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Six variations on a theme: almost planar graphs

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A graph is *apex* if it can be made planar by deleting a vertex, that is, there exists v such that $G - v$ is planar. We also define several related notions; a graph is *edge apex* if there exists e such that $G - e$ is planar, and *contraction apex* if there exists e such that G/e is planar. Additionally we define the analogues with a universal quantifier: for all v, $G - v$ is planar; for all e, $G - e$ is planar; and for all e, G/e is planar. The graph minor theorem of Robertson and Seymour ensures that each of these six notions gives rise to a finite set of obstruction graphs. For the three definitions with universal quantifiers we determine this set. For the remaining properties, apex, edge apex, and contraction apex, we show there are at least 36, 55, and 82 obstruction graphs respectively. We give two similar approaches to almost nonplanar (there exists e such that $G+e$ is nonplanar, and for all e, $G + e$ is nonplanar) and determine the corresponding minor minimal graphs.

1. Introduction

Kuratowski [1930] showed that the set of planar graphs is determined by two obstructions.

Theorem 1.1 [Kuratowski 1930; Wagner 1937]. *A graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor.*

We give the formulation in terms of minors due to Wagner [1937] to make the connection with Robertson and Seymour's [2004] graph minor theorem. We say H is a *minor* of graph G if it can be obtained by contracting edges in a subgraph of G. We can state the graph minor theorem as follows.

Theorem 1.2 [Robertson and Seymour 2004]. *In any infinite set of graphs, there is a pair such that one is a minor of the other.*

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This has two useful consequences. We say G is *minor minimal* $\mathcal{P}$ (or MM$\mathcal{P}$) if G has property $\mathcal{P}$ but no proper minor does.

Corollary 1.3. *For any graph property* $\mathcal{P}$*, there is a corresponding finite set of minor minimal* $\mathcal{P}$ *graphs.*

Corollary 1.4. *Let* $\mathcal{P}$ *be a graph property that is closed under taking minors. Then there is a finite set of minor minimal non-*$\mathcal{P}$ *graphs* S *such that for any graph* G*,* G *satisfies* $\mathcal{P}$ *if and only if* G *has no minor in* S*.*

When $\mathcal{P}$ is minor closed, we say that S is the *Kuratowski set* for $\mathcal{P}$. For example, $\{K_5, K_{3,3}\}$ is the Kuratowski set for planarity.

The graph minor theorem is not constructive, so there are only a few graph properties $\mathcal{P}$ for which we know the finite set of MM$\mathcal{P}$ graphs. In particular, there are several graph properties closely related to planarity for which this set is unknown. Our goal in this paper is to investigate the minor minimal sets for the following eight graph properties.

Definition 1.5. A planar graph is *almost nonplanar* (*AN*) if there exist two nonadjacent vertices such that adding an edge between the vertices yields a nonplanar graph. A planar graph is *completely almost nonplanar* (*CAN*) if it is not complete and adding an edge between any pair of nonadjacent vertices yields a nonplanar graph.

Let $G - v$ denote the graph resulting from deletion of vertex v and its edges in G, let $G - e$ denote the graph resulting from the deletion of edge e in G, and let G/e denote the graph resulting from the contraction of edge e in G.

Definition 1.6. A graph is *not apex* (*NA*) if, for all vertices v, $G - v$ is nonplanar. Similarly, a graph is *not edge apex* (*NE*) if, for all edges e, $G - e$ is nonplanar and *not contraction apex* (*NC*) if, for all edges e, G/e is nonplanar.

Definition 1.7. A graph G is *incompletely apex* (*IA*) if there is a vertex v such that $G - v$ is nonplanar, *incompletely edge apex* (*IE*) if there is an edge e such that $G - e$ is nonplanar, and *incompletely contraction apex* (*IC*) if there is an edge e such that G/e is nonplanar.

We call these last three properties "incomplete" in contrast to their negations. For example, we think of a graph as "completely" apex if $G - v$ is planar for every vertex v. Table 1 gives a summary of our eight definitions.

We summarize our results in Table 2. Four of the properties give Kuratowski sets (as their negation generates a minor closed set) and with the exception of NA, NE, and NC, we determine the finite set of MM$\mathcal{P}$ graphs. For the remaining three properties we give a lower bound, which is simply the number of MM$\mathcal{P}$ graphs we have found, so far.

Our paper is organized as follows. Below we conclude this introduction with a survey of the literature and provide some preliminary notions used throughout the

property	definition
AN	$\exists e$ such that $G+e$ is nonplanar, where G is planar
CAN	$\forall e,\ G+e$ is nonplanar, where G is planar, not complete
NA	$\forall v,\ G-v$ is nonplanar
NE	$\forall e,\ G-e$ is nonplanar
NC	$\forall e,\ G/e$ is nonplanar
IA	$\exists v$ such that $G-v$ is nonplanar
IE	$\exists e$ such that $G-e$ is nonplanar
IC	$\exists e$ such that G/e is nonplanar

Table 1. Comparison of the eight definitions.

graph property $\mathcal{P}$	AN	CAN	NA	NE	NC	IA	IE	IC
Is (not $\mathcal{P}$) minor closed?	no	no	yes	no	no	yes	yes	yes
number of MM$\mathcal{P}$ graphs	2	1	≥ 36	≥ 55	≥ 82	2	5	7

Table 2. Results for the eight graph properties.

paper. In Section 2 we determine the MMAN and MMCAN graphs and show that neither is a Kuratowski set. In Section 3 we give our classification of the MMIA, MMIE, and MMIC graphs, all three of which we show are Kuratowski. In Section 4 we give an overview of the MMNA graphs, which is a Kuratowski set. We classify graphs in this family of connectivity at most 1. For graphs of connectivity 2, with $\{a, b\}$ a 2-cut, we classify those for which $ab \in E(G)$, as well as those for which a component of $G-a, b$ is nonplanar. We also prove that an MMNA graph has connectivity at most 5. In total, we give explicit constructions for 36 MMNA graphs. Finally, in Section 5 we discuss MMNE and MMNC graphs, first showing these are not Kuratowski. We classify graphs of connectivity at most 1 in these two families and discuss computer searches, complete through graphs of order 9 or size 19, that yielded 55 MMNE and 82 MMNA graphs.

Apex graphs are well-studied, including results on MMNA graphs in [Ayala 2014; Barsotti and Mattman 2016; Pierce 2014]. Note that [Pierce 2014] reports on a computer search that yields 157 MMNA graphs, including all graphs through order 10 or size 21 and most of the 36 graphs we describe here. Different authors have used terms like "almost planar" or "near planar" in various ways. Here is how our definitions relate to others in the literature. Cabello and Mohar [2013] say that a graph is *near-planar* if it can be obtained from a planar graph by adding an edge. This corresponds to our definition of edge apex. Wagner [1967] defined *nearly planar* (*Fastplättbare*), which corresponds to our idea of completely apex or not IA. Two further notions of almost planar are not directly related to the properties we have defined. For Gubser [1996], a graph G is *almost planar* if for every edge e, either $G - e$ or G/e is planar. In characterizing graphs with no $K^{\aleph_0}$, Diestel, Robertson, Seymour, and Thomas say a graph G is *nearly planar* if deleting a bounded number of vertices makes G planar except for a subgraph of bounded linear width sewn onto the unique cuff of $S^2 - 1$; see [Diestel 2010, Section 12.4]. Finally, our notion of CAN is also known as *maximally planar*; see [Diestel 2010].

We conclude this introductory section with some notation and definitions, as well as a lemma, used throughout. For us, graphs are simple (no loops or double edges) and undirected. We use $V(G)$ and $E(G)$ to denote the vertices and edges of a graph. The *order* of a graph is $|V(G)|$ and $|E(G)|$ is its *size*. We use $\delta(G)$ to denote the *minimum degree* of all the vertices in G.

As mentioned earlier, $G - v$, $G - e$, and G/e denote the results of vertex deletion, edge deletion, and edge contraction, respectively. For $v, w \in V(G)$, the graph $G - v, w$ is the result of deleting two vertices and their edges. Similarly, for $e, f \in E(G)$, we define as $G - e, f$ the result of deleting two edges and $G/e, f$ the result of contracting two edges. Note that the order of deletion or contraction is arbitrary. Contracting an edge may result in a double edge. We will assume that one of the doubled edges is deleted so that G/e is again a simple graph. We use $G_1 \sqcup G_2$ to denote the disjoint union of two graphs and $G_1 \dot{\cup} G_2$ for the union identified on a single vertex. Similarly, $G_1 \ddot{\cup} G_2$ denotes the union of two graphs identified on two vertices.

In light of Kuratowski's theorem, we call K_5 and $K_{3,3}$ the *Kuratowski graphs* and also refer to them as minor minimal nonplanar or MMNP. A *Kuratowski subgraph* or *K-subgraph* of G is one homeomorphic to a Kuratowski graph. A *cut set* of graph G is a set $U \subset V(G)$ such that deleting the vertices of U and their edges results in a disconnected graph. If $|U| = k$, we call U a *k-cut*. We say G has *connectivity* k and write $\kappa(G) = k$ if k is the largest integer such that $|V(G)| > k$ and G has no l-cut for $l < k$. In particular, $\kappa(K_n) = n - 1$.

We conclude this introduction with a useful lemma. In the case that $\kappa(G) = 2$, we have $G - a, b = G_1' \sqcup G_2'$, where $\{a, b\}$ is a 2-cut. We will use G_i to denote the induced subgraph on $V(G_i') \cup \{a, b\}$. In the literature, e.g., [Mohar and Thomassen 2001], the pair (G_1, G_2) is called a separation of order 2 (since $|G_1 \cap G_2| = 2$).

Lemma 1.8. *If G is homeomorphic to K_5 or $K_{3,3}$ with cut set $\{a, b\}$ such that $G - a, b = G_1' \sqcup G_2'$, then one of G_1 and G_2 is an a-b-path.*

Proof. Since, $\kappa(K_5) = 4$ and $\kappa(K_{3,3}) = 3$, G must be a proper subdivision of a Kuratowski graph and, since they disconnect the graph, a and b are vertices on a subdivided edge of the underlying K_5 or $K_{3,3}$. This means that one of the components is simply an a-b-path. □

2. Almost nonplanar: MMAN and MMCAN graphs

In this section we classify the MMAN and MMCAN graphs. Let $K_5 - e$ denote the complete graph on five vertices with an edge deleted and $K_{3,3} - e$ the result of deleting an edge in the complete bipartite graph $K_{3,3}$. The unique MMCAN graph is $K_5 - e$ and there are two MMAN graphs, $K_5 - e$ and $K_{3,3} - e$. Neither of these are Kuratowski sets, since, for example, K_5 is a nonplanar graph (hence neither AN nor CAN) that contains the MMAN and MMCAN graph $K_5 - e$ as a minor.

Our classification of the minor minimal CAN graphs makes use of a theorem due to Mader.

Theorem 2.1 [Mader 1998]. *Any graph with n vertices and at least* $3n - 5$ *edges contains a subdivision of* K_5.

In [Diestel 2010], CAN is called *maximally planar*, and it is proved equivalent to a graph admitting a plane triangulation in Proposition 4.2.8 of that text.

Theorem 2.2. *Every plane triangulation with at least five vertices has* $K_5 - e$ *as a minor.*

Proof. Let G be a plane triangulation on at least five vertices. By Euler's formula, $|E(G)| = 3(|V(G)| - 2)$. Let G' be a nonplanar graph obtained by adding edge ab to G. Then $|E(G')| = |E(G)| + 1 = 3|V(G)| - 5$. By Mader's theorem G' has a subgraph H homeomorphic to K_5. Note that we must have $ab \in E(H)$, else H would be planar. Since H is homeomorphic to K_5, contracting appropriate edges in $H - ab$ will result in $K_5 - e$, showing that $K_5 - e$ is a minor of G. □

Corollary 2.3. *The only MMCAN graph is* $K_5 - e$.

Theorem 2.4. *The MMAN graphs are* $K_5 - e$ *and* $K_{3,3} - e$.

Proof. First note that these two graphs are MMAN. Let G be AN and let ab be the edge that is added to form the nonplanar G'. By Kuratowski's theorem G' contains a subdivision H of K_5 or $K_{3,3}$ and $ab \in E(H)$. By contracting edges, H gives $K_5 - e$ or $K_{3,3} - e$ as a minor of G. So G is MMAN only if it is one of these two. □

3. Incomplete properties: MMIA, MMIE, and MMIC graphs

In this section we classify the MMIA, MMIE, and MMIC graphs. Note that each is a Kuratowski set since the corresponding "complete" property is minor closed. In the case of the IA graphs, for example, suppose G is not IA and let H be a subgraph of G. Then for any $v \in V(H)$, the graph $H - v$ is planar since it is a subgraph of the planar graph $G - v$. Similarly if G is not IA, let $H = G/f$ for some $f \in E(G)$. Then for any $v \in V(H)$, the graph $H - v$ is planar since it is a minor of the planar graph $G - v$. This shows that the property *not* IA (also known as the completely apex property) is minor closed. Similar arguments show that not IE and not IC are also minor closed.

We next show there are exactly two MMIA graphs, $K_1 \sqcup K_5$ and $K_1 \sqcup K_{3,3}$. We begin by classifying the disconnected graphs.

Theorem 3.1. *If G is not connected and MMIA, then $G = K_1 \sqcup G_2$, where $G_2 \in \{K_5, K_{3,3}\}$.*

Proof. Note that both $K_1 \sqcup K_5$ and $K_1 \sqcup K_{3,3}$ are MMIA. If $G = G_1 \sqcup G_2$ is nonplanar with neither component empty, then K_5, or $K_{3,3}$ is a minor of one of G_1 and G_2. By minor minimality this means one of G_1 and G_2 is a Kuratowski graph, and, again by minimality, the other component can have no nontrivial proper minors, so must be simply a vertex. □

Theorem 3.2. *There are no connected MMIA graphs.*

Proof. Suppose instead that G is a connected MMIA graph. Then there is a vertex, v, such that $G - v$ is nonplanar. However, since G is connected, v must have at least one edge, e. Since when deleting a vertex we also delete all of its edges, $G - e$ must be a proper, nonplanar minor of G. However, deleting $v \in V(G - e)$ is again nonplanar so that $G - e$ is IA. This contradicts the property that G is MMIA and therefore cannot happen. □

Corollary 3.3. *There are two MMIA graphs: $K_1 \sqcup K_5$ and $K_1 \sqcup K_{3,3}$.*

Next we show there are five MMIE graphs. We begin with the disconnected examples. Note that if G has distinct edges e, e' such that $G - e, e'$ is nonplanar, then G is not MMIE. Indeed, $G - e$ is an IE proper minor.

Theorem 3.4. *If G is not connected and MMIE, then $G = K_2 \sqcup G_2$, where $G_2 \in \{K_5, K_{3,3}\}$.*

Proof. The proof is similar to that of Theorem 3.1, but now the planar component is minor minimal among graphs with edges, so K_2. □

Recall that $G_1 \dot{\cup} G_2$ denotes the union of G_1 and G_2 with one vertex in common.

Theorem 3.5. *If G is connected, MMIE, and has a cut vertex, then $G = K_2 \dot{\cup} G_2$, where $G_2 \in \{K_5, K_{3,3}\}$.*

Proof. Let G be a connected MMIE graph such that $G - v = G_1' \sqcup G_2'$. Let G_i denote the induced subgraph on $V(G_i') \cup \{v\}$. If both G_1 and G_2 are nonplanar, then G would not be MMIE since, for example, there are two distinct edges $e, e' \in E(G_2)$ such that $G - e, e'$ contains G_1 and is therefore nonplanar. If both subgraphs were planar, then G would also be planar and therefore not MMIE. So one of G_1 and G_2 is nonplanar, say G_1, and the other, G_2, is planar.

By minor minimality of G, the nonplanar G_2 is, in fact, a Kuratowski graph, and the planar G_1 is minimal among graphs with edges, i.e., K_2. □

Theorem 3.6. *If G is MMIE, then there is a unique edge e such that $G-e$ is nonplanar.*

Proof. Assume, for the sake of contradiction, that there are $e, e' \in E(G)$ such that $e \neq e'$ but $G-e$ and $G-e'$ are nonplanar. If $G-e$ is nonplanar, then there is a subgraph of $G-e$, say H, with $e \notin E(H)$, that has a K_5 or $K_{3,3}$ minor. Likewise, if $G-e'$ is nonplanar, then it has a nonplanar subgraph H' with $e' \notin E(H')$. If $H' = H$, then $e' = e$. Otherwise, $G-e, e'$ would be nonplanar, contradicting that G is MMIE. So $H' \neq H$. If $e \notin H'$, then $G-e, e'$ contains H' and will be nonplanar, contradicting that G is MMIE.

So, $e \in H'$ and, similarly, $e' \in H$. If H and H' have empty intersection, then let $e_1, e_2 \in E(H')$. This means $G-e_1, e_2$ contains H and is nonplanar. This contradicts that G is MMIE. So, H and H' have nonempty intersection. If their intersection is nonplanar, then removing e and e' will not change this intersection, and G is not MMIE. If their intersection is planar, then there must be more than one edge in H' that is not in H besides e. But, if H' has more edges besides e that are not in H it would be possible to remove another edge, $f \neq e$, without changing H. This means that $G-f, e$ is nonplanar, and contradicts that G is MMIE.

Therefore, if G is MMIE, then there is a unique edge e such that $G-e$ is nonplanar. □

Recall that a K-subgraph is one homeomorphic to K_5 or $K_{3,3}$.

Theorem 3.7. *If G is MMIE, then the edge e such that $G-e$ is nonplanar is not in a K-subgraph. Furthermore, $G-e$ is K_5 or $K_{3,3}$.*

Proof. Assume, for the sake of contradiction, that e is in a K-subgraph, H. Since no graph homeomorphic to K_5 or $K_{3,3}$ is IE, $G-e$ is planar unless $G-e$ contains some other K-subgraph, H'. However, if G contains two K-subgraphs H and H' with empty intersection, then $G-e$ will leave H' unchanged. One could then remove a second edge, $f \in E(H)$, leaving H' unchanged so that $G-e, f$ is nonplanar. This means that G cannot be MMIE since G would have an IE minor $G-e$. So, H and H' have nonempty intersection. But $H \neq H'$ since e cannot be an edge in the only K-subgraph, otherwise $G-e$ is planar.

Next, observe that any proper subgraph of a K-subgraph is planar. This means that for the K-subgraph, H', with $H \neq H'$, there must be an edge, $g \neq e$, with $g \in E(H')$ and $g \notin E(H)$. Then $G-g$ contains H and is nonplanar. This contradicts the uniqueness of the edge e and shows e is not in a K-subgraph.

Following the same argument as above, G cannot contain more than one K-subgraph. Indeed, if there were distinct K-subgraphs H and H', then either the intersection is empty or it is not, and we achieve similar contradictions as in the previous argument. So, G contains exactly one K-subgraph.

Finally, the only possible K-subgraph contained in G, call it N, must contain all edges besides e. If not, then there is an edge $e' \neq e$ such that $G - e'$ is nonplanar. This contradicts the uniqueness of e. Also, the K-subgraph N in $G - e$ must be either K_5 or $K_{3,3}$. If not, then N would be a subdivision of either K_5 or $K_{3,3}$. But, then there is a proper minor, G', of G, by contracting an edge, $e_1 \in E(N)$, which contains a K-subgraph as well. Provided e remains as an edge of G', the graph $G' - e$ is nonplanar, contradicting that G is minor minimal. On the other hand, if contracting e_1 removes e, then there must be another edge e_2 incident to e_1, with $e_2 \in E(N)$, such that e is incident to both e_1 and e_2. Since N is a subdivision of K_5 or $K_{3,3}$ and G/e_1 is nonplanar, e_1 and e_2 must be in a path of N formed by subdividing an edge of the underlying Kuratowski graph. Since e is incident to both e_1 and e_2, there exists N', another K-subgraph of G with $e \in E(N')$. This contradicts that there is only one K-subgraph of G.

So, if G is MMIE then it is made up of either K_5 or $K_{3,3}$ and an edge that is not in this K-subgraph. □

Aside from the disconnected and connectivity-1 examples above, a final way to add an edge to a K-subgraph is the graph $K_{3,3} + e$ of Figure 1, formed by adding an edge to the bipartite graph $K_{3,3}$.

Corollary 3.8. *There are five MMIE graphs*: $K_{3,3}+e$ *and* $K_2 \sqcup G_2$, $K_2 \dot{\cup} G_2$, *where* $G_2 \in \{K_5, K_{3,3}\}$.

Let $\overline{K}_5$ and $\overline{K}_{3,3}$ denote the graphs obtained from K_5 and $K_{3,3}$ by subdividing a single edge, as in Figure 1. We denote as $K_{3,3} + 2e$ the graph given by adding two edges to $K_{3,3}$, as in Figure 1.

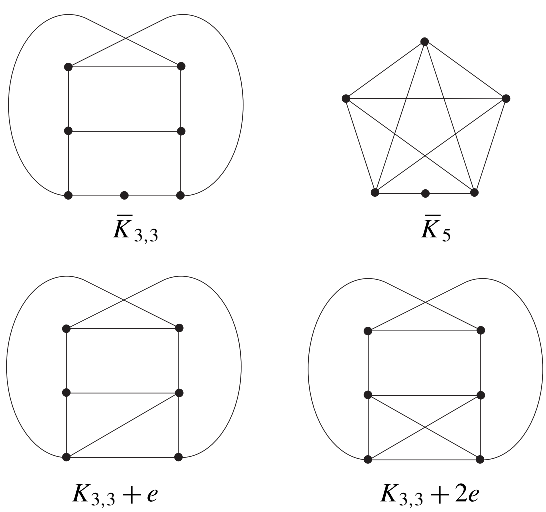

Figure 1. MMIE and MMIC graphs.

Theorem 3.9. *There are seven MMIC graphs*: $K_{3,3}+2e$ *and* $\overline{K}$, $K_2 \sqcup K$, *and* $K_2 \dot{\cup} K$ *with* $K \in \{K_5, K_{3,3}\}$.

Proof. Observe that these seven graphs are MMIC. If G is MMIC and disconnected, then G is $K_2 \sqcup K$, with K a Kuratowski graph. We omit the proof, which is similar to that for MMIE. Note that the remaining five graphs are precisely the graphs that result when a vertex of a Kuratowski graph is split.

Suppose G is MMIC and connected. Then there is an edge e such that G/e is nonplanar. Since contracting an edge will not disconnect the graph, G/e is also connected and has a K-subgraph H. If H is not a Kuratowski graph, then it has $\overline{K}_5$ or $\overline{K}_{3,3}$ as a minor, contradicting G being minor minimal. Therefore, H is Kuratowski.

If $V(H) \neq V(G/e)$, then since G/e is connected, considering any vertex in G/e beyond those in H, along with one of its edges, shows that G/e contains $K_2 \sqcup K$ or $K_2 \dot{\cup} K$, with K Kuratowski, contradicting G being minor minimal. So, $V(H) = V(G/e)$.

Now G is obtained from G/e by a vertex split. The corresponding vertex split on H gives rise to a graph H', which is one of the five graphs $K_{3,3}+2e$, $\overline{K}$, or $K_2 \dot{\cup} K$. Since G is minor minimal, $G = H'$ and is one of these five, and hence one of the seven. □

4. MMNA graphs

In this section we describe several partial results toward a classification of the MMNA graphs, with a focus on graph connectivity. In all, we describe 36 MMNA graphs, including all those of connectivity at most 1 ($\kappa(G) \leq 1$). For graphs with $\kappa(G) = 2$, where $\{a, b\}$ is a 2-cut, we classify the MMNA graphs having $ab \in E(G)$, as well as those for which a component of $G - a, b$ is nonplanar. We also show that $\kappa(G) \leq 5$ for MMNA graphs, which is a sharp bound. Since the family of apex graphs is minor closed, the MMNA graphs are a Kuratowski set.

We first bound the minimum degree, $\delta(G)$, of an MMNA graph and then classify the examples with $\kappa(G) \leq 1$.

Theorem 4.1. *The minimum vertex degree in an MMNA graph is at least* 3.

Proof. The addition or deletion of an isolated vertex or vertex of degree 1 in a planar graph will again result in a planar graph. Similarly, contracting an edge adjacent to a degree-2 vertex will not affect planarity. So if G is NA with $\delta(G) < 3$, then removing a vertex of small degree will result in a NA graph; hence G is not MMNA. □

Theorem 4.2. *There are three disconnected MMNA graphs*: $K_5 \sqcup K_5$, $K_5 \sqcup K_{3,3}$, *and* $K_{3,3} \sqcup K_{3,3}$.

Proof. First observe that these three graphs are all MMNA. On the other hand, if $G = G_1 \sqcup G_2$ is MMNA, both components must be nonplanar. Otherwise if G_1 is planar, then G_2 must be NA and is a proper minor of G, contradicting G being MMNA. So each component G_i has a K_5 or $K_{3,3}$ minor and G has one of the three candidates as a minor. Since G is minor minimal, it must be one of the three candidates. □

Theorem 4.3. *There are no MMNA graphs of connectivity* 1.

Proof. Suppose instead G is MMNA with cut vertex a. Then $G - a = G'_1 \sqcup G'_2$. If both G'_1 and G'_2 are planar, then $G - a$ is planar, contradicting that G is NA. If both are nonplanar, then G has one of the disconnected MMNA graphs as a proper minor and is not minor minimal. So, one of G'_1 and G'_2, say G'_1, is planar, and the other, G'_2, is not. Let G_i denote the induced graph on $V(G'_i) \cup \{a\}$. If G_1 is nonplanar, then together with G'_2 this gives one of the three disconnected MMNA graphs as a proper minor of G, contradicting that G is minor minimal. So G_1 is planar. But then G_2 must be NA, which again contradicts G being minor minimal. □

We can also give an upper bound on the connectivity of an MMNA graph. We first bound the minimum degree $\delta(G)$.

Theorem 4.4. *If G is MMNA, then $\delta(G) \leq 5$.*

Proof. Suppose G is MMNA and, for a contradiction, that $\delta(G) \geq 6$. Let D be the largest integer so that there are two vertices $a, b \in V(G)$ both of degree at least D. Surely, $D \geq 6$. We will argue that there are two vertices with degree at least $D+2$, contradicting our choice of D. Let $v = |V(G)|$ be the number of vertices of G. There will be $v-2$ vertices of degree at least 6 and two vertices of degree at least D. A lower bound on the number of edges of G is then $(6(v-2)+2D)/2 = 3v-6+D$.

Since G is MMNA, we can form a planar graph by deleting an edge (to get a proper minor) and then an apex vertex, which is not adjacent to the deleted edge. For if it were adjacent to the edge, the vertex deletion would also remove the edge, making G apex, a contradiction.

After deleting an edge, $G - e$ has at least $3v - 7 + D$ edges. Next delete a vertex, $a \in V(G)$ of degree d. Then the lower bound on the number of edges in the resulting planar graph is $3v - 7 + D - d$. As this graph is planar on $v - 1$ vertices, an upper bound on the number of edges is $3(v - 1) - 6$, the number of edges in a triangulation. Thus $3v - 7 + D - d \leq 3(v - 1) - 6$, which implies $d \geq D + 2$.

This means the degree of a is at least $D + 2$. However, following the argument above, if we first delete an edge incident to a, we deduce that there is a second vertex b that is again of degree at least $D + 2$. This is a contradiction since D was assumed to be the maximum such that two vertices have degree at least D. Therefore, if $\delta(G) \geq 6$, then G is not MMNA. □

Since $\kappa(G) \leq \delta(G)$, we have a bound on connectivity as an immediate corollary.

Corollary 4.5. *If G is MMNA, then $\kappa(G) \leq 5$.*

Note that K_6 is an MMNA graph of connectivity 5, so this bound is sharp. Indeed, K_6 is part of the Petersen family, a family of seven graphs shown to be MMNA by Barsotti and Mattman [2016]. Other graphs in this family provide examples of graphs of connectivity 4 ($K_{3,3,1}$) and connectivity 3 ($K_{4,4}-e$ and the Petersen graph) and the computer search of [Pierce 2014] unearthed numerous further examples with connectivity greater than 2.

Nonetheless, in the remainder of this section, we restrict attention to MMNA graphs of connectivity 2. Let us fix some notation for this situation. For G MMNA with cut set $\{a, b\}$, we have $G-a, b = G'_1 \sqcup G'_2$. Let G_i denote the induced subgraph on $V(G'_i) \cup \{a, b\}$ so that (G_1, G_2) is a separation of order 2.

Theorem 4.6. *Let G be an MMNA graph where $\kappa(G) = 2$, with cut set $\{a, b\}$. If $G-a, b = G'_1 \sqcup G'_2$, then G'_1 and G'_2 are not both nonplanar.*

Proof. Let c_a be an apex of $G-a$. By the assumption that G is MMNA, $G-a, c_a$ is planar. If $c_a = b$, we are done because $G'_1 \sqcup G'_2 = G-a, b = G-a, c_a$, which would imply both G'_1 and G'_2 are planar.

Without loss of generality, assume $c_a \in V(G'_1)$. Since none of the edges of G'_2 are in G'_1 and $a, c_a \notin V(G'_2)$, it follows that G'_2 is a subgraph of the planar graph $G-a, c_a$. Thus, G'_2 is planar. □

Theorem 4.7. *If G is MMNA and $\kappa(G) = 2$ such that $G-a, b = G'_1 \sqcup G'_2$, then, up to relabeling, $G'_1 + a$, $G'_1 + b$ are planar, and $G'_2 + a$, $G'_2 + b$ are nonplanar.*

We prove this with two lemmas.

Lemma 4.8. *$G'_1 + a$ and $G'_2 + a$ cannot both be planar.*

Proof. Let G be as described. Suppose both $G'_1 + a$ and $G'_2 + a$ are planar. Since G'_1 and G'_2 are otherwise disjoint, $G - b = (G'_1 + a) \cup (G'_2 + a)$ is the union of two planar graphs at only one vertex, with no new edges. Thus, $G - b$ is planar, which is a contradiction. So it cannot be that both $G'_1 + a$ and $G'_2 + a$ are planar. A similar argument could be made for b. □

Lemma 4.9. *$G'_1 + a$ and $G'_2 + b$ cannot both be nonplanar (up to relabeling).*

Proof. Let G be as described. Suppose both $G'_1 + a$ and $G'_2 + b$ are nonplanar. Let e be an edge between a vertex in G'_1 and the vertex b. Since G is MMNA, $G - e$ is apex. So there is a vertex v such that $(G-e) - v$ is planar. If $v = a$ then $G'_2 + b$ is a subgraph of $(G-e) - v$, which is a contradiction since $G'_2 + b$ is nonplanar. If $v \in V(G'_1)$ then again $G'_2 + b$ is a subgraph of $(G-e) - v$, which is a contradiction since $G'_2 + b$ is nonplanar. If $v = b$ then $(G-e) - v = G - v$, which implies $(G-e) - v$ is nonplanar since G is NA, so this is a contradiction. If $v \in V(G'_2)$

then $G_1' + a$ is a subgraph of $(G - e) - v$, which is a contradiction since $G_1' + a$ is nonplanar. Therefore there is no apex for $G - e$, which is a contradiction. So our assumption was wrong and one of $G_1' + a$ and $G_2' + b$ must be planar. □

We can now prove Theorem 4.7.

Proof of Theorem 4.7. Let G be as described. By the first lemma we know that at least one of $G_1' + a$ and $G_2' + a$ must be nonplanar. Without loss of generality suppose $G_2' + a$ is nonplanar. Since $G_2' + a$ is nonplanar, we know that $G_1' + b$ must be planar by the second lemma. Since $G_1' + b$ is planar, by the first lemma we know that $G_2' + b$ is nonplanar. By the second lemma this implies that $G_1' + a$ must be planar. Therefore, up to relabeling, $G_1' + a$ and $G_1' + b$ are both planar, and $G_2' + a$ and $G_2' + b$ are both nonplanar. □

Going forward, we adopt the convention suggested by Theorem 4.7 and label G_1' and G_2' such that $G_1' + a$, $G_1' + b$ are planar and $G_2' + a$, $G_2' + b$ are not. Let G be MMNA with cut set $\{a, b\}$. Our next goal is to classify such graphs in the case that ab is an edge of the graph.

Theorem 4.10. *If G is MMNA and $\kappa(G) = 2$ with cut set $\{a, b\}$ such that $ab \in E(G)$, then G_1 and G_2 are nonplanar.*

Proof. Let G_i denote the induced subgraph on $V(G_i') \cup \{a, b\}$. By Theorem 4.7, G_2 is nonplanar. For the sake of contradiction, assume G_1 is planar. Since G_2 is a proper subgraph of G, there is a vertex $v \in V(G_2)$ such that $G_2 - v$ is planar. But this means $G - v$ is planar and contradicts that G is NA.

So if G is MMNA with cut set $\{a, b\} \subset V(G)$ such that $ab \in E(G)$, then G_1 and G_2 are nonplanar. □

Theorem 4.11. *If G is MMNA and $\kappa(G) = 2$ with cut set $\{a, b\}$ such that $ab \in E(G)$, then G_1' and G_2' are both planar.*

Proof. By Theorem 4.10, G_1 is nonplanar. By Theorem 4.6, without loss of generality, G_1' is planar. Suppose G_2' is nonplanar. Then $G_1 \sqcup G_2'$ is a proper subgraph of G. Since G_1 and G_2' are both nonplanar, $G_1 \sqcup G_2'$ has a disconnected MMNA minor, contradicting that G is minor minimal. □

Theorem 4.12. *If G is MMNA with cut set $\{a, b\}$ such that $ab \in E(G)$, then $G_1 \in \{K_5, K_{3,3}\}$.*

Proof. First observe that for any $e \in E(G_1)$, the graph $G_1 - e$ must be planar. Suppose instead that there is $e' \in E(G_1)$ such that $G_1 - e'$ is nonplanar. Since $G - e'$ is apex, there is a vertex $v \in V(G)$ such that $(G - e') - v$ is planar. However, $v \notin \{a, b\}$ since $G_2' + a$ and $G_2' + b$ are nonplanar by Theorem 4.7. If $v \in V(G_1')$, then G_2 is a subgraph of $(G - e') - v$. By Theorem 4.10, since G_2 is nonplanar, $(G - e') - v$ is also nonplanar. If $v \in V(G_2')$, then $G_1 - e'$ is a subgraph of

$(G - e') - v$, and since $G_1 - e'$ is nonplanar, $(G - e') - v$ is nonplanar. So we have a contradiction and deduce that for all $e \in E(G_1)$, the graph $G_1 - e$ must be planar.

Since G_1 is nonplanar by Theorem 4.10, and since $G_1 - e$ is planar for all $e \in G_1$, it follows that G_1 consists of a K-subgraph along with some number (possibly zero) of isolated vertices. However, if G_1 is anything other than K_5 or $K_{3,3}$, then G_1 has a proper minor $N \in \{K_5, K_{3,3}\}$ formed by deleting isolated vertices or contracting edges in the K-subgraph. Then G has a proper minor G' such that N is a subgraph of G'. Since G is MMNA, there exists vertex $v \in V(G')$ that is an apex. Since N and G_2 are subgraphs of G' and both N and G_2 are nonplanar, we have that $v \in V(N) \cap V(G_2) \subset \{a, b\}$. However, $G_2 - a = G_2' + b$ and $G_2 - b = G_2' + a$ are both nonplanar (Theorem 4.7) and therefore G has a proper NA minor. This contradicts G being minor minimal.

Therefore if G is MMNA with cut set $\{a, b\}$ such that $ab \in E(G)$, then $G_1 \in \{K_5, K_{3,3}\}$. □

Theorem 4.13. *If G is MMNA with cut set $\{a, b\}$ such that $ab \in E(G)$, then there is a vertex $c \in V(G_2')$ such that every a-b-path in $G_2 - ab$ passes through c.*

Proof. Assume for the sake of contradiction that there is no such vertex c. Since G is MMNA, $G - ab$ must have some apex v. If $v \in \{a, b\}$, then $(G - ab) - v = G - v$. This would mean that G has an apex, and contradicts that G is NA. If $v \in V(G_1')$, then $(G - ab) - v$ is nonplanar as it contains $G_2' + a$, which is nonplanar by Theorem 4.7. So it must be that $v \in V(G_2')$. Then $G_1 - ab$ is a subgraph of $(G - ab) - v$. Note that $G_1 - ab \in \{K_5 - e, K_{3,3} - e\}$ since $G_1 \in \{K_5, K_{3,3}\}$ by Theorem 4.12.

Since there is no c vertex as described in the statement of the theorem, there remains an a-b-path in $(G_2 - ab) - v$. Together with $G_1 - ab$, this constitutes a nonplanar subgraph of $(G - ab) - v$, contradicting the definition of v as an apex for $G - ab$. Thus, if G is MMNA with $ab \in E(G)$, then there is a vertex c such that every a-b-path of $G_2 - ab$ passes through c. □

Theorem 4.14. *Let G be MMNA with cut set $\{a, b\}$ and $ab \in E(G)$ and let $c \in V(G_2)$ be such that every a-b-path of $G_2 - ab$ passes through c. Then $\{a, c\}$ and $\{b, c\}$ are also cut sets.*

Proof. First we show there exists some $v_2 \in V(G_2')$ such that $v_2 \neq c$, but v_2 is adjacent to a. Suppose instead that c is the only vertex in G_2' adjacent to a. Since G_2' is planar by Theorem 4.11, and since $G_2' + a$ has only one more edge than G_2', $G_2' + a$ is also planar. However, this contradicts Theorem 4.7, where $G_2' + a$ is shown to be nonplanar.

So let v_2 be a vertex of G_2' that is adjacent to a, but is not c, and take $v_1 \in V(G_1')$. We demonstrate there is no v_1-v_2-path in $G - a, c$. Since any path from a vertex in G_1' to a vertex in G_2' must pass through a or b by assumption, the supposed path from v_1 to v_2 must pass through b, since a has been deleted. However, there cannot be a path from b to v_2 that does not pass through c. Otherwise we would be able to find a path from b to v_2 and finally to a without passing through c, violating our assumption on c. We conclude that $G - a, c$ is disconnected. By an analogous argument, $\{b, c\}$ is also a cut set for G. □

In order to classify connectivity-2 MMNA graphs with $ab \in E(G)$, we need to describe G_1 in the case that $ab \notin E(G)$.

Theorem 4.15. *If G is MMNA with cut set $\{a, b\}$ such that $ab \notin E(G)$, then $G_1 \in \{K_5 - e, K_{3,3} - e, K_{3,3}\}$ and $G_1 + ab$ is nonplanar.*

Proof. Let $G - a, b = G'_1 \sqcup G'_2$ and let G_i denote the subgraph induced by vertices $V(G'_i) \cup \{a, b\}$. If G_1 is nonplanar, then G_1 has a K-subgraph N. Form a new graph, H, by replacing G_1 with N. It is clear that $a, b \in V(N)$ because if not, then G contains two disjoint K-subgraphs ($G'_2 + a$ and $G'_2 + b$ are nonplanar, Theorem 4.7) and therefore has a proper MMNA minor.

We can see that H is NA. Take $v \in V(H)$. If $v \in V(N - a, b)$, then $G'_2 + a$ is a subgraph of $H - v$ so $H - v$ is nonplanar. If $v \in V(G'_2)$, then N is a subgraph of $H - v$ so $H - v$ is nonplanar. And if $v \in \{a, b\}$, then either $G'_2 + a$ or $G'_2 + b$ is a subgraph of $H - v$ and therefore $H - v$ is nonplanar. Thus, H is NA. Since G is minor minimal, $G_1 = N$. As G is MMNA it has no degree-2 vertices and since $ab \notin E(G)$, we have $G_1 = K_{3,3}$ in this case.

Suppose next that G_1 is planar. Assume for the sake of contradiction $G_1 + ab$ is planar and replace G_1 with the edge ab to form a new graph H'. Equivalently, $H' = G_2 + ab$. We observe that for every $v \in V(H')$, the graph $H' - v$ is nonplanar. If $v \in \{a, b\}$, then $G'_2 + a$ or $G'_2 + b$ is a subgraph of $H' - v$, which is then nonplanar. On the other hand if $v \in V(G'_2)$, then since G is NA, $G - v$ has a K-subgraph M. However, if $|\{a, b\} \cap V(M)| < 2$, then since G_1 is planar, M lies wholly in G_2 and we may delete G'_1 without changing M. That is, M is a subgraph of $H' - v$. If $|\{a, b\} \cap V(M)| = 2$, then by Lemma 1.8, a and b are vertices in a path of M. Since $G_1 + ab$ is planar, we may replace G_1 by ab to create a new K-subgraph B in $H' - v$. Therefore H' is NA. However, as H' is a proper minor of G, this is a contradiction. We conclude $G_1 + ab$ is nonplanar.

Finally, observe that $G_1 + ab$ is a K-subgraph. Otherwise, we may replace it with a K-subgraph contained in $G_1 + ab$ to get a proper minor of G that is NA. Since an MMNA graph cannot have vertices of degree 2 or less, $G_1 + ab \in \{K_5, K_{3,3}\}$.

This shows if G is MMNA with cut set $\{a, b\}$ such that $ab \notin E(G)$, then we have $G_1 \in \{K_5 - e, K_{3,3} - e, K_{3,3}\}$. □

Theorem 4.16. *If G is MMNA, $\kappa(G) = 2$ with cut set $\{a, b\}$, and $ab \in E(G)$, then G is one of the nine graphs shown in Figure 2.*

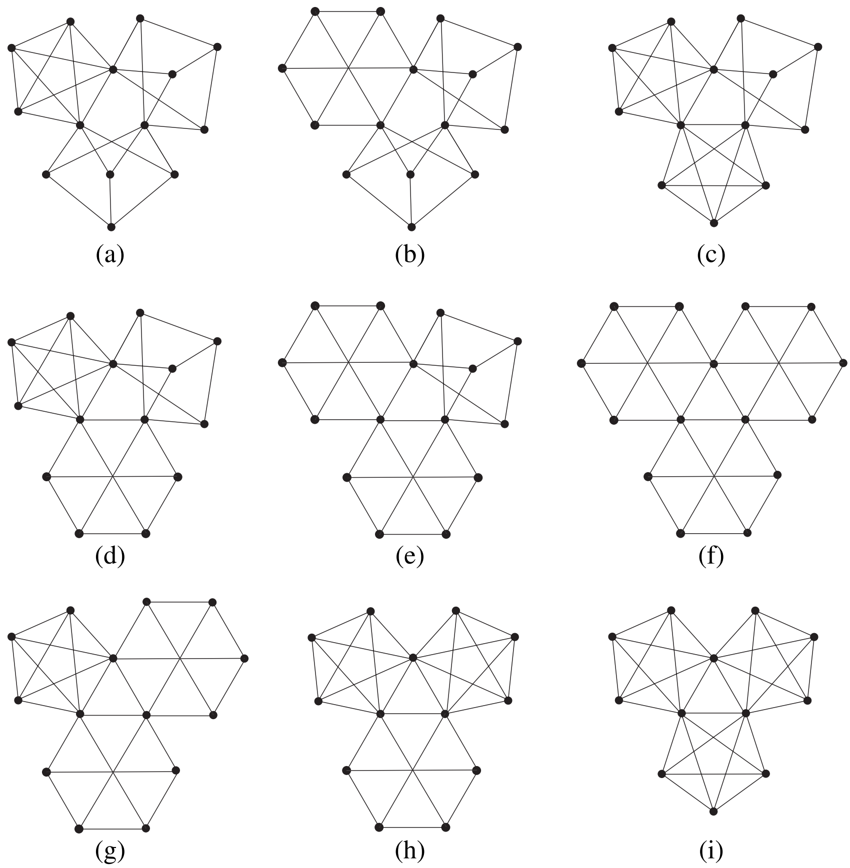

Figure 2. The nine MMNA graphs with $ab \in E(G)$.

Proof. It is straightforward to verify that the nine graphs are MMNA. Let G be MMNA, $\kappa(G) = 2$ with cut set $\{a, b\}$, and $ab \in E(G)$. By Theorems 4.13 and 4.14, there exists a vertex c such that $\{a, c\}$ and $\{b, c\}$ are also 2-cuts for G. Let H_1 play the role of G_1 for the $\{a, c\}$ cut set. That is, $G - a, c = H_1' \sqcup J_1'$ with $H_1' + a$ and $H_1' + c$ planar (see Theorem 4.7). Similarly, let H_2 be the G_1 for the $\{b, c\}$ cut set. By Theorem 4.12, $G_1 \in \{K_{3,3}, K_5\}$ and by that theorem and Theorem 4.15, $H_i \in \{K_{3,3}, K_{3,3} - e, K_5, K_5 - e\}$.

Note that, if H_1 is $K_{3,3} - e$ or $K_5 - e$, then $G - b$ is planar and similarly for H_2. Thus, $H_1, H_2 \in \{K_{3,3}, K_5\}$. There are three cases depending on whether $ac, bc \in E(G)$ or not.

First suppose that ab is the only one of ab, bc, and ac present in the graph. As above, G_1, H_1 and H_2 are each either $K_{3,3}$ or K_5. However, by Theorem 4.15, this means $H_1 = H_2 = K_{3,3}$. So, there are exactly two graphs (graphs (a) and (b) in Figure 2) of this type, depending on whether G_1 is K_5 or $K_{3,3}$.

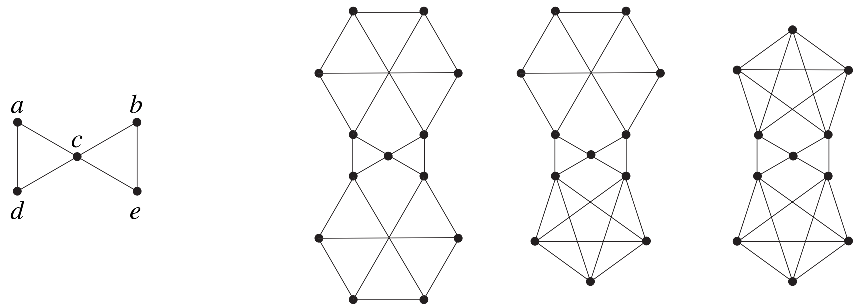

Figure 3. Bowtie graphs.

Next suppose that exactly one of ac and bc, say ac, is in the graph. As in the previous case H_2 must be $K_{3,3}$. There are three graphs (graphs (c), (d), and (e) of Figure 2) of this type as $\{G_1, H_1\}$ is either $\{K_5, K_5\}$, $\{K_5, K_{3,3}\}$, or $\{K_{3,3}, K_{3,3}\}$.

Finally, suppose all three edges ab, ac and bc are in the graph. Then, as above, G_1, H_1, and H_2 are each either $K_{3,3}$, or K_5. There are four graphs of this type, shown as graphs (f) through (i) of Figure 2. For example, such a graph has between zero and three K_5's.

This shows that the nine graphs of Figure 2 are the graphs where G is MMNA, $\kappa(G) = 2$ with cut set $\{a, b\}$, and $ab \in E(G)$. □

Henceforth, we can assume $ab \notin E(G)$. By Theorem 4.15, this means $G_1 \in \{K_5 - e, K_{3,3} - e, K_{3,3}\}$. We will say that G is a *bowtie* if the neighborhood of a, b in G_2 is as shown in Figure 3 (left). That is, a and b have degree 2 in G_2 and c has degree 4. Although d and e have additional neighbors in G_2 besides $\{a, c\}$ and $\{b, c\}$ respectively, $de \notin E(G_2)$.

Theorem 4.17. *If G is a bowtie MMNA graph, then G is one of the three graphs shown in Figure 3 (right).*

Proof. It is straightforward to verify that the three graphs in the figure are MMNA. Let G be a bowtie MMNA graph. Then, referring to Figure 3 (left), $\{d, e\}$ is a cut set as well. Let H_1 play the role of the G_1 for the $\{d, e\}$ cut set. By Theorem 4.15, G_1 and H_1 are both drawn from $\{K_{3,3}, K_{3,3} - e, K_5 - e\}$.

We will argue that neither G_1 nor H_1 is $K_{3,3}$. For the sake of contradiction, assume instead $G_1 = K_{3,3}$. Notice G_1 and G_2' are disjoint, and nonplanar. So, G has a proper NA minor, $G_1 \sqcup G_2'$, which contradicts that G is to be minor minimal.

So, G_1 and H_1 are both in $\{K_{3,3} - e, K_5 - e\}$, where ab is the missing edge, e, and the only possibilities are the three graphs shown in Figure 3 (right). □

Let G be MMNA with cut set $\{a, b\}$ such that $ab \notin E(G)$. We say G is of $(2, 2, c)$ *type* if, in G_2, the vertices a and b are of degree 2 and have c common neighbors. For example, a bowtie graph is of $(2, 2, 1)$ type.

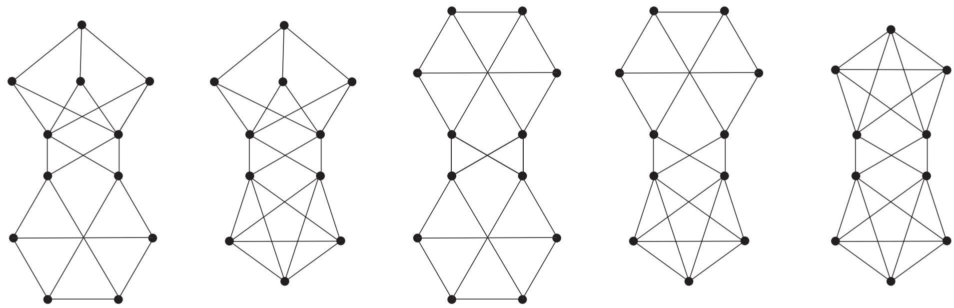

Figure 4. Graphs of type (2, 2, 2).

Theorem 4.18. *If G is MMNA and of* (2, 2, 2) *type, then G is one of the five graphs shown in Figure 4.*

Proof. It is straightforward to verify that the five graphs are MMNA. Let G be MMNA with cut set $\{a, b\}$ and of (2, 2, 2) type. Let $\{c, d\}$ be the common neighbors of a and b in G_2. Note that $cd \notin E(G)$, as otherwise G must be one of the nine graphs of Theorem 4.16 and none of those are (2, 2, 2) type.

By Theorem 4.15, and using symmetry, $G_1, G'_2 \in \{K_{3,3}, K_{3,3} - e, K_5 - e\}$. However, they cannot both be $K_{3,3}$, as otherwise $G_1 \sqcup G'_2$ is a proper NA subgraph, which contradicts that G is minor minimal. So at most one of the subgraphs can be $K_{3,3}$. This leaves the five possibilities shown in Figure 4. □

Theorem 4.19. *Suppose G is MMNA and of connectivity* 2 *with $G_1 \in \{K_5 - e, K_{3,3} - e\}$. Then there is no vertex, other than a and b, common to all a-b-paths in G_2.*

Proof. Assume, for the sake of contradiction, that $G_1 \in \{K_5 - e, K_{3,3} - e\}$ and there is a vertex $c \in V(G'_2)$ that lies on every a-b-path in G_2. Then, as in Theorem 4.14, $\{a, c\}$ and $\{b, c\}$ are 2-cuts for G, and as in the proof of Theorem 4.16, we can let H_1 play the role of the G_1 for the $\{a, c\}$ cut and similarly H_2 for the $\{b, c\}$ cut and, by Theorems 4.12 and 4.15, both H_1 and H_2 are drawn from $\{K_5, K_{3,3}, K_5 - e, K_{3,3} - e\}$. Then $G - c$ is planar, contradicting that G is NA.

Therefore, if G is MMNA, of connectivity 2 with $G_1 \in \{K_5 - e, K_{3,3} - e\}$, then there is no vertex, other than a and b, common to all a-b-paths in G_2. □

Theorem 4.20. *Let G be MMNA with $\kappa(G) = 2$ and $ab \notin E(G)$, where $\{a, b\}$ is a 2-cut. If G'_2 is nonplanar, then there are independent a-b-paths in G_2.*

Proof. By Theorem 4.15, $G_1 \in \{K_5 - e, K_{3,3}, K_{3,3} - e\}$. However, if $G_1 = K_{3,3}$ then, together with G'_2, this constitutes a pair of disjoint K-subgraphs, which would mean G has a proper disconnected NA minor, a contradiction. So $G_1 \in \{K_5 - e, K_{3,3} - e\}$ and we can apply Menger's theorem and Theorem 4.19. □

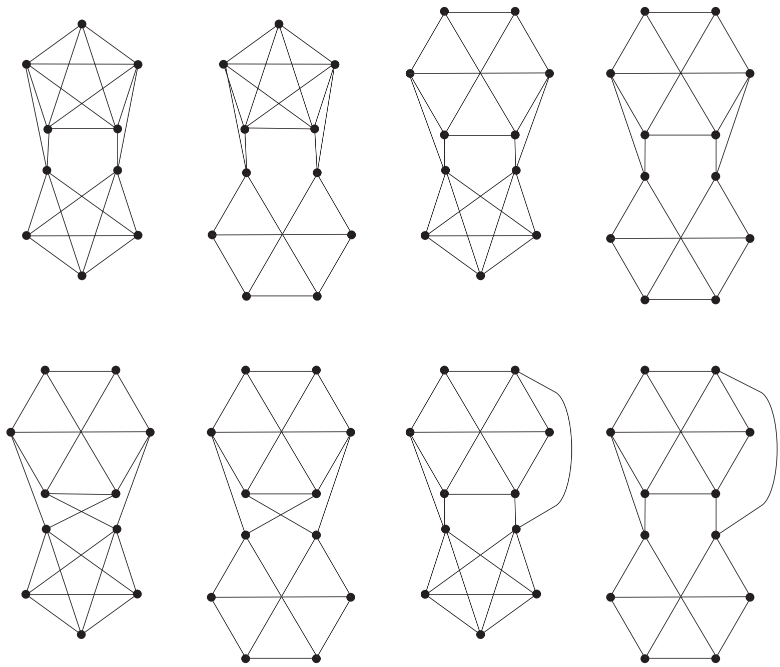

Figure 5. Graphs of type (2, 2, 0).

Theorem 4.21. *If G is MMNA of* (2, 2, 0) *type and* $G_2' \in \{K_5, K_{3,3}\}$, *then G is one of the eight graphs in Figure 5.*

Proof. Notice that the eight graphs in the figure are MMNA. Suppose G is MMNA of (2, 2, 0) type with G_2' a Kuratowski graph. By Theorem 4.15, $G_1 \in \{K_5 - e, K_{3,3}, K_{3,3} - e\}$. However, G_1 cannot be $K_{3,3}$ because then, together with G_2' it forms a disconnected MMNA minor of G. We continue by examining the ways to construct G_2.

To construct G_2, we consider how to add the vertices a and b to G_2'. Let a have neighbors $v_1, v_2 \in V(G_2')$ and let $v_3, v_4 \in V(G_2')$ be the neighbors of b. Since G is of (2, 2, 0) type, $\{v_1, v_2\} \cap \{v_3, v_4\} = \varnothing$. Up to symmetry, there is only one way to attach a and b to K_5. This gives two of the graphs in the figure, as G_1 is either $K_5 - e$ or $K_{3,3} - e$.

In $K_{3,3}$, the vertices are split into two parts A and B, each of three vertices. Then the four vertices v_i, $i = 1, \ldots, 4$, are either divided with two in each part, or else with three in one part and the fourth in the other. In the first case, there are two subcases: either $\{v_1, v_2\} \subset A$ (and $\{v_3, v_4\} \subset B$) or else $|\{v_1, v_2\} \cap A| = |\{v_1, v_2\} \cap B| = 1$ (and similarly for $\{v_3, v_4\}$). These three choices for G_2 along with the two choices for G_1, either $K_5 - e$ or $K_{3,3} - e$, account for the remaining six graphs in Figure 5. □

Theorem 4.22. *If G is MMNA of* (2, 2, 1) *type and* $G_2' \in \{K_5, K_{3,3}\}$, *then G is one of the eight graphs of Figure 6.*

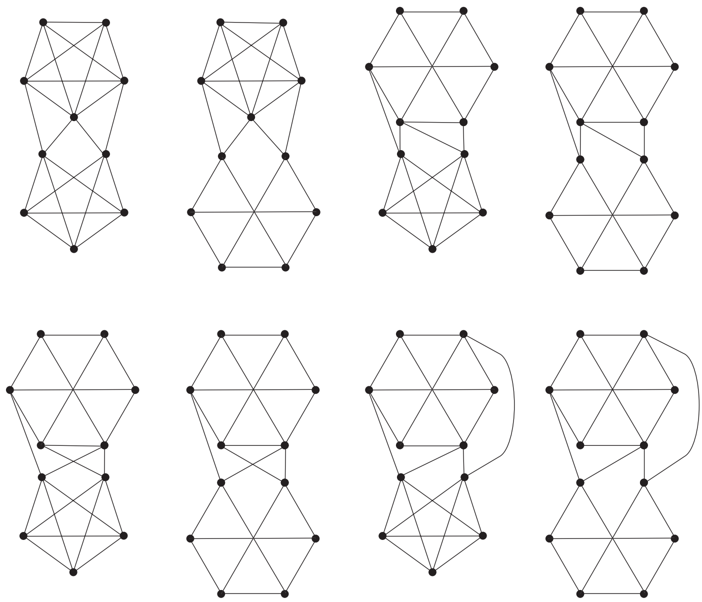

Figure 6. Graphs of type (2, 2, 1).

Proof. The proof is similar to that for (2, 2, 0) type. If G_2' is a Kuratowski graph, then G_1 cannot be $K_{3,3}$, as that would result in a proper NA minor. So $G_1 \in \{K_5 - e, K_{3,3} - e\}$. If $G_2' = K_5$, up to symmetry there is only one way to form G_2 and this gives two graphs in the figure, as G_1 is either $K_5 - e$ or $K_{3,3} - e$.

If $G_2' = K_{3,3}$, there are three ways to form G_2. Together, a and b have three neighbors in G_2', which can either all lie in one part or else be split with a single vertex in one part and the remaining two in the other. In this second case, there are two further subcases since the vertex that is alone in its part can either be the common neighbor or not. Together with these three choices for G_2, there are two choices for G_1, either $K_5 - e$ or $K_{3,3} - e$. This gives the remaining six graphs of Figure 6. □

We conclude this section with a classification of the MMNA graphs of connectivity 2, with 2-cut $\{a, b\}$ such that $G - a, b$ has a nonplanar component. By Theorem 4.11 we must have $ab \notin E(G)$, and by Theorem 4.7, G_1' is planar. In other words, if there is a nonplanar component, it must be G_2'. So far, we have constructed 21 graphs with nonplanar G_2', the three bowtie graphs of Theorem 4.17, two of the (2, 2, 2) graphs (the two to the left of Figure 4), and eight each of (2, 2, 0) type (Theorem 4.21) and (2, 2, 1) type (Theorem 4.22). This is in fact a complete listing of the graphs with G_2' nonplanar, as we now show.

Theorem 4.23. *Let G be MMNA with $\kappa(G) = 2$ and 2-cut $\{a, b\}$ such that $G - a, b$ has a nonplanar component. Then G is of $(2, 2, c)$ type with $c = 0$, 1, or 2 and appears in one of Figures 3 (right), 4, 5, or 6.*

Proof. Assume the hypothesis. As remarked above, if $\{a, b\}$ is a 2-cut, this implies $ab \notin E(G)$ and G'_2 is nonplanar. Let H'_2 be a K-subgraph of G'_2. Since $ab \notin E(G)$, combining Theorems 4.15 and 4.2, we have $G_1 \in \{K_5 - e, K_{3,3} - e\}$. By Theorem 4.20 there are independent a-b-paths in G_2, call them P_1 and P_2. Since, by Theorem 4.15, $G_1 + ab$ is nonplanar, P_1 and P_2 each have vertices in common with H'_2. (Otherwise, G has disjoint nonplanar subgraphs and therefore a disconnected NA minor, by Theorem 4.2, contradicting G being minor minimal.) By contracting edges if necessary, we have a minor of G for which the vertices of P_i are $a, a_i, \ldots, b_i, b$ with $a_i, b_i \in V(H_2)$, $i = 1, 2$. Then there are several cases that correspond to $(2, 2, c)$ type, where $c = 0, 1, 2$.

Suppose first that $a_1 = b_1$ and $a_2 = b_2$ so that G is of $(2, 2, 2)$ type. By contracting edges in H'_2 if needed, we recognize that G has one of the five graphs of Theorem 4.18 as a minor. Since G is MMNA, G is one of these five graphs and since G'_2 is nonplanar, G must be one of the two graphs with $G'_2 = K_{3,3}$ (i.e., the two to the left of Figure 4). In other words G is of $(2, 2, 2)$ type and appears in one of the figures, as required.

The rest of the argument is a little technical and we introduce some notation to simplify the exposition. The K-subgraph H'_2 is a subdivision of K_5 or $K_{3,3}$ and, along with vertices of degree 2, has five or six vertices of higher degree that we will call *branch vertices*. Corresponding to the edges of K_5 or $K_{3,3}$, the branch vertices are connected by paths that we call 2-*paths* whose internal vertices are all of degree 2.

To continue the argument, suppose next that, say, $a_1 = b_1$, but $a_2 \neq b_2$. By contracting edges in H'_2 if necessary, we can arrange that at least two of the three vertices a_1, a_2, and b_2 become branch vertices of the K-subgraph. If all three can be made branch vertices, then, by further edge contractions, if necessary, we see that one of the eight $(2, 2, 1)$ graphs of Theorem 4.22 is a minor of G. Since G is MMNA, this means G is one of the $(2, 2, 1)$ graphs, with $G'_2 \in \{K_5, K_{3,3}\}$ appearing in Figure 6, as required. If not, we can assume that it is a_1 that remains as a degree-2 vertex of H'_2. For, if it is a_2 or b_2 that remains, we can contract edges to make $a_2 = b_2$ and return to the previous case. With a_1 as a degree-2 vertex in G'_2, we recognize that, perhaps by further edge contractions, G has a bowtie graph as a minor. Since G is MMNA, G is a bowtie graph. That is G is of $(2, 2, 1)$ type and appears in Figure 3 (right), as required.

Finally, suppose $a_1 \neq b_1$ and $a_2 \neq b_2$. If all four can be made distinct branch vertices by edge contractions in H'_2, then G has a $(2, 2, 0)$ minor, so G is a $(2, 2, 0)$ graph with $G'_2 \in \{K_5, K_{3,3}\}$ appearing in Figure 5, as required.

Next, suppose at most three can be made into branch vertices and, without loss of generality, suppose it is a_1 that remains as a degree-2 vertex in H'_2. This means a_1 lies on a 2-path between two of b_1, a_2, and b_2. If the path ends at b_1, by further edge contractions in H_2', we can realize $a_1 = b_1$ as a branch vertex and return to an earlier case. So, we can assume that a_1 is on a 2-path between a_2 and b_2. Use the part of the 2-path between a_1 and b_2 to form a new a-b-path P_1' (i.e., $a_1' = a_1$ and $b_1' = b_2$) and use a path in H_2' between the branch vertices a_2 and b_1 that avoids the branch vertex b_2 to construct an independent a-b-path P_2' (i.e., P_2' has $a_2' = a_2$ and $b_2' = b_1$). Now we can contract edges in P_1' to identify $a_1' = a_1$ and $b_1' = b_2$ to return to the earlier case where $a_1 = b_1$. This completes the argument when at most three of the vertices can be moved to branch vertices.

Finally, suppose that at most two of the vertices can be made into branch vertices of H_2' by contracting edges, if needed. There are two subcases. If a_1 and b_1 are the branch vertices, then a_2 and b_2 are degree-2 vertices on a 2-path between a_1 and b_1. Here we can further contract edges in H_2' to identify a_2 and b_2, which returns us to an earlier case. In the second subcase, without loss of generality, it is a_1 and a_2 that are the branch vertices of H_2'. Assuming we cannot easily contract edges to identify a_1 and b_1 or a_2 and b_2, it must be that the 2-path from a_1 to a_2 passes first through b_2 and then through b_1. In this case, we replace P_1 and P_2 by the independent paths P_1', which uses the 2-path from a_1 to b_2 (so $a_1' = a_1$ and $b_1' = b_2$), and P_2', which uses the 2-path from a_2 to b_1 (then $a_2' = a_2$ and $b_2' = b_1$). By further edge contractions, we return to our first case where $a_1 = b_1$ and $a_2 = b_2$. □

Together, the three bowtie graphs and the eight of Figure 6 give eleven MMNA graphs of (2, 2, 1) type. In total we have found three disconnected MMNA graphs, nine where $ab \in E(G)$, as well as eight, eleven, and five, respectively when G is of type $(2, 2, c)$ for $c = 0, 1, 2$, respectively. This gives a total of 36 MMNA graphs.

5. MMNE and MMNC graphs

In this section we classify MMNE and MMNC graphs of connectivity, $\kappa(G)$, at most 1. For MMNE graphs we also show $\kappa(G) \leq 5$ and determine the graphs with $\kappa(G) = 2$ and minimum degree at least 3. We conclude the section by describing a computer search that found 55 MMNE and 82 MMNC graphs.

We begin by observing that the MMNE and MMNC graphs are not Kuratowski sets as the opposite properties are not minor closed. Recall that NE is an abbreviation for not edge apex. The opposite property is *edge apex*, meaning there is an $e \in E(G)$ so that $G - e$ is planar. We call such an edge an *apex edge*. Similarly, the opposite of NC is *contraction apex*, meaning there is an edge e such that G/e is planar. We call e a *contraction apex*.

Theorem 5.1. *Deleting an edge of an edge apex graph results in an edge apex graph. Contracting an edge of an edge apex graph results in an edge apex graph unless the edge that is contracted is the* only *apex edge.*

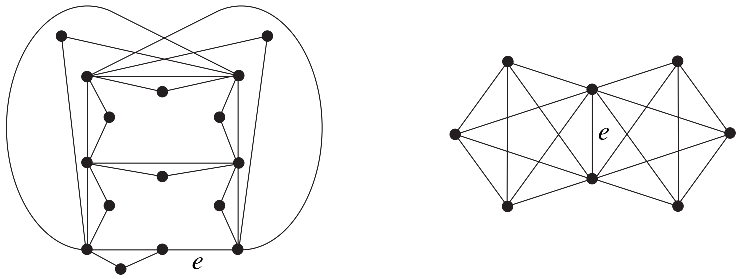

Figure 7. Examples showing that the sets of MMNE and MMNC graphs are not Kuratowski sets.

Proof. Suppose that G is edge apex, so it contains an edge e such that $G-e$ is planar. Let G' be the result of deleting some edge f in G. If $f \neq e$, consider $G'-e$ and note that $G'-e = G-e, f$, which is a minor of $G-e$. Graph $G-e$ is planar, so $G'-e$ is also planar, and e is an apex edge for G', which is therefore edge apex. Otherwise, if $f = e$, then G' would be planar and so would also be edge apex.

Now suppose that G contains at least two edges e_1 and e_2 ($e_1 \neq e_2$) such that both $G-e_1$ and $G-e_2$ are planar. Let f be an arbitrary edge in G and let G'' be the result of contracting edge f in G. Without loss of generality, suppose that $f \neq e_1$. Consider the graph $G''-e_1$, where if e_1 is incident to f in G then e_1 is incident to the vertex formed by contracting f in G''. Note that this graph $G''-e_1$ is a minor of $G-e_1$. But $G-e_1$ is planar, and since planarity is closed under taking minors, the graph $G''-e_1$ is planar. So edge e_1 is an apex edge of G''. □

Theorem 5.2. *The set of graphs that are edge apex is not closed under taking minors.*

Proof. Let G be the graph in Figure 7 (left). This graph can be described as $K_{3,3}$ with all but one edge replaced by a triangle, and with that one edge subdivided into an edge e and another edge to be replaced by a triangle. This graph is edge apex with e as the unique apex edge. However, G/e is $K_{3,3}$ with every edge replaced by a triangle, so G/e is not edge apex. □

Theorem 5.3. *Contracting an edge of a contraction apex graph results in a contraction apex graph. Deleting an edge of a contraction apex graph results in a contraction apex graph unless the edge that is deleted is the* **only** *contraction apex.*

Proof. Suppose that G is contraction apex, so it contains an edge e such that G/e is planar. Let G' be the result of contracting some edge f in G. If $f \neq e$, consider G'/e and note that $G'/e = G/e, f$, which is a minor of G/e. Graph G/e is planar, so G'/e is also planar, and e is a contraction apex for G', which is therefore a contraction apex graph. Otherwise, if $f = e$, then G' would be planar and so would also be contraction apex.

Now suppose that G contains at least two edges e_1 and e_2 ($e_1 \neq e_2$) such that both G/e_1 and G/e_2 are planar. Let f be an arbitrary edge in G and let G'' be the result of deleting edge f in G. Without loss of generality, suppose that $f \neq e_1$. Consider the graph G''/e_1 and note that it is a minor of G/e_1. But G/e_1 is planar, and since planarity is closed under taking minors, the graph G''/e_1 is planar. So edge e_1 is a contraction apex of G''. □

Theorem 5.4. *The set of graphs that are contraction apex is not closed under taking minors.*

Proof. Define the graph G as two copies of K_5 that share a common edge e; see Figure 7 (right). We show that G is contraction apex, but has a minor that is NC. Indeed, contracting the common edge, $G/e = K_4 \dot{\cup} K_4$, which is planar. Note that this is the unique contraction apex of G.

Now define the subgraph G' as $G - e$. Label the two subgraphs isomorphic to $K_5 - e$ as G'_1 and G'_2. Without loss of generality, suppose we contract an edge f in G'_2. Notice that we are left with $G'_1 = K_5 - e$, and a path through G'_2 that connects the two degree-3 vertices of G'_1. Thus, G'/f has a subgraph homeomorphic to K_5 and is nonplanar. By symmetry, whatever edge $f \in E(G')$ we choose, G'/f is nonplanar. Thus G' is NC. □

We next classify the disconnected and connectivity-1 MMNE and MMNC graphs, which turn out to be the same sets.

Theorem 5.5. *The disconnected MMNE graphs are* $K_5 \sqcup K_5$, $K_5 \sqcup K_{3,3}$, *and* $K_{3,3} \sqcup K_{3,3}$.

Proof. First observe that these three graphs are MMNE. Let G be MMNE and disconnected. Suppose one of G_1, G_2 is planar, say G_1. Then let $e_1 \in E(G_1)$, and note that $G - e_1$ is not NE and nonplanar. Let e_2 be the edge whose removal from $G - e_1$ gives a planar graph. Since G_1 is planar, it must be that e_2 is in $E(G_2)$. But, since G_1 is planar, this means that removing e_2 from G gives the disconnected union of the planar G_1 and a planar minor of G_2. So, this graph, $G - e_2$, is planar, which is a contradiction since G is NE. So it must be that G_1 and G_2 are both nonplanar. Thus one of the graphs generated by $G_1 \sqcup G_2$, where $G_1, G_2 \in \{K_5, K_{3,3}\}$, must be a minor of G. Since G is minor minimal, G must be one of these three graphs. □

Theorem 5.6. *The disconnected MMNC graphs are* $K_5 \sqcup K_5$, $K_5 \sqcup K_{3,3}$, *and* $K_{3,3} \sqcup K_{3,3}$.

Proof. First observe that these three graphs are MMNC. Let G be MMNC and disconnected. Suppose one of G_1, G_2 is planar, say G_1. Then let $e_1 \in E(G_1)$, and note that $G - e_1$ is not NC and nonplanar. Then there is an edge $e_2 \in E(G - e_1)$ such that $(G - e_1)/e_2$ is planar. Since G_1 is planar, it must be that e_2 is in $E(G_2)$. But, since G_1 is planar, this means that contracting e_2 in G gives the disconnected union of the planar G_1 and a planar minor of G_2. This graph G/e_2 is planar, which is a contradiction since G is NC. So it must be that G_1 and G_2 are both nonplanar. Then one of the graphs $G = G_1 \sqcup G_2$, with $G_i \in \{K_5, K_{3,3}\}$, is a minor of G. Since G is minor minimal, it is one of those three graphs. □

Corollary 5.7. *Let G be disconnected. The following are equivalent*: *G is MMNA*; *G is MMNE*; *G is MMNC.*

Recall that $G_1 \dot\cup G_2$ is the union of G_1 and G_2 with one vertex identified.

Theorem 5.8. *If G is MMNE and* $\kappa(G) = 1$ *then* $G = G_1 \dot\cup G_2$, *where* $G_1, G_2 \in \{K_5, K_{3,3}\}$, *and they share exactly one vertex.*

Proof. First observe that these three graphs are MMNE. Let $G = G_1 \dot\cup G_2$ and suppose for the sake of contradiction that one of G_1 and G_2, say G_1, is planar. Let e be an edge of G_1. Then $G - e$ is not NE and nonplanar. Let f be the apex edge of $G - e$. Since G_1 is planar, f must lie in $E(G_2)$. Since $G_2 - f$ is a subgraph of the planar $G - e, f$, it must itself be planar. Note that $G - f = G_1 \cup (G_2 - f)$ is the union of two planar graphs that share at most one vertex, which is clearly planar. This is a contradiction, since G is NE. So both G_1 and G_2 are nonplanar. So G has one of the graphs $G_1 \dot\cup G_2,\ G_1,\ G_2 \in \{K_5, K_{3,3}\}$ as a minor. Since these graphs are NE and G is minor minimal, G must be one of these three graphs. □

Theorem 5.9. *If G is MMNC and* $\kappa(G) = 1$ *then* $G = G_1 \dot\cup G_2$, *where* $G_1, G_2 \in \{K_5, K_{3,3}\}$, *and they share exactly one vertex.*

Proof. First observe that these three graphs are MMNC. Let $G = G_1 \dot\cup G_2$ and suppose for the sake of contradiction that one of G_1 and G_2, say G_1, is planar. Let e be an edge of G_1. Then $G - e$ is not NC and nonplanar. Let $f \in E(G - e)$ be the contraction apex of $G - e$; that is, $(G - e)/f$ is planar. Since G_1 is planar, f must lie in G_2. Since G_2/f is a subgraph of the planar $(G - e)/f$, it must itself be planar. Note that $G/f = G_1 \cup (G_2/f)$ is the union of two planar graphs that share at most one vertex, which is clearly planar. This is a contradiction, since G is NC.

Thus, both G_1 and G_2 are nonplanar. So G has one of the graphs $G_1 \dot\cup G_2$ with $G_1, G_2 \in \{K_5, K_{3,3}\}$ as a minor. Since these graphs are NC and G is minor minimal, G must be one of these three graphs. □

Corollary 5.10. *Let G have connectivity* 1. *Then G is MMNE if and only if it is MMNC.*

Recall that there are no MMNA graphs of connectivity 1. In particular, for each of $K_5 \dot\cup K_5$, $K_5 \dot\cup K_{3,3}$, and $K_{3,3} \dot\cup K_{3,3}$, the cut vertex is an apex. We next classify the MMNE graphs of connectivity 2 under the assumption that the minimum degree, $\delta(G)$, is at least 3. We will argue that there are exactly six such graphs and we begin with the observation that those graphs are indeed MMNE. As discussed at the end

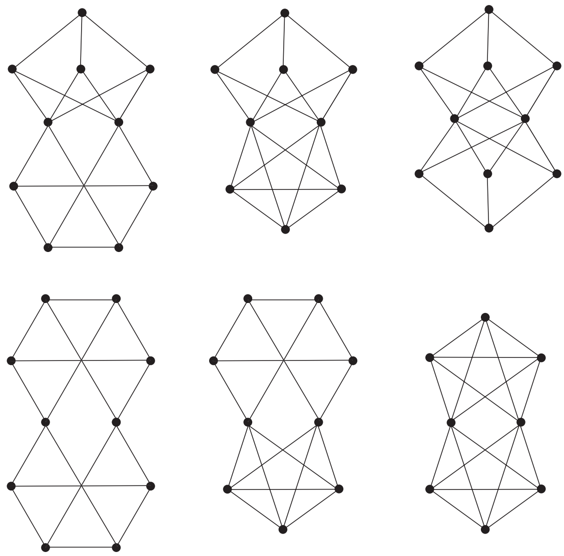

Figure 8. The six MMNE graphs of connectivity 2 with $\delta(G) \geq 3$.

of this section, based on a computer search, these again coincide with the MMNC examples of connectivity 2 with $\delta(G) \geq 3$. In addition to being both MMNE and MMNC, these 12 graphs with $\kappa(G) \leq 2$ are exactly the obstructions, of connectivity at most 2, to embedding a graph in the projective plane; see [Mohar and Thomassen 2001, Section 6.5].

Theorem 5.11. *The six graphs of Figure 8 are MMNE.*

Note that these graphs are of the form $G_1 \ddot{\cup} G_2$ with $G_i \in \{K_5 - e, K_{3,3}, K_{3,3} - e\}$, i.e., the union of G_1 and G_2 identified on two vertices.

Proof. Let G be one of the six graphs and e denote an arbitrary edge of G. It is easy to verify that each $G - e$ is nonplanar, so G is NE. We must also show that no minor of G is NE. We first observe that for each choice of e, there is another edge f such that $G - e, f$ is planar. That is, $G - e$ is not NE. Also, there is an edge g such that $(G/e) - g$ is planar, which shows G/e is not NE.

By Theorem 5.1, deleting or contracting further edges continues to give minors of G that are not NE, so long as we do not contract the unique apex edge in a graph. Working around this obstacle is not difficult as we very quickly come to planar minors. Planarity *is* closed under taking minors and a planar graph is not NE. □

A key step in the classification is the observation that ab is not an edge of G.

Lemma 5.12. *If G is MMNE, $\kappa(G) = 2$ with cut set $\{a, b\}$, and $\delta(G) \geq 3$, then ab is not an edge in G.*

Proof. Let G be as described. Let $G - a, b = G'_1 \sqcup G'_2$ and let G_i be the induced subgraph of G on the vertices $V(G'_i) \cup \{a, b\}$. For a contradiction, suppose that ab is an edge in G. There are three cases to consider depending on which of G_1 and G_2 is planar. If both are planar, then G is the union of two planar graphs that share an edge and therefore is planar. This contradicts G being MMNE.

Next suppose exactly one of G_1 and G_2 is planar, say G_1. If $e \in E(G_2)$ is an edge other than ab, then $G_2 - e$ must be nonplanar. For otherwise, $G - e$, the union of two planar graphs, G_1 and $G_2 - e$ along ab, is planar contradicting G being NE. If $G_2 - ab$ is also nonplanar, then G_2 is a proper subgraph that is NE, which contradicts G being minor minimal. So, $G_2 - ab$ is planar.

This means that $G - ab$ is the union of the planar $G_1 - ab$ and the planar $G_2 - ab$, joined at two vertices. However, since G is NE, $G - ab$ is nonplanar, so it has a subgraph homeomorphic to K_5 or $K_{3,3}$. Using Lemma 1.8, we know that the subgraph must use only a path through one of G_1, G_2, and nothing else in that component. This means that one of G^*_i is an edge away from containing a K-subgraph, where G^*_i denotes $G_i - ab$. Since G_1 is planar, it must be G^*_2 that contains a subdivision of K_5 or $K_{3,3}$ with an edge removed. Thus, G_2 has a subgraph homeomorphic to K_5 or $K_{3,3}$ that uses the edge ab.

Replace G^*_1 by the path of Lemma 1.8 to form a subgraph H of G. We claim that H is NE. Indeed, deleting $e \in E(G^*_2)$ leaves $H - e$ with the nonplanar subgraph $G_2 - e$. Deleting ab or an edge in the G^*_1 path leaves an a-b-path that completes a K-subgraph in G^*_2. Since G is minor minimal, G must be H. However, H has at least one degree-2 vertex, contradicting $\delta(G) \geq 3$.

Finally, we have the case where G_1 and G_2 are both nonplanar. Here there are three subcases to consider depending on which of $G^*_1 = G_1 - ab$ and $G^*_2 = G_2 - ab$ is planar.

Suppose first that both G^*_1 and G^*_2 are planar. In this case, each of G_1 and G_2 has a K-subgraph that contains ab. It follows that one of the graphs of Theorem 5.11 is a proper minor of G, contradicting the minor minimality of G.

In the subcase where both G^*_1 and G^*_2 are nonplanar, let e be the apex edge of $G - ab$. Since the only edge common to G^*_1 and G^*_2 is ab, the edge e is in exactly one of G^*_1 and G^*_2. Whichever it is not in will constitute a nonplanar subgraph of $G - ab, e$, which is a contradiction.

Finally, assume exactly one of G^*_1 and G^*_2 is planar, say G^*_1. As above, G^*_1 planar and G_1 not implies G_1 contains a K-subgraph including ab as an edge. On the other hand, since G^*_2 is nonplanar, it has a K-subgraph H. Let $M = G_1 \cup H$ and, for a contradiction, suppose that M is a proper minor. Then M must have an apex edge. However, if we remove an edge e from G_1, then H remains, meaning $M - e$ is nonplanar. If we remove e from H (which shares no edges with G_1 since it is a subgraph of G^*_2), then G_1 remains, meaning $M - e$ is still nonplanar. Therefore, no matter what edge we remove from M, we cannot make it planar and M is NE. However, M is a minor of G, so this contradicts G being MMNE. Therefore, H is not a proper minor of G_2^*, so G_2^* is a subdivision of K_5 or $K_{3,3}$. A similar argument (replacing H by K_5 or $K_{3,3}$) shows, in fact, G_2^* *is* K_5 or $K_{3,3}$ and not just a subdivision. However, since ab is not an edge of G_2^*, then G_2^* must be $K_{3,3}$.

Thus $G_2^* = K_{3,3}$ and G_1 contains a subdivision of $K_{3,3}$ or K_5 that includes ab as an edge. This means G includes one of the graphs of Theorem 5.11 as a proper minor and is not minor minimal.

This completes the last subcase of the last case and shows that ab is not an edge of G. □

For G of connectivity 2 with cut set $\{a, b\}$, we have $G - a, b = G_1' \sqcup G_2'$. We will use G_i to denote the induced subgraph on $V(G_i') \cup \{a, b\}$.

Lemma 5.13. *If G is MMNE, $\kappa(G) = 2$, and G_1 and G_2 are both nonplanar, then $G_1 = G_2 = K_{3,3}$.*

Proof. Let G be as described. First suppose for the sake of contradiction that G_1 is nonplanar but not $K_{3,3}$. Note that G_1 cannot be K_5 because $ab \notin E(G)$ by Lemma 5.12. So G_1 has some nonplanar proper minor H, and $H \cup G_2$ is a proper minor of G. Since there are no edges between H and G_2, the apex edge of $H \cup G_2$ must be in exactly one of H or G_2. Whichever one does not contain the apex edge will be a nonplanar subgraph even when the edge is removed, contradicting the fact that G is MMNE. Therefore $G_1 = K_{3,3}$. A symmetrical argument can be made for G_2. □

Lemma 5.14. *If G is MMNE, $\kappa(G) = 2$, with cut set $\{a, b\}$, $\delta(G) \geq 3$, and both G_1 and G_2 are planar, then $G_i \in \{K_5 - e, K_{3,3} - e\}$ with ab as the missing edge.*

Proof. Let G be as described. For a contradiction, assume that $G_1 + ab$ is planar. Since G is NE, for every $e \in E(G)$, the graph $G - e$ is nonplanar and, therefore, has a K-subgraph, H. By Lemma 1.8 and our assumption that $G_1 + ab$ is planar, $H \cap G_1$ is an a-b-path. In particular $G_2 + ab$ is nonplanar.

Note that there are edge-disjoint a-b-paths P_1 and P_2 in G_1. If not, say every a-b-path goes through the edge e'. Then $G - e'$ must be planar as, by Lemma 1.8, a K-subgraph of $G - e'$ would either use a path in G_1, which is not possible as all such paths pass through e', or else use a path in G_2, which is not possible since $G_1 + ab$ is planar. The contradiction shows there are edge-disjoint paths P_1 and P_2.

This means we can construct a proper minor M of G by adding a triangle on ab. That is, $V(M) = V(G_2) \cup \{c\}$ and $E(M) = E(G_2) \cup \{ab, bc, ac\}$. Since G is NE, for any $e \in E(G_2)$, the graph $G - e$ is nonplanar with a K-subgraph that uses only a path in G_1. So, $M - e$ is also nonplanar. On the other hand, if we delete any e in $\{ab, ac, bc\}$, we are left with a subgraph of $M - e$ homeomorphic to $G_2 + ab$. So $M - e$ is again nonplanar. Then M is a proper NE minor of G contradicting G being minor minimal.

We conclude $G_1 + ab$ is nonplanar. A similar argument shows $G_2 + ab$ is nonplanar as well. Then G must have one of the NE graphs $G_1 \ddot{\cup} G_2$ with $G_i \in \{K_5 - e, K_{3,3} - e\}$ as a minor. Since G is minor minimal, G is a graph of this form. □

Lemma 5.15. *If G is MMNE, $\kappa(G) = 2$, $\delta(G) \geq 3$, G_1 is planar, and G_2 is nonplanar, then $G_1 \in \{K_5 - e, K_{3,3} - e\}$, sharing two vertices and no edges with $G_2 = K_{3,3}$.*

Proof. Let G be as described. For a contradiction, suppose $G_1 + ab$ is planar. Then $G_2 + ab$ must be NE. Indeed, if we delete ab, we are left with the nonplanar G_2. Let $e \in E(G_2)$. Since G is NE, $G - e$ is nonplanar and has a K-subgraph K. If K uses at most one of $\{a, b\}$, then K lies entirely in G_2 and avoids e. So, $(G_2 + ab) - e$ is nonplanar in this case. On the other hand, if $\{a, b\} \subset V(K)$, then, by Lemma 1.8 and since $G_1 + ab$ is planar, the part of K in G_1 is an a-b-path. So using edge ab instead, K remains as a K-subgraph of $(G_2 + ab) - e$, which is again nonplanar. However, $G_2 + ab$ being NE contradicts G being minor minimal. We conclude $G_1 + ab$ is nonplanar.

This means G_1 has one of $K_5 - e$ and $K_{3,3} - e$ as a minor with the missing edge corresponding to ab. Replace G_1 by its minor $K_5 - e$ or $K_{3,3} - e$, call it H, to form $M = H \cup G_2$, a minor of G. We claim M is again NE. Indeed, if we delete $e \in E(H)$, the graph G_2 shows $M - e$ is nonplanar. For $e \in E(G_2)$, we know $G - e$ has a K-subgraph K. If K sees at most one of a and b, it must lie entirely in G_2 (since H is planar) and $M - e$ is nonplanar. If $\{a, b\} \subset V(K)$, then, by Lemma 1.8, K is simply a path on one side of the 2-cut. If K is a path in G_1, then replace that by a path in H to recognize K as a subgraph of $M - e$, which is therefore nonplanar. On the other hand, if K is a path in G_2, this path avoids e. So, we can use H along with that path to again find a nonplanar subgraph of $M - e$. Since G is minor minimal, $G = M$ and $G_1 \in \{K_5 - e, K_{3,3} - e\}$ as required.

Now, G_2 being nonplanar has a K-subgraph K. Also, there must be an a-b-path P in G_2, as otherwise G has connectivity 1. Moreover, both K and $G_1 \cup P$ are nonplanar, and so they must overlap, as otherwise G has a proper disconnected MMNE minor. This means P passes through K and, by contracting edges in P if necessary, we can assume G has a minor with $\{a, b\} \subset V(K)$. From this, form the minor $M = G_1 \cup K$. If K is a subdivision of K_5, Then M and hence G has the MMNA graph $G_1 \ddot{\cup} (K_5 - e)$ as a proper minor, which is a contradiction. So, K is a subdivision of $K_{3,3}$. After contracting edges, G either has the MMNA $G_1 \ddot{\cup} (K_{3,3} - e)$ as a proper minor, which is a contradiction, or else G has $G_1 \ddot{\cup} K_{3,3}$ as a minor, where a and b are in the same part of $K_{3,3}$. Since G was minor minimal, we conclude $G = G_1 \ddot{\cup} K_{3,3}$. In other words, as required, $G_2 = K_{3,3}$, sharing two vertices and no edge with $G_1 \in \{K_5 - e, K_{3,3} - e\}$. □

Theorem 5.16. *If G is MMNE*, $\kappa(G) = 2$, *and* $\delta(G) \geq 3$, *then G is one of the six graphs of Figure 8.*

Proof. We showed that these six graphs are MMNE in Theorem 5.11. Lemma 5.13 immediately gives that if G_1 and G_2 are both nonplanar, then they are both $K_{3,3}$. Lemmas 5.14 and 5.15 complete the other parts of the proof. In total, these account for six graphs: one from Lemma 5.13, three from Lemma 5.14, and two from Lemma 5.15. □

The restriction on the minimum degree in the last theorem is necessary. Indeed, there are many MMNE graphs with $\delta(G) = 2$ (meaning $\kappa(G) \leq 2$). For example, contracting edge e of Figure 7 (left) results in an MMNE graph that is formed by replacing each edge of $K_{3,3}$ with a triangle. Similarly, replacing each edge of K_5 with a triangle also yields an MMNE graph. Further examples of MMNE graphs with a degree-2 vertex are the first seven listed in Section A.1 of the Appendix.

We remark that these examples arise in part due to our insistence that edge contraction lead to a simple graph. Contracting an edge of a degree-2 vertex in a triangle gives a (multi)graph with a doubled edge. Our convention is to delete one of the doubled edges to return to a simple graph.

We next show that $\delta(G) = 2$ is the minimum for MMNE graphs.

Theorem 5.17. *The minimum vertex degree in an MMNE graph is at least* 2.

Proof. The addition or deletion of an isolated vertex or vertex of degree 1 in a planar graph will again result in a planar graph. So if G is NE with $\delta(G) < 2$, then removing a vertex of degree 0 or 1 will result in a NE graph; hence G is not MMNE. □

Although we cannot completely classify the $\delta(G) = 2$ MMNE graphs, we show that degree-2 vertices must occur as part of a triangle.

Theorem 5.18. *In an MMNE graph, the neighbors of a degree-*2 *vertex are themselves neighbors.*

Proof. Let G be an NE graph with a degree-2 vertex v with neighbors a and b. For a contradiction, suppose ab is not an edge of G. Perhaps G is MMNE so that every proper minor of G is not NE. Let $H = G/av$ be the graph that results from contracting edge av in G. Since G is MMNE, there must be some edge e in H such that $H - e$ is planar. Note that e cannot be the newly formed edge ab in H, else, since degree-1 vertices have no impact on the planarity of a graph, $G - av$ would also be planar, contradicting G being MMNE. Consider the graph $G - e$. Note that $G - e$ and $H - e$ are homeomorphic, so since $H - e$ is planar, $G - e$ is also planar. But this contradicts G being MMNE. □

If graph G has a triangle abc, a ∇Y move on G means forming a new graph G' with one additional vertex v (i.e., $V(G') = V(G) \cup \{v\}$) and replacing the edges

ab, ac, and bc with va, vb, vc. So, G' has the same number of edges as G and one additional vertex. Pierce [2014] shows that ∇Y often preserves NA, as was originally observed by Barsotti in unpublished work. (The bowtie graphs of Figure 3 are examples where ∇Y does not preserve NA.) Here we give a similar result for NE graphs.

Theorem 5.19. *Given an NE graph G with triangle t, let G' be the result of performing a ∇Y move on triangle t in G, and let v be the vertex added in G'. Graph G' is NE if and only if $G' - e_i$ is nonplanar for each e_i incident to v.*

Proof. If G' is NE, then $G' - e_i$ is nonplanar by definition. Conversely suppose that $G' - e_i$ is nonplanar for each e_i incident to v. Perhaps G' is not NE, so there is $e \in E(G')$ such that $G' - e$ is planar. Note that e cannot be incident to v. Since e is not part of triangle t, performing a ∇Y move on $G - e$ will result in $G' - e$, so ∇Y on $G - e$ is also planar. Note that undoing the ∇Y transform on this graph will preserve its planarity. However, graph $G - e$ being planar contradicts G being NE. $\square$

We next give an upper bound on the connectivity of MMNE graphs. We first observe that the minimum degree $\delta(G)$ is bounded by 5.

Theorem 5.20. *If G is MMNE, then $\delta(G) \leq 5$.*

Proof. Suppose G is MMNE with $\delta(G) \geq 6$ and let $n = |V(G)|$. We can assume $n \geq 6$, as G must be nonplanar and the only nonplanar graph with five or fewer vertices is K_5, which is not MMNE. Since $\delta(G) \geq 6$, a lower bound on $|E(G)|$ is $6n/2 = 3n$. Now since G is MMNE, there exist two edges e and f such that $G - e, f$ is a planar graph with at least $3n - 2$ edges. However, a planar graph on n vertices can have no more than $3n - 6$ edges, the number of edges in a planar triangulation. The contradiction shows there is no MMNE graph with $\delta(G) \geq 6$. $\square$

As $\kappa(G) \leq \delta(G)$, we have a bound on the connectivity as an immediate corollary.

Corollary 5.21. *If G is MMNE, then $\kappa(G) \leq 5$.*

Finally, we observe a nice connection between MMNE and MMNA graphs.

Theorem 5.22. *If G is MMNE, then G is MMNA or apex.*

Proof. Suppose G is MMNE and NA. We will argue that G is in fact MMNA. For this, let H be a proper minor. Since G is MMNE, H is edge apex. This means either H is already planar, or else there is an edge e such that $H - e$ is planar. In the latter case, if v is a vertex of e, then $H - v$ is again planar. This shows that H is apex, as required. $\square$

Results of computer searches. In addition to the results above, we have found other examples of MMNE and MMNC graphs through brute-force computer searches. Our code is available at https://github.com/mikepierce/MMGraphFunctions/tree/master/brute-force-search. See the file `Brute-Force-Search.nb` for documentation.

The algorithms underlying the searches are fairly straightforward. First we generate a list of all the graphs that we are going to search using the gtools that are available with the `nauty and Traces` graph theory software [McKay and Piperno 2014]. Specifically, we use the gtools `geng` and `planarg` to produce all connected, nonplanar graphs of minimum vertex degree at least 2 that either have fewer than 20 edges or that have fewer than 10 vertices. The commands used to generate these graphs in `bash` are the following:

```
    $ for i in {6..9}; do
geng -c -d2 ${i} | planarg -v > ${i}v.txt
done
    $ for i in {10..16}; do
geng -c -d2 ${i} 0:17 | planarg -v > ${i}v,(0-17)e.txt
geng -c -d2 ${i} 18   | planarg -v > ${i}v,(18)e.txt
geng -c -d2 ${i} 19   | planarg -v > ${i}v,(19)e.txt
done
```

This brute force search was carried out on a standard laptop computer with 4 GB of memory and an Intel Core i3-350M 2.266 GHz processor. The graphs to be searched were split among many different files so that the search could be run in more manageable segments and so that we did not overflow the laptop's memory. We chose to limit our search to graphs with fewer than 20 edges or fewer than 10 vertices due to time constraints. There are a total of 158 505 connected, nonplanar graphs that have 9 vertices and a minimum vertex degree of at least 2. Searching these graphs took about five hours. Since there are 9 229 423 such graphs on 10 vertices, searching these would take more than ten days. Similarly it took about three days to search all 7 753 990 connected, nonplanar graphs that have 19 edges and a minimum vertex degree of at least 2, so searching all 44 858 715 similar graphs on 20 edges is not feasible.

Next we reformat these graphs in each file produced to be read into Wolfram Mathematica. Then we use Mathematica functions to iterate over this list of graphs one file at a time and pull out any that are found to be either MMNE or MMNC. The code in Mathematica was run on a single Mathematica kernel (no attempt was made to parallelize the search in Mathematica). An overview of the method of testing if a graph G is MMNE is as follows, and an analogous method is used to test if a graph is MMNC:

(1) For each $e \in E(G)$, if $G - e$ is planar return false.

(2) Build all the simple minors of G (the graphs in $\{G - e, G/e \mid e \in E(G)\}$) and remove any duplicates (under isomorphism). If for any of these graphs there is no edge f such that $G - f$ is planar, return false.

(3) Take $S = \{G\} \cup \{G - e \mid e \in E(G)\}$. While $S \neq \varnothing$:

(a) Reset S to the result of contracting each edge of each graph in S.
(b) Remove all planar graphs and duplicate graphs from S.
(c) If there exists $G \in S$ such that $G - e$ is nonplanar for each $e \in E(G)$ then return false.

(4) Return true.

We need step (3) explicitly because both of the properties edge apex and contraction apex are *not* closed under taking graph minors as shown in Theorems 5.2 and 5.4.

In addition to the 12 MMNE graphs that have been considered in this section, the brute-force search has found 15 more examples of MMNE graphs (listed in Section A.1 of the Appendix). Notable graphs in this list are $K_{4,3}$, $K_6 - e$, the rook's graph on 9 vertices, and some examples of MMNE graphs with degree-2 vertices. The brute-force search also found new examples of MMNC graphs in addition to the six graphs considered in this section. In particular, the computer demonstrated that the six MMNE graphs of connectivity 2 in Figure 8 are also MMNC. Along with these graphs there are 69 other MMNC graphs on 19 or fewer edges or 9 or fewer vertices. Section A.2 of the Appendix is an abridged list of these graphs (those on 17 or fewer edges or 9 or fewer vertices).

Beyond a simple brute-force search, we also conducted a more intelligent graph search using the knowledge that performing ∇Y and Y∇ moves on a graph has the potential to preserve the NE or NC property of that graph; see Theorem 5.19. The idea is that the ∇Y or Y∇ families of an MMNE or MMNC graph may contain new MMNE or MMNC graphs. The details of the methodology of this search, as well as the Mathematica code, can be found in [Pierce 2014]. In total, we have found 55 MMNE graphs and 82 MMNC graphs, and we suspect that there are many more of each. Tables 3 and 4 below give a classification of the MMNE and MMNC graphs we have found organized by graph size.

graph size ($\|E(G)\|$)	≤ 11	12	13	14	15	16	17	18	19	20
number of MMNE graphs	0	1	0	2	0	2	3	11	6	≥ 2

graph size ($\|E(G)\|$)	21	22	23	24	25	26	27	28	29	30
number of MMNE graphs	≥ 13	≥ 7	≥ 4	≥ 2	≥ 0	≥ 0	≥ 1	≥ 0	≥ 0	≥ 1

Table 3. The number of MMNE graphs we have found grouped by size. Note that this is a complete classification based on graph size up to and including size 19.

graph size ($\lvert E(G)\rvert$)	$\leq$11	12	13	14	15	16	17	18	19	20
number of MMNC graphs	0	1	0	0	1	6	14	32	25	≥ 3

Table 4. The number of MMNC graphs we have found grouped by size. Note that this is a complete classification based on graph size with the exception of size 20.

Appendix: Edge lists of graphs found through computer searches

A.1. ***MMNE graphs.*** The following 15 MMNE graphs are the result of a computer search conducted on the set of graphs that have 19 or fewer edges or 9 or fewer vertices, and that all have a minimum vertex degree of at least 2. These graphs, together with eleven other graphs considered explicitly in the paper (i.e., all but $K_5 \sqcup K_5$, which has order 10 and size 20) make up all 26 MMNE graphs on 19 or fewer edges or on 9 or fewer vertices. (Note that Table 3 gives 25 graphs of size 19 or less. Adding the graph $K_5 \dot{\cup} K_5$, of order 9 and size 20, is what brings the total to 26.)

{(1, 8), (1, 9), (2, 4), (2, 7), (2, 8), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6),
(4, 8), (5, 6), (5, 7), (5, 9), (6, 7), (6, 9), (7, 9), (8, 9)}

{(1, 6), (1, 7), (2, 5), (2, 7), (3, 7), (3, 8), (3, 9), (4, 5), (4, 6), (4, 8),
(4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (8, 9)}

{(1, 8), (1, 9), (2, 6), (2, 7), (2, 9), (3, 5), (3, 7), (3, 9), (4, 5), (4, 6),
(4, 9), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8), (7, 8), (8, 9)}

{(1, 8), (1, 9), (2, 7), (2, 10), (3, 6), (3, 8), (3, 10), (4, 6), (4, 7), (4, 9),
(5, 6), (5, 7), (5, 8), (6, 9), (6, 10), (7, 8), (7, 10), (8, 9), (9, 10)}

{(1, 9), (1, 10), (2, 7), (2, 8), (2, 10), (3, 7), (3, 8), (3, 9), (4, 6), (4, 8),
(4, 10), (5, 6), (5, 7), (5, 9), (6, 7), (6, 8), (7, 10), (8, 9), (9, 10)}

{(1, 6), (1, 9), (2, 7), (2, 8), (3, 6), (3, 7), (3, 10), (4, 5), (4, 6), (4, 7),
(4, 10), (5, 8), (5, 9), (5, 10), (6, 9), (7, 8), (8, 9), (8, 10), (9, 10)}

{(1, 8), (1, 10), (2, 4), (2, 8), (2, 9), (3, 4), (3, 5), (3, 9), (4, 5), (4, 6),
(5, 7), (5, 10), (6, 7), (6, 8), (6, 9), (7, 9), (7, 10), (8, 10), (9, 10)}

{(1, 6), (1, 7), (1, 9), (2, 7), (2, 8), (2, 9), (3, 6), (3, 8), (3, 9), (4, 5),
(4, 8), (4, 9), (5, 6), (5, 7), (5, 9), (6, 8), (7, 8)}

{(1, 7), (1, 8), (1, 9), (2, 6), (2, 8), (2, 9), (3, 6), (3, 7), (3, 9), (4, 6),
(4, 7), (4, 8), (5, 6), (5, 7), (5, 8), (5, 9)}

{(1, 6), (1, 7), (1, 8), (2, 5), (2, 7), (2, 8), (3, 4), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8)}

{(1, 6), (1, 7), (1, 9), (2, 5), (2, 7), (2, 8), (3, 7), (3, 8), (3, 9), (4, 5), (4, 6), (4, 8), (4, 9), (5, 7), (5, 9), (6, 7), (6, 8), (8, 9)}

{(1, 4), (1, 7), (1, 8), (2, 3), (2, 7), (2, 8), (3, 5), (3, 6), (4, 5), (4, 6), (5, 7), (5, 8), (6, 7), (6, 8)}

{(1, 5), (1, 6), (1, 7), (2, 5), (2, 6), (2, 7), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 7)}

{(1, 6), (1, 7), (1, 8), (1, 9), (2, 4), (2, 5), (2, 8), (2, 9), (3, 4), (3, 5), (3, 6), (3, 7), (4, 7), (4, 9), (5, 6), (5, 8), (6, 9), (7, 8)}

{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)}

A.2. ***MMNC graphs.*** The following 22 MMNC graphs are the result of a computer search conducted on the set of graphs that have 17 or fewer edges or 9 or fewer vertices, and that all have a minimum vertex degree of at least 2.

{(1, 9), (1, 12), (2, 8), (2, 11), (3, 6), (3, 7), (4, 5), (4, 10), (5, 11), (5, 12), (6, 9), (6, 11), (7, 8), (7, 12), (8, 10), (9, 10)}

{(1, 6), (1, 10), (2, 5), (2, 9), (3, 4), (3, 6), (3, 8), (4, 5), (4, 7), (5, 10), (6, 9), (7, 9), (7, 11), (8, 10), (8, 11), (9, 11), (10, 11)}

{(1, 6), (1, 10), (2, 7), (2, 8), (2, 9), (3, 6), (3, 8), (3, 9), (4, 7), (4, 9), (4, 10), (5, 7), (5, 8), (5, 10), (6, 7), (8, 10), (9, 10)}

{(1, 9), (1, 10), (2, 3), (2, 6), (2, 7), (3, 4), (3, 5), (4, 7), (4, 10), (5, 6), (5, 9), (6, 8), (6, 10), (7, 8), (7, 9), (8, 9), (8, 10)}

{(1, 9), (1, 11), (2, 9), (2, 10), (3, 4), (3, 6), (3, 11), (4, 5), (4, 10), (5, 8), (5, 9), (6, 7), (6, 9), (7, 10), (7, 11), (8, 10), (8, 11)}

{(1, 9), (1, 11), (2, 9), (2, 10), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 9), (5, 11), (6, 10), (7, 8), (7, 9), (8, 10), (8, 11), (10, 11)}

{(1, 4), (1, 11), (2, 6), (2, 9), (3, 5), (3, 6), (3, 7), (4, 5), (4, 9), (5, 10), (6, 11), (7, 9), (7, 10), (8, 9), (8, 10), (8, 11), (10, 11)}

{(1, 9), (1, 11), (2, 4), (2, 5), (2, 6), (3, 5), (3, 6), (3, 7), (4, 8), (4, 9), (5, 11), (6, 10), (7, 9), (7, 10), (8, 10), (8, 11), (10, 11)}

{(1, 10), (1, 11), (2, 3), (2, 7), (2, 9), (3, 6), (3, 8), (4, 5), (4, 9), (4, 10), (5, 8), (5, 11), (6, 7), (6, 11), (7, 10), (8, 10), (9, 11)}

$\{(1,8),(1,9),(2,6),(2,12),(3,5),(3,11),(4,11),(4,12),(5,7),(5,9),(6,7),(6,8),(7,10),(8,11),(9,12),(10,11),(10,12)\}$

$\{(1,9),(1,11),(2,5),(2,12),(3,4),(3,12),(4,8),(4,9),(5,7),(5,9),(6,7),(6,8),(6,11),(7,10),(8,10),(10,12),(11,12)\}$

$\{(1,4),(1,8),(1,9),(2,3),(2,8),(2,9),(3,4),(3,6),(3,9),(4,5),(4,8),(5,6),(5,7),(5,9),(6,7),(6,8),(7,8),(7,9)\}$

$\{(1,4),(1,8),(1,9),(2,4),(2,7),(2,9),(3,4),(3,6),(3,9),(5,6),(5,7),(5,8),(5,9),(6,7),(6,8),(7,8)\}$

$\{(1,5),(1,6),(1,8),(2,3),(2,4),(2,7),(3,6),(3,10),(4,5),(4,10),(5,9),(6,9),(7,9),(7,10),(8,9),(8,10)\}$

$\{(1,5),(1,6),(1,8),(2,3),(2,4),(2,7),(3,6),(3,10),(4,5),(4,9),(5,10),(6,9),(7,9),(7,10),(8,9),(8,10)\}$

$\{(1,2),(1,9),(1,10),(2,7),(2,8),(3,8),(3,9),(3,10),(4,7),(4,9),(4,10),(5,7),(5,8),(5,10),(6,7),(6,8),(6,9)\}$

$\{(1,2),(1,4),(1,10),(2,3),(2,9),(3,4),(3,7),(4,8),(5,7),(5,8),(5,10),(6,7),(6,8),(6,9),(7,10),(8,9),(9,10)\}$

$\{(1,5),(1,6),(1,7),(2,5),(2,6),(2,7),(3,5),(3,6),(3,7),(4,5),(4,6),(4,7)\}$

$\{(1,2),(1,4),(1,7),(1,9),(2,3),(2,6),(2,8),(3,5),(3,6),(3,9),(4,5),(4,7),(4,8),(5,8),(5,9),(6,8),(6,9),(7,8),(7,9)\}$

$\{(1,6),(1,7),(1,8),(1,9),(2,4),(2,5),(2,8),(2,9),(3,4),(3,5),(3,6),(3,7),(4,7),(4,9),(5,6),(5,8),(6,9),(7,8)\}$

$\{(1,5),(1,6),(1,7),(1,8),(2,3),(2,4),(2,7),(2,8),(3,4),(3,6),(3,8),(4,5),(4,8),(5,6),(5,7),(6,7)\}$

$\{(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$

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2018 vol. 11 no. 3

A mathematical model of treatment of cancer stem cells with immunotherapy 361
Zachary J. Abernathy and Gabrielle Epelle
RNA, local moves on plane trees, and transpositions on tableaux 383
Laura Del Duca, Jennifer Tripp, Julianna Tymoczko and Judy Wang
Six variations on a theme: almost planar graphs 413
Max Lipton, Eoin Mackall, Thomas W. Mattman, Mike Pierce, Samantha Robinson, Jeremy Thomas and Ilan Weinschelbaum
Nested Frobenius extensions of graded superrings 449
Edward Poon and Alistair Savage
On G-graphs of certain finite groups 463
Mohammad Reza Darafsheh and Safoora Madady Moghadam
The tropical semiring in higher dimensions 477
John Norton and Sandra Spiroff
A tale of two circles: geometry of a class of quartic polynomials 489
Christopher Frayer and Landon Gauthier
Zeros of polynomials with four-term recurrence 501
Khang Tran and Andres Zumba
Binary frames with prescribed dot products and frame operator 519
Veronika Furst and Eric P. Smith

1944-4176(2018)11:3;1-6