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# The tropical semiring in higher dimensions

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We discuss the generalization, in higher dimensions, of the tropical semiring, whose two binary operations on the set of real numbers together with infinity are defined to be the minimum and the sum of a pair, respectively. In particular, our objects are closed convex sets, and for any pair, we take the convex hull of their union and their Minkowski sum, respectively, as the binary operations. We consider the semiring in several different cases, determined by a recession cone.

## Introduction

The tropical semiring is $\langle \mathbb{R} \cup \{\infty\}, \oplus, \odot \rangle$, with the two operations defined by

$$x \oplus y = \min(x, y) \quad \text{and} \quad x \odot y = x + y.$$

The fact that this is a semiring comes from the lack of inverses under $\oplus$, as the additive neutral object is infinity. The multiplicative neutral object, i.e., under the operation $\odot$, is zero. Inspired by [Speyer and Sturmfels 2009, p. 165], we generalize the tropical semiring to higher dimensions. In particular, our elements are polyhedra, or more generally, closed convex sets, in $\mathbb{R}^n$ with a fixed recession cone, i.e., the directions in which the set recedes, and the two operations are defined by taking the convex hull of the union and by the Minkowski sum. Indeed, when $n = 1$ and the recession cone is $\mathbb{R}_+ = \{\xi : \xi \geq 0\}$, then this definition reduces to the tropical semiring [Maclagan and Sturmfels 2015; Speyer and Sturmfels 2009] as described above: the real numbers $x$ and $y$ represent the sets of solutions to the inequalities $t \geq x$ and $t \geq y$, respectively; i.e., they correspond to the polyhedra in $\mathbb{R}$ given by the positive rays with vertices at $x$, $y$. In particular, for each, the recession cone is the nonnegative ray emanating from the origin, or $\mathbb{R}_+$. Clearly, the union of these two sets is represented by the inequality $t \geq \min(x, y)$ and likewise, the Minkowski sum is given by the inequality $t \geq x + y$. Careful consideration must be given to the neutral objects in this setting.

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As suggested in [Speyer and Sturmfels 2009], the set of convex polyhedra in $\mathbb{R}^n$ with fixed recession cone will form a semiring. We explore this idea in detail, considering various recession cones. In particular, we first consider the case of bounded polyhedra, i.e., convex polytopes, in $\mathbb{R}^n$. In this case, the common recession cone is $\{0\}$ and the properties follow quite nicely. Furthermore, we can generalize this case to that of compact (convex) sets in $\mathbb{R}^n$. These proofs are the content of the second section.[1] Prior to that, we provide the necessary background on recession cones and asymptotic cones, and include examples to demonstrate the possible pathology of $\oplus$ and $\odot$ if the recession cone is not fixed. The main portion of the paper is dedicated to establishing the axioms of the various semirings, and most especially, those dealing with the closure of the two operations. The final section of the paper considers unbounded closed convex sets. We demonstrate the semirings of closed convex polyhedra and general convex sets, both with recession cone equal to the nonnegative orthant $\mathbb{R}^n_+$.

## 1. Background: polyhedra, recession cones, and asymptotic cones

Some general references for the material in this section are [Rockafellar 1970; Ziegler 1995; Border 1985; 2002].

**Definition 1.1** [Rockafellar 1970, p. 10]. A subset $P$ of $\mathbb{R}^n$ is *convex* if it satisfies the following property: for every $x, y \in P$ and $\lambda \in \mathbb{R}$, $0 < \lambda < 1$, the element $\lambda x + (1-\lambda)y$ is in $P$.

**Fact 1.2** [Rockafellar 1970, §2]. Given a subset $S$ of $\mathbb{R}^n$, the *convex hull* of $S$, denoted by $\operatorname{conv} S$, is the intersection of all the convex sets containing $S$. It is the smallest convex set containing $S$. In particular, it is the set of all convex combinations of the elements of $S$; i.e.,

$$\operatorname{conv} S = \{\lambda_1 s_1 + \cdots + \lambda_k s_k : s_i \in S,\ \lambda_i \geq 0,\ \lambda_1 + \cdots + \lambda_k = 1,\ k \in \mathbb{N}\}.$$

**Definition 1.3** [Rockafellar 1970, p. 61]. Given a nonempty convex set $P$ in $\mathbb{R}^n$, the *recession cone* is the set of all $y \in \mathbb{R}^n$ such that $p + y \in P$ for all $p \in P$. Denoted by $0^+P$, the recession cone is the set of all directions in which $P$ recedes, i.e., is unbounded.

**Fact 1.4** [Rockafellar 1970, Theorem 8.4]. A nonempty closed convex set $P$ in $\mathbb{R}^n$ is bounded if and only if its recession cone $0^+P$ consists of the zero vector alone.

**Example 1.5.** In the case of $n = 2$, the following sets have recession cone equal to the first quadrant of the plane $\mathbb{R}^2_+ = \{\boldsymbol{x} = (\xi_1, \xi_2) : \xi_1 \geq 0, \xi_2 \geq 0\}$.

(1) $P = \{(x, y) : x \geq -5,\ y \geq -18,\ y \geq -\frac{5}{3}x + 2\}$;

[1] Section 2 and part of Section 3 are the basis for Norton's undergraduate thesis for the Honors College at the University of Mississippi.

(2) $Q = \{(x, y) : x \geq -3,\ y \geq -15,\ y \geq -6x - 16,\ y \geq -\tfrac{1}{2}x - 8\}$;

(3) [Rockafellar 1970, Example p. 62] $\{(x, y) : x > 0,\ y \geq 1/x\}$.

**Definition 1.6** [Rockafellar 1970, p. 170; Ziegler 1995, p. 28; Aliprantis and Border 2006, p. 232]. A *polyhedral convex set* in $\mathbb{R}^n$ is a set which can be expressed as the intersection of some finite collection of closed half spaces; i.e., it is the set of solutions to some finite system of inequalities $A\boldsymbol{x} \leq \boldsymbol{b}$. A *convex polytope* is a bounded polyhedron; i.e., the convex hull of a finite set.

**Fact 1.7** [Ziegler 1995, Proposition 1.12]. If $P$ is a polyhedral convex set in $\mathbb{R}^n$, then $0^+P$ is the set of solutions to the system of inequalities $A\boldsymbol{x} \leq \boldsymbol{0}$.

**Definition 1.8** [Rockafellar 1970, p. 162]. A point $x$ in a convex set $P$ is an *extreme point* if the only way to express $x$ as the convex combination $(1-\lambda)y + \lambda z$ for $y, z \in P$ and $0 < \lambda < 1$ is by taking $y = z = x$. Denote the set of extreme points of $P$ by $\operatorname{ext}(P)$.

**Fact 1.9** [Rockafellar 1970, Corollary 19.1.1]. If $P$ is a polyhedral convex set, then $\operatorname{ext}(P)$ is finite.

In Example 1.5, the first two sets are polyhedra (see Figure 2), but the third one is not. The finite system of inequalities associated to $P$ is

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -\frac{5}{3} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 5 \\ 18 \\ -2 \end{bmatrix},$$

so that $0^+P = \{(x, y) : x \geq 0,\ y \geq 0\}$, and $\operatorname{ext}(P) = \{(-5, \frac{31}{3}),\ (12, -18)\}$.

For a set that is not convex, there is a generalization of the notion of a recession cone. While we only consider convex sets, this new cone is relevant since the two definitions coincide when the convex set is closed; hence we may apply related results in the literature in our cases. We apply this material in the last section.

**Definition 1.10** [Border 1985, Definition 2.34]. The *asymptotic cone* of a set $P$ in $\mathbb{R}^n$, denoted by $\boldsymbol{A}P$, is the set of all possible limits of sequences of the form $\{\alpha_i x_i\}_i$, where each $x_i \in P$, $\alpha_i > 0$, and $\alpha_i \to 0$.

Some properties of the asymptotic cone will be necessary to our proof:

**Fact 1.11** [Debreu 1959, §1.9; Border 2002, Lemma 4]. The following hold for sets $E$, $F$ in $\mathbb{R}^n$:

(1) $\boldsymbol{A}E$ is a cone.

(2) $\boldsymbol{A}E \subseteq \boldsymbol{A}F$ if $E \subseteq F$.

(3) $0^+E \subseteq \boldsymbol{A}E$.

(4) $\boldsymbol{A}E \subseteq \boldsymbol{A}(E + F)$.

(5) $\boldsymbol{A}E$ is closed.

(6) $\boldsymbol{A}E$ is convex if $E$ is convex.

(7) $0^+E = \boldsymbol{A}E$ if $E$ is closed and convex.

(8) $\boldsymbol{A}E + \boldsymbol{A}F \subseteq \boldsymbol{A}(E+F)$ if $E+F$ is convex.

(9) A set $E$ is bounded if and only if $\boldsymbol{A}E = \{0\}$.

**Fact 1.12** [Shveidel 2001, proof of Theorem 2.3]. For a set $P \subseteq \mathbb{R}^n$, we have $\boldsymbol{A}P = \boldsymbol{A}\overline{P}$.

**Example 1.13** [Woo 2013]. In $\mathbb{R}^2$, let

$$P = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\} \cup \{(x, y) : 0 \le x < 1,\ y \ge 1\}.$$

Although $P$ is unbounded, $0^+P = \{0\}$; however, $P$ is not closed (see Fact 1.7). On the other hand, $0^+\overline{P} = \{(0, y) : y \ge 0\} = \boldsymbol{A}\overline{P} = \boldsymbol{A}P$.

As the above definitions and results are important to establishing the closure of the operation $\oplus$, the following definition and result are helpful in establishing the closure of the operation $\odot$.

**Fact 1.14** [Schneider 2014, Theorem 1.1.2]. Let $P$, $Q$ be convex subsets of $\mathbb{R}^n$. Then $\operatorname{conv}(P) = P$, and the Minkowski sum $P + Q$ of $P$ and $Q$ is convex. In particular, if $P$, $Q$ are nonempty, then $P + Q = \{p + q : p \in P,\ q \in Q\}$, and $P + \varnothing = \varnothing$.

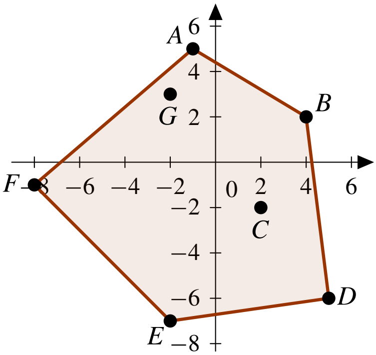
**Definition 1.15** [Debreu 1959, 1.9 m., p. 22]. The cones $C_1, C_2, \ldots, C_k$ in $\mathbb{R}^n$ are *positively semi-independent* if, for any $c_i \in C_i$, the condition $c_1 + c_2 + \cdots + c_k = 0$ implies that each $c_i = 0$.

**Fact 1.16** [Border 2002, Theorem 8]. For closed and convex sets $E, F \subseteq \mathbb{R}^n$ whose asymptotic cones $\boldsymbol{A}E$ and $\boldsymbol{A}F$ are positively semi-independent, the Minkowski sum $E + F$ is closed and $\boldsymbol{A}(E+F) \subseteq \boldsymbol{A}E + \boldsymbol{A}F$.

**Example 1.17** [Border 2002, Example 2]. In $\mathbb{R}^2$, set $E = \{(x, y) : x > 0,\ y \ge 1/x\}$ and $F = \{(x, y) : x < 0,\ y \ge -1/x\}$. Note that both $E$ and $F$ are closed sets, but $E + F = \{(x, y) : y > 0\}$, which is not closed.

Finally, Carathéodory's theorem (see, e.g., [Schneider 2014, Theorem 1.1.4]) will be helpful when considering the elements of convex sets.

**Carathéodory's theorem.** *If a point $x$ lies in the convex (hull of a) set $P \subseteq \mathbb{R}^n$, then $x$ can be written as a convex combination of no more than $n+1$ points in $P$; i.e., there are $p_0, p_1, \ldots, p_n \in P$ and $\lambda_i \ge 0$ such that $\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1$ and $x = \lambda_0 p_0 + \cdots + \lambda_n p_n$.*



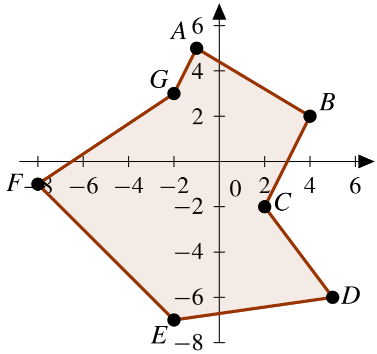




**Figure 1.** A convex polytope (left) and a nonconvex set (right) in $\mathbb{R}^2$.

## 2. The tropical semiring in higher dimensions: the bounded case

***The semiring of convex polytopes.*** Recall that a convex polytope in $\mathbb{R}^n$ is a bounded polyhedral set; i.e., the convex hull of a finite number of points in $\mathbb{R}^n$. See Figure 1. In particular, these sets are those convex polyhedra in $\mathbb{R}^n$ with recession cone equal to the zero vector.

**Theorem 2.1.** *The set of all convex polytopes $P$, $Q$ in $\mathbb{R}^n$, with operations shown below, is a semiring*:

$$P \oplus Q = \operatorname{conv}(P \cup Q) \quad P \odot Q = P + Q = \{p+q : p \in P,\ q \in Q\}. \tag{2-1}$$

*Proof.* Let $P$, $Q$, $R$ be convex polytopes in $\mathbb{R}^n$. Note that the empty set satisfies the convexity property vacuously, and as the solution set of any inconsistent system, it is a polytope. In particular, if $P$, $Q$ are nonempty, set $P = \operatorname{conv}(p_1, \dots, p_s)$ and $Q = \operatorname{conv}(q_1, \dots, q_t)$.

**Claim 2.1A.** *The set of all convex polytopes in $\mathbb{R}^n$ under the operation of $\oplus$ is a commutative monoid.*

- The operation $\oplus$ is closed; i.e., $\operatorname{conv}(P \cup Q)$ is a convex polytope:[2] First of all, $P \oplus \varnothing = \operatorname{conv}(P \cup \varnothing) = \operatorname{conv}(P) = P$, as $P$ is convex, and likewise for $\varnothing \oplus Q$. Moreover, $\varnothing \oplus \varnothing = \varnothing$. Thus, we may assume that $P$, $Q$ are both nonempty. We will show that $\operatorname{conv}(P \cup Q) = \operatorname{conv}(p_1, \dots, p_s, q_1, \dots, q_t)$. Let $z \in \operatorname{conv}(P \cup Q)$. By Carathéodory's theorem, $z = \sum_{i=0}^{n} \lambda_i y_i$, where each $\lambda_i \ge 0$, $\sum_{i=0}^{n} \lambda_i = 1$ and $y_i \in P \cup Q$. For each $y_i \in P$, one can write $y_i = \sum_{j=1}^{s} \delta_{ij} p_j$, where $\delta_{ij} \ge 0$ for

[2]This fact appears in several books without proof. Therefore, we provide an argument, for the benefit of the undergraduate reader. (Likewise, for some other proofs in this section.) For algorithms that compute the convex hull of a finite set of points in the plane, for example, Graham's scan and Jarvis's march, see, e.g., [Cormen et al. 2001, Chapter 33, Section 3].

all $j$ and $\sum_{j=1}^{s} \delta_{ij} = 1$. If all $y_i \in P$, then

$$\begin{aligned} z &= \sum_{i=0}^{n} \Bigl( \lambda_i \sum_{j=1}^{s} \delta_{ij} p_j \Bigr) \\ &= \sum_{j=1}^{s} \Bigl( \sum_{i=0}^{n} \lambda_i \delta_{ij} \Bigr) p_j \in \operatorname{conv}(p_1, \dots, p_s) \subseteq \operatorname{conv}(p_1, \dots, p_s, q_1, \dots, q_t); \end{aligned}$$

it is similar if all $y_i \in Q$. Thus, let $m \in \mathbb{N},\ m < n$, such that $y_0, \dots, y_{m-1} \in P \setminus Q$ and $y_m, \dots, y_n \in Q$. Then

$$z = \sum_{i=0}^{n} \lambda_i y_i = \sum_{i=0}^{m-1} \Bigl( \sum_{j=1}^{s} \lambda_i \delta_{ij} p_j \Bigr) + \sum_{i=m}^{n} \Bigl( \sum_{k=1}^{t} \lambda_i \delta_{ik} q_k \Bigr)$$

is a convex combination of $\{p_1, \dots, q_t\}$; hence, $\operatorname{conv}(P \cup Q) \subseteq \operatorname{conv}(p_1, \dots, q_t)$. Since the containment $\supseteq$ is clear, $\operatorname{conv}(P \cup Q) = \operatorname{conv}(p_1, \dots, p_s, q_1, \dots, q_t)$, and the latter, by Definition 1.6, is a polytope.

- The operation $\oplus$ is associative; i.e., $(P \oplus Q) \oplus R = P \oplus (Q \oplus R)$: Regarding $(P \oplus Q) \oplus R = P \oplus (Q \oplus R)$, we wish to prove

$$\operatorname{conv}[\operatorname{conv}(P \cup Q) \cup R] = \operatorname{conv}[P \cup \operatorname{conv}(Q \cup R)]. \tag{2-2}$$

If any one or more of the sets is the empty set, then it is easy to see that the equality holds. Otherwise, it suffices to show that each of these sets is equal to $\operatorname{conv}(P \cup Q \cup R)$. Consider the set on the left. Since $P \cup Q \cup R \subseteq \operatorname{conv}(P \cup Q) \cup R$, we have $\operatorname{conv}(P \cup Q \cup R) \subseteq \operatorname{conv}[\operatorname{conv}(P \cup Q) \cup R]$.

Conversely, as $\operatorname{conv}(P \cup Q)$, $R \subseteq \operatorname{conv}(P \cup Q \cup R)$, we have $\operatorname{conv}(P \cup Q) \cup R \subseteq \operatorname{conv}(P \cup Q \cup R)$. Take the convex hull of both sides: $\operatorname{conv}[\operatorname{conv}(P \cup Q) \cup R] \subseteq \operatorname{conv}(P \cup Q \cup R)$. This establishes that $\operatorname{conv}(P \cup Q \cup R) = \operatorname{conv}[\operatorname{conv}(P \cup Q) \cup R]$. The argument for $\operatorname{conv}[P \cup \operatorname{conv}(Q \cup R)]$ is analogous; hence we have (2-2).

- The operation $\oplus$ is commutative: order does not matter in unions of sets.

- There exists a neutral object $\mathcal{O}$ for addition such that for any convex polytope $P$ in $\mathbb{R}^n$, $P \oplus \mathcal{O} = \mathcal{O} \oplus P = P$: take $\mathcal{O}$ to be the empty set $\varnothing$, since $\operatorname{conv}(P \cup \varnothing) = P$.

**Claim 2.1B.** *The set of all convex polytopes in $\mathbb{R}^n$ under the operation of $\odot$ is a commutative monoid.*

- The operation $\odot$ is closed; i.e., $P + Q$ is a convex polytope: First of all, $P \odot \varnothing = \varnothing$ since $P + \varnothing = \varnothing$ in Minkowski addition, and likewise for $\varnothing \odot Q$. Moreover, $\varnothing \odot \varnothing = \varnothing$. Thus, we may assume that $P$, $Q$ are both nonempty. We will show that $P + Q = \operatorname{conv}(\{p_j + q_k : 1 \le j \le s,\ 1 \le k \le t\})$, as per the hint in [Aliprantis and Border 2006, proof of Lemma 5.124]. Let $p \in P$ and $q \in Q$. Write $p = \sum_{j=1}^{s} \lambda_j p_j$

and $q = \sum_{k=1}^{t} \mu_k q_k$, where $\lambda_j, \mu_k \geq 0$ and $\sum_{j=1}^{s} \lambda_j = 1 = \sum_{k=1}^{t} \mu_k$. Then,

$$\begin{aligned} p+q &= \sum_{j=1}^{s} \lambda_j p_j + \sum_{k=1}^{t} \mu_k q_k \\ &= \left(\sum_{k=1}^{t} \mu_k\right) \sum_{j=1}^{s} \lambda_j p_j + \left(\sum_{j=1}^{s} \lambda_j\right) \sum_{k=1}^{t} \mu_k q_k = \sum_{j=1}^{s} \sum_{k=1}^{t} \lambda_j \mu_k (p_j + q_k) \end{aligned}$$

is a convex combination of $\{p_j + q_k : 1 \leq j \leq s,\ 1 \leq k \leq t\}$.

Conversely, let $\sum_{i=0}^{n} \lambda_i (x_i + y_i)$ be a convex combination of $\{p_j + q_k : 1 \leq j \leq s, 1 \leq k \leq t\}$; i.e., $x_i = p_j$ for some $j$ and $y_i = q_k$ for some $k$, $\lambda_i \geq 0$, and $\sum_{i=0}^{n} \lambda_i = 1$. Then

$$\sum_{i=0}^{n} \lambda_i (x_i + y_i) = \sum_{i=0}^{n} \lambda_i x_i + \sum_{i=0}^{n} \lambda_i y_i,$$

where the first sum is in $\mathrm{conv}(p_1, \ldots, p_s)$ and the second sum is in $\mathrm{conv}(q_1, \ldots, q_t)$. Thus, $P + Q = \mathrm{conv}(\{p_j + q_k | 1 \leq j \leq s, 1 \leq k \leq t\})$, and the latter, by Definition 1.6, is a convex polytope.

- The operation $\odot$ is associative: addition in $\mathbb{R}^n$ is associative.
- The operation $\odot$ is commutative: addition in $\mathbb{R}^n$ is commutative.
- There exists a neutral object $\mathcal{I}$ for multiplication such that for any convex polytope $P$ in $\mathbb{R}^n$, $P \odot \mathcal{I} = \mathcal{I} \odot P = P$: Take $\mathcal{I}$ to be $\mathrm{conv}(\{\mathbf{0}\}) = \{\mathbf{0}\}$, which is a convex polytope by Definition 1.6, and the common recession cone of all nonempty convex polytopes $P$ in $\mathbb{R}^n$. Then $P + \mathbf{0} = P$, by definition of $0^+ P$, and $\varnothing + \mathbf{0} = \varnothing$.

**Claim 2.1C.** *The operation $\odot$ is distributive over $\oplus$; i.e.,*

$$P \odot (Q \oplus R) = (P \odot Q) \oplus (P \odot R).$$

We wish to establish that $P + \mathrm{conv}(Q \cup R) = \mathrm{conv}[(P + Q) \cup (P + R)]$. If $P = \varnothing$ or more than two of the sets are empty, then both expressions equal $\varnothing$, and if only $Q = \varnothing$ or only $R = \varnothing$, then both expressions equal $P + R$ or $P + Q$ respectively. Thus, assume all three are nonempty.

First of all, take $p + z$, where $p \in P$ and $z \in \mathrm{conv}(Q \cup R)$. Then $z = \sum_{i=0}^{n} \lambda_i y_i$, where $\lambda_i \geq 0$, $\sum_{i=0}^{n} \lambda_i = 1$, and $y_i \in Q \cup R$. Therefore, we have

$$p + z = 1p + \sum_{i=0}^{n} \lambda_i y_i = \left(\sum_{i=0}^{n} \lambda_i\right) p + \sum_{i=0}^{n} \lambda_i y_i = \sum_{i=0}^{n} \lambda_i (p + y_i).$$

The elements $p + y_j$ are in $P + Q$ or $P + R$, and possibly both. Therefore, the last expression is in $\mathrm{conv}[(P+Q) \cup (P+R)]$; i.e., $p + z \in \mathrm{conv}[(P+Q) \cup (P+R)]$. Since $p$ and $z$ are arbitrary, we have $P + \mathrm{conv}(Q \cup R) \subseteq \mathrm{conv}[(P + Q) \cup (P + R)]$.

Conversely, since $P+Q$, $P+R \subseteq P+\operatorname{conv}(Q\cup R)$, it follows that $P+\operatorname{conv}(Q\cup R)$ contains $(P+Q)\cup(P+R)$. Take the convex hull of both sides:

$$\operatorname{conv}[(P+Q)\cup(P+R)]\subseteq\operatorname{conv}[P+\operatorname{conv}(Q\cup R)]=\operatorname{conv}(P)+\operatorname{conv}[\operatorname{conv}(Q\cup R)],$$

where the equality follows by Fact 1.14. Now since both terms in the last sum are convex, the expression simplifies to $P+\operatorname{conv}(Q\cup R)$. This establishes the other inclusion, and therefore, $P+\operatorname{conv}(Q\cup R)=\operatorname{conv}[(P+Q)\cup(P+R)]$.

**Claim 2.1D.** *The additive neutral object $\mathcal{O}$ is an absorbing element for $\odot$; i.e., for any convex polytope $P$ in $\mathbb{R}^n$, $\mathcal{O}\odot P=P\odot\mathcal{O}=\mathcal{O}$.*

This follows from the fact that, in Minkowski addition, $\varnothing+P=\varnothing$. □

***The semiring of convex compact sets.*** In this section, we generalize the above work with convex polytopes to general convex compact subsets of $\mathbb{R}^n$. Of import is the Heine–Borel theorem (see, e.g., [Aliprantis and Border 2006, Theorem 3.19]):

**Heine–Borel theorem.** *Subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded.*

**Proposition 2.2.** *The set of all compact convex sets $P$, $Q$ in $\mathbb{R}^n$, with the operations as in (2-1), is a semiring.*

*Proof.* We note that the arguments for many of the claims above do not change. In particular, the empty set is compact; hence it remains the neutral element under $\oplus$. However, closure of the two operations must be considered. Therefore, let $P$, $Q$ be compact convex sets in $\mathbb{R}^n$.

- The operation $\oplus$ is closed; i.e., $\operatorname{conv}(P\cup Q)$ is a compact convex set: The union of finitely many compact sets is compact. Thus, $P\cup Q$ is compact. Next, the convex hull of a compact set in $\mathbb{R}^n$ remains compact (see, e.g., [Aliprantis and Border 2006, Corollary 5.18]); thus, $\operatorname{conv}(P\cup Q)$ is a compact convex set.

- The operation $\odot$ is closed; i.e., $P+Q$ is a compact convex set: As per [Border 2002, Corollary 11], the summation of a closed set and a compact set is closed. As such, $P\odot Q=P+Q$ is closed, and convex. Moreover, $P+Q$ is bounded since $P$, $Q$ are bounded. Apply the Heine–Borel theorem. □

## 3. The tropical semiring in higher dimensions: the unbounded case

***The semiring of convex polyhedra.*** We consider the set of convex polyhedra in $\mathbb{R}^n$ with the operations $\oplus$ and $\odot$ as in (2-1). Although convex polyhedra are necessarily closed (see, e.g., [Rockafellar 1970, Theorem 19.1]), the convex hull of the union of two convex polyhedral sets need not be polyhedral or closed, as evinced by Example 3.1 below, that is, if their recession cones do not coincide. Therefore, we restrict our sets to those with the same recession cone, namely the nonnegative orthant $\mathbb{R}^n_+ = \{\boldsymbol{x} = (\xi_1, \ldots, \xi_n) : \xi_1 \geq 0, \ldots, \xi_n \geq 0\}$. This restriction is a generalization of the nonnegative ray in the tropical semiring when $n = 1$.

**Example 3.1** [Rockafellar 1970, p. 177]. In $\mathbb{R}^2$, let $P = \{(-1, 0)\}$ and $Q = \{(x, y) : x, y \geq 0\}$. Then $\operatorname{conv}(P \cup Q) = \{(-1, 0)\} \cup \{(x, y) : -1 < x,\ 0 \leq y\}$, which is neither polyhedral nor closed. However, $0^+P$ is the origin, while $0^+Q = Q = \mathbb{R}^2_+$.

**Proposition 3.2.** *Let $\mathcal{P}$ be the set of all convex polyhedra in $\mathbb{R}^n$ with recession cone equal to the nonnegative orthant $\mathbb{R}^n_+$. Then $\langle \mathcal{P} \cup \{\varnothing\}, \oplus, \odot \rangle$, with operations defined in (2-1), is a semiring.*

*Proof.* It suffices to address the issues regarding closure of the two operations for convex polyhedra $P$, $Q$ in $\mathbb{R}^n$ with recession cone equal to $\mathbb{R}^n_+$, and the multiplicative neutral object, since the earlier arguments for the remaining properties apply here.

- The operation $\oplus$ is closed; i.e., $\operatorname{conv}(P \cup Q)$ is a convex polyhedron in $\mathbb{R}^n$ with recession cone equal to $\mathbb{R}^n_+$: Since $\operatorname{conv}(P \cup Q)$ is convex, it remains to establish that $\operatorname{conv}(P \cup Q)$ is polyhedral with a recession cone equal to the nonnegative orthant. The fact that the recession cone of $\operatorname{conv}(P \cup Q)$ is equal to $\mathbb{R}^n_+$ follows from [Rockafellar 1970, Theorem 9.8.1]; therefore, it only remains to show that $\operatorname{conv}(P \cup Q)$ is polyhedral. By Definition 1.6, $P$ is the irredundant intersection of some finite collection of closed half spaces, including those of the form $\{\boldsymbol{x} : \langle \boldsymbol{x}, (0, \ldots, 0, 1, 0, \ldots, 0) \rangle \geq a_i\}$ for some $a_i \in \mathbb{R}$, i.e., $x_i \geq a_i$, since $0^+P = \mathbb{R}^n_+$. Likewise, $0^+Q = \mathbb{R}^n_+$; hence, for each $i$, the half-spaces defining $Q$ include $x_i \geq c_i$ for some $c_i \in \mathbb{R}$. Thus, every element of $P \cup Q$ satisfies the set of inequalities

$$\{\boldsymbol{x} : \langle \boldsymbol{x}, (0, \ldots, 0, 1, 0, \ldots, 0) \rangle \geq \min(a_i, c_i)\}.$$

Moreover, if $z \in \operatorname{conv}(P \cup Q) \setminus (P \cup Q)$, then $z$ is in the finite region bounded by the (necessarily finite set of) extreme points of $P$ and $Q$. See Figure 2 for an example. Thus, $\operatorname{conv}(P \cup Q) = \operatorname{conv}(\operatorname{ext}(P) \cup \operatorname{ext}(Q)) + \mathbb{R}^n_+$, and the latter, by [Ziegler 1995, Theorem 1.2], is polyhedral.

- The operation $\odot$ is closed; i.e., $P + Q$ is a convex polyhedron with recession cone equal to $\mathbb{R}^n_+$: By [Rockafellar 1970, Corollary 19.3.2], the Minkowski sum of two polyhedral convex sets in $\mathbb{R}^n$ is polyhedral, and it is convex. Therefore, it remains to show that $0^+(P + Q) = \mathbb{R}^n_+$. Since polyhedral convex sets are closed, their recession cones are equal to their asymptotic cones. Hence by Fact 1.11(8), $\boldsymbol{A}P + \boldsymbol{A}Q \subseteq \boldsymbol{A}(P + Q)$. Next, as $\boldsymbol{A}P = \boldsymbol{A}Q = \mathbb{R}^n_+$, it follows that if $y \in \boldsymbol{A}P \setminus \{\boldsymbol{0}\}$, then $-y \notin \boldsymbol{A}Q$. In other words, $\boldsymbol{A}P$ and $\boldsymbol{A}Q$ are positively semi-independent, as per Definition 1.15. Thus, by Fact 1.16, $\boldsymbol{A}(P + Q) \subseteq \boldsymbol{A}P + \boldsymbol{A}Q$ and the result follows.

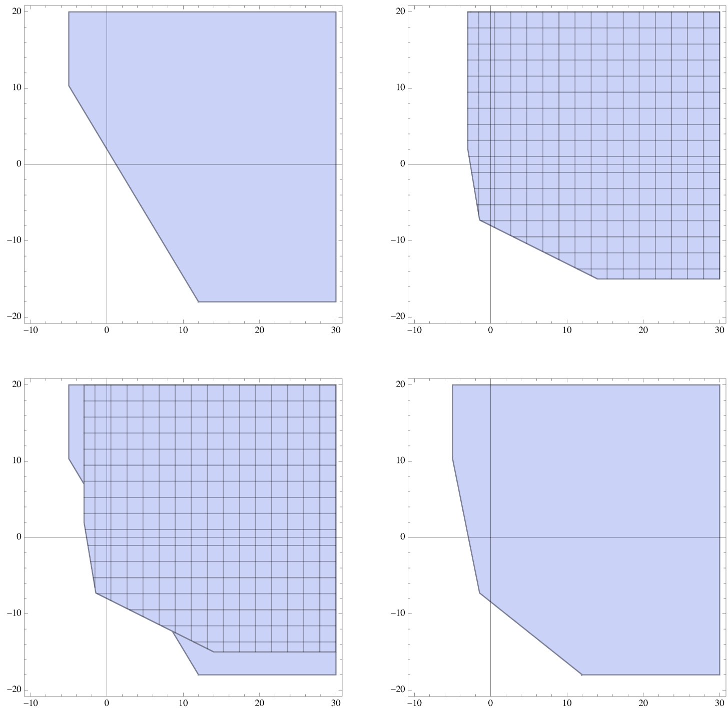



**Figure 2.** The graphs of polyhedra $P$ (top left) and $Q$ (top right) from Example 1.5, and $P \cup Q$ (bottom left) and $\operatorname{conv}(P \cup Q)$ (bottom right).

• There exists a neutral object $\mathcal{I}$ for multiplication: Take $\mathcal{I}$ to be $\mathbb{R}^n_+$, which is not only an element of $\mathcal{P}$, but also the common recession cone of all nonempty polyhedra $P$ in $\mathcal{P}$. Thus $P+\mathbb{R}^n_+ = P$, by the definition of $0^+P$, and $\varnothing+\mathbb{R}^n_+ = \varnothing$. $\square$

**Remark 3.3.** While the set of real numbers $\mathbb{R}^1$ is in one-to-one correspondence with the set of all nonempty closed convex polyhedra in the real number line, the same is not true for $\mathbb{R}^n$ when $n \geq 2$. As mentioned in the Introduction, $r \leftrightarrow [r, \infty)$, in the case that $n = 1$, but an ordered pair $(r_1, r_2)$ does not correspond to a unique closed convex polyhedron in $\mathbb{R}^2$.

***The semiring of closed convex sets with a fixed recession cone.*** Finally, we generalize the above work to closed convex subsets of $\mathbb{R}^n$ with a fixed recession cone $C$. As evinced in Example 1.13, pathology arises if the convex sets are not assumed to be closed. However, despite taking two convex sets that are closed, neither the convex hull of the union nor the Minkowski sum need be closed, as demonstrated by Examples 3.1 and 1.17, respectively, that is, if their recession cones do not coincide. Moreover, our earlier work hints at the possible necessity of taking $C$ such that $AC \cap (-AC) = \{\mathbf{0}\}$.

**Theorem 3.4.** *Let $\mathcal{S}$ be the set of all closed convex sets in $\mathbb{R}^n$ with fixed recession cone $C$ satisfying either of the conditions below*:

(1) $AC \cap (-AC) = \{\mathbf{0}\}$.

(2) *$C$ is a closed half-space containing the origin.*

*Then $\langle \mathcal{S} \cup \{\varnothing\}, \oplus, \odot \rangle$, with operations defined in (2-1), is a semiring.*

*Proof.* Again, the earlier arguments for the most of the properties apply here; therefore, we address the issues regarding closure of the two operations for closed convex sets $P$, $Q$ of $\mathbb{R}^n$ with fixed recession cone $C$ satisfying either of the two conditions. The fact that $\operatorname{conv}(P \cup Q)$ is a closed convex subset in $\mathbb{R}^n$ with recession cone $C$ follows from [Rockafellar 1970, Theorem 9.8.1]. If $C$ satisfies condition (1), then we may apply our previous argument. If $C$ satisfies condition (2), then $P$ and $Q$ are parallel to $C$, and hence so is $P + Q$. The result follows. □

To tie this theorem to our earlier work, we make note of the following:

**Corollary 3.5.** *The empty set, together with the set of all closed convex sets in $\mathbb{R}^n$ with recession cone equal to $\mathbb{R}^n_+$, and operations defined in (2-1), is a semiring.*

**Remark 3.6.** The set of all closed convex sets in $\mathbb{R}^n$ with recession cone $C$ equal to $\mathbb{R}^n$ is the trivial semiring $\{C\}$.

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