog outside the confines of its appearance in graphs [Archdeacon and Sagols 2002]. We aim to get the ball rolling on spherical links.

An *n-link* $\mathcal{L}$ in the 2-sphere is a disjoint collection of q embedded 1-spheres and $n-q$ embedded 0-spheres. Two links are equivalent if there is a spherical isotopy taking one to the other. Throughout this paper we use standard notation for a k-sphere: S^k. When speaking of spherical links, it does not make topological sense to call 0-spheres "components", since an entire S^0 is not connected. Henceforth we will refer to an S^1 or an S^0 as a *piece* of an n-link. We will call a spherical embedding of 1-spheres a *nesting*. Note that when we refer to nestings and nests in this paper, we are working with entities distinct from those in [Archdeacon and Sagols 2002].

We must now consider what constitutes a split spherical link. Note that the following definition only makes sense after we have chosen which pairs of points form 0-spheres: An n-link $\mathcal{L}$ is *split* if there exists an embedding ϕ of S^1 in $S^2 - \mathcal{L}$ such that each component of $S^2 - \phi(S^1)$ contains at least one piece of $\mathcal{L}$ and each piece of $\mathcal{L}$ is entirely contained in one such component. Otherwise, $\mathcal{L}$ is nonsplit.

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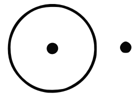

Figure 1. A nonsplit 2-link.

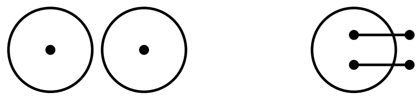

Figure 2. A 3-link with two circles and a 3-link with one circle.

Although there is only one type of nonsplit spherical 2-link, when we look at n-linking for $n > 2$, we can have different numbers of disjoint 1-spheres and 0-spheres. For example, we could have the two types of nonsplit spherical 3-links as in Figure 2.

We will find that the enumeration of n-link-types becomes more richly complex as n increases. Before finding all n-links for $n \leq 8$ in Section 3, we lay down the necessary and sufficient conditions for any spherical embedding of q circles and 2ℓ points to form a nonsplit $(q+\ell)$-link (given appropriate S^0 identifications). When considering such links, it will be helpful not only to think about nestings with points, but also to associate a weighted tree $\mathcal{T}$. To construct $\mathcal{T}$, first consider the nesting $\mathcal{N}$. If we identify a vertex on each circle, this embedding is a plane graph of disjoint loops, so the dual graph will be a tree in which each vertex is an open component of $S^2 - \mathcal{N}$. To account for embedded points, we give each vertex a weight equal to the number of points in the corresponding region.

The weighted tree $\mathcal{T}$ corresponds to a nesting with unpaired points, but we want to work with links; we will need to consider what happens to the tree after we make S^0 identifications. To make an identification, we will choose two vertices that each have weight at least 1, add an edge between them, and reduce their weights each by one (Figure 3).

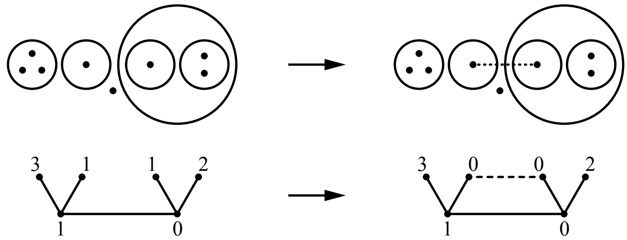

Figure 3. A nesting with points and its corresponding tree as we make an S^0 identification.

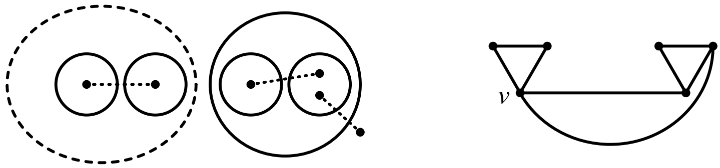

Figure 4. The vertex v corresponds with the region of the splitting circle.

If we do this until each vertex has weight 0, the resulting multigraph $G_{\mathcal{T}}$ will represent a link (unique if we distinguish the original tree edges from the S^0 identification edges). How can we tell from the graph if the link is split? Certainly a loop in the graph represents a split S^0. Any other type of split link, in which both components of $S^2 - \phi(S^1)$ (as in the split definition) have some positive number of circles, occurs if and only if there is a cut vertex in the multigraph (Figure 4).

We have now built up enough background to state our main result in dual ways.

Theorem 1.1. *Suppose we have a weighted tree $\mathcal{T}$ with q edges and total weight 2ℓ. In the corresponding embedding of q circles (with nesting $\mathcal{N}$) and 2ℓ points, it is possible to identify 0-spheres so that we have a nonsplit spherical $(q+\ell)$-link if and only if all of the following conditions are satisfied:*

(1) *Each leaf has weight at least one. That is, we must embed at least one point in each simply connected region of $S^2 - \mathcal{N}$.*

(2) *No vertex v is assigned a weight greater than $\ell - \deg(v) + 1$. That is, we can embed no more than $\ell - \kappa + 1$ points in a region of $S^2 - \mathcal{N}$ that has fundamental group $\mathbb{Z} * \cdots * \mathbb{Z}$, where $\mathbb{Z}$ appears $\kappa - 1$ times.*

(3) *Given any vertex of degree κ, the other vertices have total weight summing to at least $2(\kappa - 1)$. In other words, given a region as in (2), we must embed at least $2(\kappa - 1)$ points in the remaining regions.*

With this result, we can tell which embeddings of n (1 and 0)-spheres will form a nonsplit n-link. However, enumeration will require distinguishing links from one another on the sphere, which we only address for $n \leq 8$ in this paper.

Future directions

All the enumeration in this paper was done by hand; code will probably be necessary to enumerate spherical n-links for $n \geq 9$. As there is a one-to-one correspondence between nestings and unlabeled trees, much of the code will probably be similar to what is used in the problem of enumerating unlabeled trees (see [Harary 1969; Sloane 2006]).

While our results regard embeddings in S^2, it would be interesting to see how tabulations differ on different surfaces; for example, while a spherical embedding yields a correspondence between nestings and unlabeled trees, in the plane the correspondence is between nestings and rooted trees.

Our necessary and sufficient conditions depend on "appropriate" S^0 identifications. What happens if we make the worst possible S^0 identifications; that is, given a nesting with an even number of disjointly embedded points, what is the minimal nonsplit n-link among all possible S^0 pairings?

We could seek to generalize our result in a combinatorial manner; instead of looking at 0-spheres (i.e., pairs of points), we could look at triples, quadruples, or λ-tuples of points.

Because of the Jordan–Brouwer separation theorem [Guillemin and Pollack 1974], our results generalize to higher dimensions. The same necessary and sufficient conditions and link enumerations apply to embeddings of k-spheres and 0-spheres in S^{k+1}, since the dual weighted tree construction will still be well-defined. Perhaps this result has applications. It would also be interesting to investigate enumerating other types of higher-dimensional linking with spheres of different dimensions.

2. Proof of Theorem 1.1

In the following lemmas, we will switch between thinking about nestings and weighted trees. The following concepts will be useful when working with nestings.

Suppose we have a nesting $\mathcal{N}$. If we single out an open region in $S^2 - \mathcal{N}$ there will be some number of embedded circles that form holes in the region. We will call each such circle, along with all pieces in its interior, a *nest* (see Figure 5).

Suppose we have a nesting $\mathcal{N}$ and single out an open region R. Let each nest relative to R have a corresponding vertex. We add an edge between vertices if there is an S^0 identification "connecting" the nests. We will denote any graph resulting from this process as H_R.

Lemma 2.1. *The conditions of Theorem 1.1 are necessary for a* $(q+\ell)$*-link.*

Proof. Condition (1) is obvious; if a leaf v has weight 0, no matter how we construct $G_{\mathcal{T}}$ from $\mathcal{T}$, the vertex v will still have degree 1 and so the resulting graph cannot be 2-connected. Now suppose we single out a region R. The number κ of nests relative to R is equal to the number of vertices in H_R. To ensure a nonsplit link, we must make S^0 identifications so that H_R is connected; minimally, we will thus

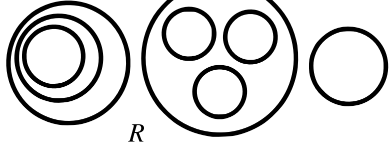

Figure 5. There are three nests relative to the region R.

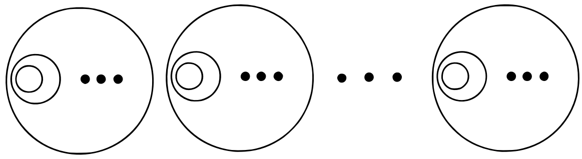

Figure 6. A simple κ-nesting.

need $(\kappa - 1)$ 0-spheres and thus $2(\kappa - 1)$ points in the non-R regions. This proves the necessity of (3). Condition (2) follows: since we need $2(\kappa - 1)$ points among the non-R regions to connect them and since we are avoiding split 0-spheres, R cannot have over half of the remaining $2\ell - 2(\kappa - 1)$ points. □

We now prove sufficiency in a specific base case before proving it in general. In this proof we will primarily refer to nestings rather than weighted trees. We define a *simple κ-nesting* to be an embedding of q 1-spheres in S^2 that can achieve the arrangement of simple nests as in Figure 6 through spherical isotopy. The corresponding tree is a (possibly topologically nonreduced) star.

Given a simple nesting $\mathcal{N}$, we will call the κ simply connected regions of $S^2 - \mathcal{N}$ *innermost* (corresponding to leaves). We call the region with fundamental group $\mathbb{Z} * \cdots * \mathbb{Z}$ (where $\mathbb{Z}$ appears $\kappa - 1$ times) *outermost*. Any other region (though there need not be any regions beyond the innermost and outermost ones) in a simple κ-nesting is annular, with fundamental group $\mathbb{Z}$. When we refer to nests in a simple nesting, we will always work relative to the outermost region, denoting H_R as just H.

Lemma 2.2. *The conditions in Theorem 1.1 are sufficient for a nonsplit $(q+\ell)$-link in a simple κ-nesting $\mathcal{N}$.*

Proof. We will first find a way to link the circles and then the 0-spheres. Given that we use exactly $2(\kappa - 1)$ points in the former and that no innermost or annular region has more than $\ell - \kappa + 1$ (i.e., more than half of the) unpaired points after the process, the latter will follow easily. Because we want H to be connected while only matching $(\kappa - 1)$ 0-spheres, it is imperative to avoid cycles during the construction. We now state our algorithm for linking all q circles and $2(\kappa - 1)$ of the 2ℓ points given an embedding that follows the conditions of Theorem 1.1:

(1) Pick a region R with the most unpaired points; in case of ties, let R be in a nest N_R with the most total unpaired points. Pair one point from R (the *selector*) with a point (the *selected*) in another nest. If possible, let the selected point come from as-yet unchosen innermost region, making sure such a pairing does not induce a cycle in H. If our choice of R leads inevitably to either a cycle or a pairing that does not include an as-yet unchosen innermost region when such a thing exists, we adjust our choice of our selector region R_1 to be in a different nest-component (i.e., a collection of nests whose vertices in

H are in a different component from the vertex corresponding to N_R). Let R_1 have the most unpaired points of the regions in different nest-components from N_R, preferably in a nest with the most total points. Then pair a selector point from R_1 with a point in an as-yet unchosen innermost region.

(2) Mark off this S^0 so the points are disregarded for the rest of the algorithm.

(3) Repeat steps 1–2 until $(\kappa - 2)$ 0-spheres have been paired off.

(4) If each of the $(\kappa - 2)$ 0-spheres contains a point from an annular region, match a point each from the two remaining innermost regions for the last S^0. If not, follow steps 1–2 for the last S^0.

In this algorithm, we form exactly $(\kappa - 1)$ 0-spheres, so it remains to prove:

(a) that we are indeed allowed to choose points in the first step without inducing cycles given only the conditions in the theorem,

(b) that the algorithm results in a nonsplit $(q+\kappa-1)$-link, and

(c) that no innermost or annular region is left at the end of the construction with more than $\ell - \kappa + 1$ unpaired points.

(a) At some point in the algorithm, let R_0 be our initial choice for R in Step (1) and let the vertex v represent R_0's nest in H. Suppose we have not yet had to switch R. If an innermost region I in a nest whose vertex is disconnected from v does not yet have a matched point, we can match a point from R_0 with one from I without inducing a cycle. Now suppose that every innermost region in $\mathcal{N}$ has matched points. Because of the rules for choosing the initial R in each step, every nest-component will have extra points; we can use one such point in a distinct nest-component to pair with one of the R_0's without inducing a cycle.

Now suppose we are in the remaining situation: The nests with vertices in components disconnected from v each have matched innermost regions, but at least one nest (with corresponding vertex u) in v's connected component C in H has no matched points in its innermost region. Note that u and v are not necessarily distinct, but we will not have to deal with this contingency until we prove (c).

Consider the nesting corresponding to C. Because we still have an unmatched innermost region, all prior matchings had their selected points in distinct innermost regions. Thus, since C is a tree (being connected with no cycles) and the nest corresponding to u has an unmatched innermost region, the rest of the nests corresponding to C's non-u vertices must have matched innermost regions. In fact, since we assumed all the non-C nests had matched innermost regions, the u-nest is the only one without a match in all of $\mathcal{N}$. Ergo when we switch R, we will not have to do it again for the rest of the construction. Note also that when we switch the selector region, we have only one choice for the region of the selected point: it must be a point in the innermost region of the u-nest.

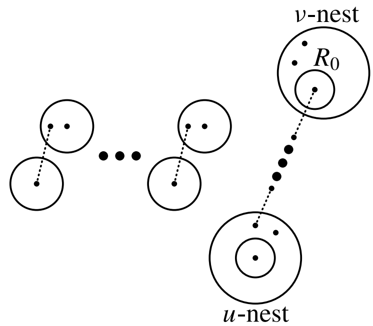

Figure 7. A simplified diagram of the situation when we have to switch R.

In addition, since the C-nesting has extra points and the nest-component containing R_1, which may no longer have extra points after deposing R_0, is now matched with the C-nesting, we can proceed as usual: all the remaining nest-components have extra points. Thus it is possible to follow our construction given the conditions of Theorem 1.1.

(b) The somewhat strict stipulations in the algorithm have a great payoff: since H has $\kappa - 1$ edges and no cycles, it is a tree, and thus connected. Since we also make sure that every innermost region has a point matched with one in another nest, it follows immediately that all the circles are linked nontrivially.

(c) We will consider four cases to prove that no innermost or annular region is left with more than $\ell - \kappa + 1$ unpaired points. Before delving in, however, we make note that since we have already paired $2(\kappa - 1)$ of the 2ℓ points, we will be left with $2\ell - 2\kappa + 2$ unpaired points; a region is left with more than $\ell - \kappa + 1$ unpaired points after the algorithm if and only if it has two or more unpaired points than any other region in $\mathcal{N}$:

(i) Suppose we finish the algorithm with an innermost region I having more than $\ell - \kappa + 1$ points left over and we never had to switch R. Since I ends with at least two more points than any other region, since the algorithm only matches a point at a time from any one region, and since I never had its "R" status revoked for the "special case" stipulations in the construction, I must have been R at each step. When we add in the $\kappa - 1$ points we matched in I, we find that I must have started with more than ℓ points, a contradiction.

(ii) Suppose we finish the algorithm with an annular region A having more than $\ell - \kappa + 1$ points left over and we never had to switch R. This case and its corresponding argument are an analog to those of (i) except for the stipulation in Step (4) of the construction; no matter, for when we add the $\kappa - 2$ matched points to A's total, we find that A must have started with more than $\ell - 1$ points, another contradiction.

(iii) Now suppose we have to switch R at some point in the algorithm and, letting R_0, v, u, and C be as above, v is distinct from u. Let m be the number of unpaired points in R_0 at this step. Since the u-nest has an unmatched innermost region, by the rules of the algorithm, it could never have been a selected nest. But since u is connected to v, the u-nest must have had an annular selector region A that at some prior step in the algorithm had a number of unpaired points greater than or equal to R_0's then-number of unpaired points. It follows that R_0 and A (and any other appropriate regions) traded off being R according to the usual rules, implying that m must be no more than one greater than the number of points in A.

When we apply the switch, the number of unmatched points in A and in R_0 remains the same. If the construction is not yet finished, we can continue in the usual way (the stipulation in Step (4) will not apply since we have already matched all the innermost regions), with R_0, A, and any other appropriate regions trading off as R. However, no matter what, we will not have an end situation in which a region has at least two more points than any other region. Thus, no annular or innermost region is left with more than $\ell - \kappa + 1$ points.

(iv) Lastly, suppose we have to switch R at some point in the algorithm and $v = u$. Let m be as in (iii). We can assume that m is strictly greater than the number of points in any other region; if there were equality, we wouldn't risk ending the algorithm with m having two more points than any other region. Note that there is at least one innermost region I in each nest-component (distinct from the C-nesting) that trades off being R with R_0 until the R switching step. Hence, m is exactly one greater than the number of points in at least one other region at the R switching step. We can narrow our focus to the case where there is only one component distinct from C. If there were not, in the step after switching R, a point in R_0 would pair with a point in another component (which has a region with $m-1$ points), thus preventing R_0 from finishing the algorithm with two more points than any other region.

In the case of only two components, the R switching step is the last step of the construction. Thus, we must show that $m \leq \ell - \kappa + 1$. Consider the situation at the beginning of the R switching step. Because the v-nest still has an unmatched innermost region, it has strictly more than m points. Thus, by the rules of choosing R in case of ties, the nest containing I must have some annular region with a points, where $a \geq 1$. Figure 8 illustrates the situation. At this stage we need at least $m + (m-1) + a + 1 + 2(\kappa - 2) = 2m + a + 2(\kappa - 2)$ of the 2ℓ total points. Now suppose that $m \geq \ell - \kappa + 2$. Then we have at least $2(\ell - \kappa + 2) + a + 2(\kappa - 2) = 2\ell + a$ points. But since $a \geq 1$, we have reached a contradiction. Thus no region in this case is left with more than $\ell - \kappa + 1$ points after the construction.

It now only remains to show that we can pair up the unmatched points so that there are no split 0-spheres. To do so, we use the following algorithm:

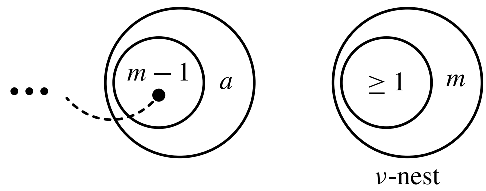

Figure 8. The situation before switching R when $v = u$.

(1) Pick two regions A and B, each having a number of points greater than or equal to that of any other region in $S^2 - \mathcal{N}$.

(2) Form an S^0 from a point in A and a point in B.

(3) If there are still unpaired points, return to the first step. If not, we are done.

Suppose that we have followed through with this algorithm but still have at least one split S^0 in some region $\mathcal{R}$. Note that $\mathcal{R}$ is the only region left with unpaired points; if there were others we could continue with the algorithm. In addition, if we run the algorithm backwards, one of the two most recently matched points must have come from $\mathcal{R}$. In fact, this is true at each step: since $\mathcal{R}$ has at least two more points than any other region at the end of each step, it must have been one of the regions with the most points at the beginning of any step. Thus, if we run the algorithm all the way back ($\ell - \kappa + 1$ steps), counting the number of points in $\mathcal{R}$ along the way, we find that $\mathcal{R}$ must have started this second algorithm with at least $\ell - \kappa + 2$ points, a contradiction.

Thus it is possible to find a nonsplit (ℓ+q)-link in a simple κ-nesting given the conditions of Theorem 1.1. □

We can now show sufficiency for any nesting with points.

Lemma 2.3. *The conditions of Theorem 1.1 are sufficient for a* (q+ℓ)*-link*

Proof. We will use induction on the number of vertices in the weighted tree. Lemma 2.2 covered the base case, so assume that the conditions of the theorem are sufficient for a (q−1+ℓ)-link (i.e., on any tree with q vertices). Let $\mathcal{T}_0$ be a weighted tree with $q + 1$ vertices that follows the conditions of Theorem 1.1.

Let v_0 be a leaf of $\mathcal{T}_0$ with weight μ_1 and let u_0 be the vertex adjacent to v_0, with weight μ_2. Now suppose we delete v_0 from $\mathcal{T}_0$ and absorb its weight into u_0. From this we get a new weighted tree $\mathcal{T}_1$, where $u_1 \in V(\mathcal{T}_1)$ used to be u_0. Note that $\deg(u_1) = \deg(u_0) - 1$. Obviously this move preserves the first condition of Theorem 1.1 in $\mathcal{T}_1$. Suppose first that the move preserves the second condition: u_1's weight, $\mu_1 + \mu_2$, is less than or equal to $\ell - \deg(u_1) + 1 = \ell - \deg(u_0) + 2$.

We first aim to show that $\mathcal{T}_1$ follows the third condition given that it follows the second. Let $\mu_3 = 2\ell - (\mu_1 + \mu_2)$. We want to show that $\mu_3 \geq 2(\deg(u_1) - 1) =$

$2(\deg(u_0)-2)$. Because $\mathcal{T}_0$ follows the rules, we have

$$\mu_1+\mu_3 \geq 2(\deg(u_0)-1), \tag{1}$$

and because of our assumption on $\mathcal{T}_1$,

$$\mu_1+\mu_2 \leq \ell-\deg(u_0)+2. \tag{2}$$

By the bound given by (2) and the definition of μ_3, we have $\mu_3 \geq \ell+\deg(u_0)-2$, so if $\ell \geq \deg(u_0)-2$, we're in the clear. Henceforth assume that $\ell \leq \deg(u_0)-3$. From (1), we have $\mu_3 \geq 2(\deg(u_0)-1)-\mu_1$. Using the upper bound on μ_1 from (2) and the one on ℓ, we obtain

$$\begin{aligned}
\mu_3 &\geq 2(\deg(u_0)-1)-(\ell-\deg(u_0)+2-\mu_2)\\
&= 2(\deg(u_0)-2)-\ell+\deg(u_0)+\mu_2\\
&\geq 2(\deg(u_0)-2)-(\deg(u_0)-3)+\deg(u_0)+\mu_2\\
&= 2(\deg(u_0)-2)+3+\mu_2\\
&> 2(\deg(u_0)-2),
\end{aligned}$$

which we sought.

We have thus shown that if we choose a v_0 to delete such that $\mathcal{T}_1$ follows the second condition, it will follow all the conditions. Thus, we can use the inductive assumption to add edges to $\mathcal{T}_1$ using the weights so that the resulting multigraph $G_{\mathcal{T}_1}$ is 2-connected and contains no loops. When we add v_0 back (along with edge v_0u_0), we transfer μ_1 of u_1's added edges to v_0. This operation will certainly not create any loops. We now show that it preserves 2-connectivity. Consider any vertex w that is not v_0 or u_0: the operation preserves the two internally disjoint paths between any two vertices that are not v_0 or u_0, so if we were to delete w, all the other non-(u_0 or v_0) vertices would remain connected. But u_0 and v_0 would also be connected to the rest of the graph since they are connected to each other and at least one other non-w vertex. Now suppose we delete u_0 from $\mathcal{T}_0$. Again, all the non-v_0 vertices will still be connected. But v_0 will also be connected to the rest of the graph since it is adjacent to at least one non-u_0 vertex. Lastly, suppose we delete v_0: the rest of the graph is still connected by the $\mathcal{T}_1$ edges. Thus, the multigraph $G_{\mathcal{T}_0}$ induced by $G_{\mathcal{T}_1}$ is 2-connected and without loops and thus determines a nonsplit planar $(q+\ell)$-link

It now only remains to show that we can pick a v_0 to remove such that u_1 has weight less than or equal to $\ell-\deg(u_1)+1$. Suppose we cannot find such a v_0. Let λ be the number of leaves in $\mathcal{T}_0$ and let $\kappa=\max\{\deg(v) : v \in V(\mathcal{T}_0)\}$. Since we have already shown the result for simple nestings in Lemma 2.2, we can assume $\kappa \leq \lambda-1$, that $\lambda \geq 4$, and that there are at least two u_0s we could have depending on our choice of v_0. Also, since any u_0 has weight less than or equal to $\ell-\deg(u_0)+1$ and the corresponding u_1 has weight greater than or equal to $\ell-\deg(u_0)+3$, each leaf must have weight at least 2. Thus the total weight of $\mathcal{T}_0$ is at least $2(\ell-\kappa+3)+2(\lambda-2) \geq 2\ell-2\kappa+6+2(\kappa-1) = 2\ell+4$, a contradiction. Thus we are able to choose a "nice" v_0 such that the inductive hypothesis holds and is inherited by the larger tree. □

3. Enumeration

We mentioned in the Introduction that there is only one nonsplit spherical 2-link and there are two types of nonsplit spherical 3-links. We have now proven which embeddings will form nonsplit links given appropriate S^0 identifications. However, enumeration encompasses even more complications: we must determine whether an embedding is unique up to spherical isotopy. In addition, we have a couple different ways to count links: we can simply count the allowable embeddings or we can count how many ways we can identify 0-spheres appropriately within an embedding (Figure 9). In the link diagrams found in the online supplement, if there is more than one allowable S^0 identification for an embedding, we will write how many total identifications there are next to its image. Note that there are four distinct nonsplit 4-links; 11 distinct embeddings and 12 distinct 5-links; 32 distinct embeddings and 39 total 6-links; 105 total embeddings and 158 total 7-links; and 354 embeddings and 723 8-links.

To show rigorously how many allowable S^0 pairings there are in an embedding, one fact is particularly helpful: The number of S^0 identifications between two regions R_1 and R_2 is greater than or equal to $p(R_1) + p(R_2) - m$, where $p(R)$ is the number of unmatched points in R and m is half the number of unmatched points left in a nesting. This fact is easily proven. Let $r = p(R_1) + p(R_2)$. If $r \leq m$, the claim is trivially true. Suppose $r > m$. Then there are not enough points in the rest of the nesting to fully match with points in R_1 and R_2 without inducing split 0-spheres; we must match a point in R_1 with one in R_2. Now r has decreased by two and m has decreased by one. If $r-2 > m-1$, we again match a point from R_1 with one from R_2. We must iterate k times, where $r-2(k-1) > m-(k-1)$ and $r-2k \leq m-k$: that is when $k = r-m$, which we sought.

The above allows us to reduce larger cases to smaller cases. We also use commonsense techniques, such as choosing a region that can be distinguished from others (usually one with one point) to determine exhaustively the matching possibilities or utilizing symmetry without loss of generality.

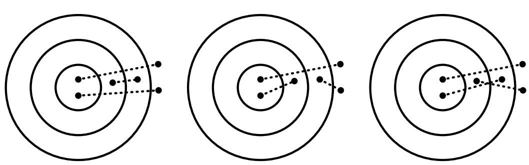

Figure 9. This is the same embedding, but there are three different 6-links here.

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burkhm2@uw.edu *Mathematics Department, University of Washington, Seattle, WA, United States*

foisyjs@potsdam.edu *Department of Mathematics, SUNY Potsdam, Potsdam, NY, United States*

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