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On a connection between local rings and their associated graded algebras

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We study a class of local rings and a local adaptation of the homogeneous property for graded rings. While the rings of interest satisfy the property in the local case, we show that their associated graded k-algebras do not satisfy the property in the graded case.

1. Introduction and preliminaries

Let $Q = k[\![X_1, X_2, \dots, X_n]\!]$ denote the power series ring in n variables over the field k. Let J be an ideal in Q. For an element $b \in J$, the initial form of b is the homogeneous finite sum of lowest-degree terms of b, denoted by b^*. Let $Q^{\mathrm{g}} = k[X_1, X_2, \dots, X_n]$ denote the polynomial ring in n variables over the field k. The initial ideal of J is the ideal in Q^{g} generated by all of the initial forms of J and is denoted by $\mathrm{In}(J)$. That is,

$$\mathrm{In}(J) = \left\{ \textstyle\sum_{i=1}^{m} a_i b_i^* \;\middle|\; a_i \in Q^{\mathrm{g}},\ b_i \in J,\ 1 \le i \le m \right\}.$$

Computations in $\mathrm{In}(J)$ are not always straightforward. The following example is intended to help illustrate some of the nuances of $\mathrm{In}(J)$.

Example 1.1. Let $Q = k[\![X, Y]\!]$ and $J = (x^2 + y^3, xy)$. Since

$$(x^2 + y^3)(-x^2 y + x^4 y^5 + x^{13} + \cdots) \quad \text{and} \quad xy(x^3 + xy^3 - x^5 y^4 + \cdots)$$

are in J, we have that the initial form of their sum

$$(-x^4 y + x^4 y + x^2 y^4 - x^2 y^4 + x^6 y^5 - x^6 y^5 + x^4 y^8 + x^{15} + x^{13} y^3 + \cdots)^* = x^4 y^8$$

is in $\mathrm{In}(J)$.

Describing $\mathrm{In}(J)$ is not as simple as finding the initial forms of the generators of J. The next example is adapted from [Eisenbud 1995], although similar examples can be found in several other texts.

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Example 1.2. Let $Q = k[\![X, Y]\!]$ and $J = (x^2 + y^3, xy)$. Then $(x^2 + y^3)^* = x^2$ and $(xy)^* = xy$, but $\operatorname{In}(J) = (x^2, y^4, xy)$. In Lemma 2.5, we provide a method to prove this fact for a more general class of rings.

Let R be a commutative local ring with maximal ideal $\mathfrak{m}$ and residue field k. By the Cohen structure theorem, the completion of any local ring can be written as a quotient of a regular local ring by an ideal. Hence, if R is a complete local ring then $R = Q/J$, where $J \subseteq (X_1, X_2, \dots, X_n)^2$.

Definition 1.3. Let R be a complete local ring with a minimal Cohen presentation $R = Q/J$, where $J = (f_1, f_2, \dots, f_l)$ with $f_i \in Q$ for $1 \le i \le l$. If f_i^* has degree t for each i then R is t-homogeneous.

In [Hoffmeier and Şega 2017] the authors give a more general version of the above definition. They go on to show that knowing a ring is t-homogeneous is helpful for identifying various homological properties. Indeed, Theorem 2.5 of that paper establishes that the t-homogeneous property plays an important role connecting these homological traits of local rings.

Let $J = (f_1, f_2, \dots, f_l) \subseteq Q^{\mathrm{g}}$ be the ideal generated by polynomials f_i in Q^{g} for $1 \le i \le l$. If each of the f_i is homogeneous of degree t then the quotient $R = Q^{\mathrm{g}}/J$ is a t-homogeneous graded k-algebra.

The associated graded ring of R with respect to the maximal ideal is the direct sum

$$R^{\mathrm{g}} = \bigoplus_{i \ge 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

This notation is consistent with Q^{g}. That is, for the local ring $Q = k[\![X_1, X_2, \dots, X_n]\!]$, we have $Q^{\mathrm{g}} = k[X_1, X_2, \dots, X_n]$. Furthermore, if $R = Q/J$ then $R^{\mathrm{g}} = Q^{\mathrm{g}}/\operatorname{In}(J)$.

We now state [Hoffmeier and Şega 2017, Lemma 1.3], which also provides further motivation for the terminology given in Definition 1.3.

Lemma 1.4. *Let R be a complete local ring. If R^{g} is a t-homogeneous k-algebra, then R is a t-homogeneous local ring.*

Hoffmeier and Şega [2017, Remark 1.4] also provide a counterexample to show that the converse of the lemma does not hold. We now reproduce this example.

Example 1.5. Let $Q = k[\![X, Y]\!]$, $J = (x^2 + y^3, xy)$, and $R = Q/J$. Then R is 2-homogeneous. However, $R^{\mathrm{g}} = Q^{\mathrm{g}}/\operatorname{In}(J) = k[X, Y]/(x^2, y^4, xy)$, which is not 2-homogeneous.

It is significant that the converse of Lemma 1.4 does not hold. Otherwise, the t-homogeneous property of a local ring R would depend only on its associated graded k-algebra R^{g}, making the connections between the homological properties of R alluded to above (stated in [Hoffmeier and Şega 2017, Theorem 2.5]) also related to R^{g}. The main goal of this note is to identify a larger class of rings for which the converse of the lemma fails, which consequently further distinguishes the homological nature of local rings from properties of their associated graded k-algebras. We achieve this in the next section by generalizing Example 1.5.

Further motivation for our result is the fact that Example 1.5 is stated without proof in [Hoffmeier and Şega 2017] and is therefore further explained by the proof of our more general result.

Remark 1.6. Connections between a local ring and its associated graded algebra have been well documented throughout the literature of commutative algebra. For example, if R^{g} is Cohen–Macaulay then R is Cohen–Macaulay and if R^{g} is Gorenstein then R is Gorenstein; see, e.g., [Achilles and Avramov 1982]. The text [Bruns and Herzog 1993] also states several of these results and is a good reference for other topics that appear in this note. In his survey on the subject, Fröberg [1987] states that "A local ring is at least as nice as its associated graded ring." Our results provide another example that makes the inequality Fröberg alludes to strict.

2. Unassociated t-homogeneous local rings

In this section we prove our main result. We begin with a definition.

Definition 2.1. Let R be a t-homogeneous local ring. If R^{g} is not a t-homogeneous graded k-algebra then we say that R is unassociated t-homogeneous.

Theorem 2.2. *Let $J = (x^2 + y^t, xy) \subseteq Q = k[\![X, Y]\!]$ with $t \geq 3$ and set $R = Q/J$. Then R is unassociated 2-homogeneous.*

Remark 2.3. Note that by setting $t = 3$ in Theorem 2.2, we recover the result in Example 1.5.

We now provide two lemmas which will be used in the proof of the theorem.

Lemma 2.4. *Let $J = (x^2 + y^t, xy) \subseteq Q = k[\![X, Y]\!]$ with $t \geq 3$. Then y^t is not in* $\mathrm{In}(J)$.

Proof. Suppose $y^t \in \mathrm{In}(J)$. Then

$$y^t = \sum_{i=1}^{m} a_i b_i^*,$$

where $a_i \in Q^{\mathrm{g}}$, $b_i \in J$, and $1 \leq i \leq m$. For each i, let $b_i = c_i(x^2 + y^t) + d_i(xy)$ with $c_i, d_i \in Q$. Hence,

$$y^t = \sum_{i=1}^{m} a_i (c_i(x^2 + y^t) + d_i(xy))^*.$$

Since the sum equals y^t, the terms of the sum that are factors of xy either cancel or are dropped by taking the lowest-degree terms. Therefore,

$$y^t = \sum_{i=1}^{m} a_i (c_i(x^2 + y^t))^*.$$

Since $t \geq 3$, we have $(c_i(x^2+y^t))^* = c_i^* x^2$ for each i, where c_i^* is the finite sum of lowest-degree terms of c_i. Hence

$$y^t = \sum_{i=1}^{m} a_i c_i^* x^2,$$

which is a contradiction. □

Lemma 2.5. *Let* $J = (x^2 + y^t, xy) \subseteq Q = k[\![X, Y]\!]$ *as in Lemma 2.4. Then* $\mathrm{In}(J) = (x^2, y^{t+1}, xy)$.

Proof. First, we show that $(x^2, y^{t+1}, xy) \subseteq \mathrm{In}(J)$. It is sufficient to show that $x^2, y^{t+1}, xy \in \mathrm{In}(J)$, which is clear since

$$x^2 = (x^2)^*, \quad xy = (xy)^*, \quad y^{t+1} = (y(x^2+y^t) - x(xy))^*.$$

Next, we show that $\mathrm{In}(J) \subseteq (x^2, y^{t+1}, xy)$. Let $g \in \mathrm{In}(J)$. Then

$$g = a_1 F_1^* + a_2 F_2^* + \cdots + a_n F_n^*,$$

where $a_i \in k[X, Y]$ and $F_i \in J$ for $1 \leq i \leq n$. Therefore, it suffices to show if $F \in J$ then $F^* \in (x^2, y^{t+1}, xy)$. Let $\alpha, \beta \in k[\![X, Y]\!]$ such that $F = \alpha(x^2+y^t) + \beta xy$. Then

$$F^* = (\alpha x^2 + \alpha y^t + \beta xy)^* = ax^2 + by^t + cxy$$

for some $a, b, c \in k[X, Y]$. If $b = 0$ then $F^* = ax^2 + cxy \in (x^2, y^{t+1}, xy)$.

Assume $b \neq 0$. Since F^* is homogeneous, b is homogeneous and may be written as $b = px^n + qy$, where $p \in k$, $q \in k[X, Y]$, and n is a nonnegative integer. Therefore,

$$F^* = ax^2 + cxy + px^n y^t + qy^{t+1}.$$

If $p = 0$ then we again have the needed form.

Assume $p \neq 0$ and consider two cases for n.

Case (i): Assume $n \geq 1$. Then $px^n y^t = px^{n-1} y^{t-1}(xy)$. Hence,

$$F^* = ax^2 + qy^{t+1} + (c + px^{n-1} y^{t-1})xy$$

has the needed form.

Case (ii): Assume $n = 0$. Since we have already shown $(x^2, y^{t+1}, xy) \subseteq \mathrm{In}(J)$, we have

$$F^* - ax^2 - qy^{t+1} - cxy = px^n y^t \in \mathrm{In}(J).$$

Since $n = 0$ and $p^{-1} \in k$ we have $y^t \in \mathrm{In}(J)$, which contradicts Lemma 2.4. Therefore, case (ii) does not occur. □

Remark 2.6. A common approach to working with $\mathrm{In}(J)$ is to invoke the use of Gröbner bases. However, we opt for the more elementary method presented above.

We are now ready to prove the theorem.

Proof of Theorem 2.2. Since R is artinian, it is complete. Since $(x^2 + y^t)^* = x^2$ and $(xy)^* = xy$, we know R is 2-homogeneous. As noted above,

$$R^{\mathrm{g}} = Q^{\mathrm{g}} / \operatorname{In}(J).$$

By Lemma 2.5, $\operatorname{In}(J) = (x^2, y^{t+1}, xy)$. Hence, R^{g} is not a graded k-algebra and the theorem follows. □

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