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Christopher Frayer and Landon Gauthier

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A tale of two circles: geometry of a class of quartic polynomials

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Let $\mathcal{P}$ be the family of complex-valued polynomials of the form $p(z) = (z-1)(z-r_1)(z-r_2)^2$ with $|r_1| = |r_2| = 1$. The Gauss–Lucas theorem guarantees that the critical points of $p \in \mathcal{P}$ will lie within the unit disk. This paper further explores the location and structure of these critical points. For example, the unit disk contains two "desert" regions, the open disk $\{z \in \mathbb{C} : |z - \frac{3}{4}| < \frac{1}{4}\}$ and the interior of $2x^4 - 3x^3 + x + 4x^2y^2 - 3xy^2 + 2y^4 = 0$, in which critical points of p cannot occur. Furthermore, each c inside the unit disk and outside of the two desert regions is the critical point of at most two polynomials in $\mathcal{P}$.

1. Introduction

Given a complex-valued polynomial $p(z)$, the Gauss–Lucas theorem guarantees that its critical points lie in the convex hull of its roots. Critical points of polynomials of the form

$$p(z) = (z-1)(z-r_1)(z-r_2)$$

with $|r_1| = |r_2| = 1$ are studied in [Frayer et al. 2014]. For such a polynomial, a critical point almost always determines p uniquely, and the unit disk contains a *desert*, the open disk $\{z \in \mathbb{C} : |z - \frac{2}{3}| < \frac{1}{3}\}$, in which critical points of p cannot occur.

This paper extends the results of [Frayer et al. 2014] to a family of polynomials of the form

$$\mathcal{P} = \{p : \mathbb{C} \to \mathbb{C} : p(z) = (z-1)(z-r_1)(z-r_2)^2,\ |r_1| = |r_2| = 1\}.$$

We used GeoGebra to investigate the critical points of $p(z) = (z-1)(z-r_1)(z-r_2)^2$. In Figure 1, we set r_1 and r_2 in motion around the unit circle and traced the loci of the critical points with the color gray. Much to our surprise, the unit disk contained

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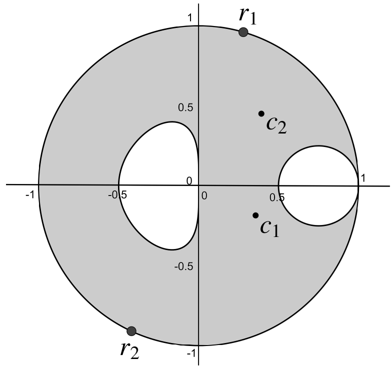

Figure 1. Letting the roots vary and tracking the loci of the critical points yields a very surprising result.

two desert regions. In this paper we determine the boundary equations of the desert regions and characterize the critical points of polynomials in $\mathcal{P}$.

2. Preliminary information

Circles tangent to the line $x = 1$ will appear frequently throughout this paper. We let T_α denote the circle of diameter α passing through 1 and $1-\alpha$ in the complex plane. That is,

$$T_\alpha = \left\{z \in \mathbb{C} : \left|z - \left(1 - \tfrac{1}{2}\alpha\right)\right| = \tfrac{1}{2}\alpha\right\}.$$

For example, T_2 is the unit circle. A key result from [Frayer et al. 2014] will be used to analyze critical points of a polynomial in $\mathcal{P}$.

Theorem 1 [Frayer et al. 2014]. *Suppose $f(z) = (z-1)(z-r_1)\cdots(z-r_n)$ with $|r_k| = 1$ for each k. Let $c_1, c_2, \ldots, c_n$ denote the critical points of $f(z)$, and suppose that $1 \neq c_k \in T_{\alpha_k}$ for each k. Then*

$$\sum_{k=1}^{n} \frac{1}{\alpha_k} = n. \tag{1}$$

An additional fact of interest is related to fractional linear transformations. Functions of the form

$$f(z) = e^{i\theta}\frac{z-\alpha}{\bar{\alpha}z - 1}$$

with $|\alpha| < 1$ are the only one-to-one analytic mappings of the unit disk onto itself [Saff and Snider 1993, p. 334]. Therefore, the only fractional linear transformations sending the unit circle to the unit circle are of the form $f(z)$ or $1/f(z)$. In either case, writing $e^{i\theta} = e^{i\theta/2}/e^{-i\theta/2}$ leads to the following result.

Theorem 2. *A fractional linear transformation T sends the unit circle to the unit circle if and only if*

$$T(z) = \frac{\bar{\alpha} z - \bar{\beta}}{\beta z - \alpha}$$

for some $\alpha, \beta \in \mathbb{C}$ with $|\alpha/\beta| \neq 1$.

3. Critical points

A polynomial of the form

$$p(z) = (z-1)(z-r_1)(z-r_2)^2 \in \mathcal{P}$$

has three critical points: one trivial critical point at the repeated root r_2, and two additional critical points. Differentiation yields

$$p'(z) = (z-r_2)(4z^2 - (3r_1 + 2r_2 + 3)z + r_1 r_2 + 2r_1 + r_2).$$

Definition 3. We define the *nontrivial* critical points of p to be the two roots of

$$q(z) = 4z^2 - (3r_1 + 2r_2 + 3)z + r_1 r_2 + 2r_1 + r_2.$$

We begin by analyzing a few special cases for future reference.

Example 4. Let $p \in \mathcal{P}$ have a nontrivial critical point at $z = 1$. Then p must have a repeated root at $z = 1$. Therefore, $p \in \mathcal{P}$ has a nontrivial critical point at $z = 1$ if and only if $p(z) = (z-1)^2(z-r)^2$ or $p(z) = (z-1)^3(z-r)$ for some $r \in T_2$.

Now that we know which polynomials in $\mathcal{P}$ have a nontrivial critical point at $c = 1$, we will assume that $c \neq 1$ as necessary throughout the remainder of the paper.

Example 5. Let $p \in \mathcal{P}$ have a nontrivial critical point at $c \in T_2$, where $c \neq 1$. The Gauss–Lucas theorem implies that c is a root of p. In order for c to be a nontrivial critical point, p must have a triple root at c. Therefore, $p \in \mathcal{P}$ has a nontrivial critical point at $c \in T_2$, where $c \neq 1$, if and only if $p(z) = (z-1)(z-c)^3$. In this case, $p'(z) = 4(z-1)^2\bigl(z - \bigl(\frac{3}{4} + \frac{1}{4}c\bigr)\bigr)$ and the other nontrivial critical point, $\frac{3}{4} + \frac{1}{4}c \in T_{1/2}$, lies on the line segment $\underline{1c}$. In fact, whenever p has two distinct roots, due to repeated roots, then the critical points of p lie on the line segment between the two roots.

The Gauss–Lucas theorem guarantees that the nontrivial critical points of $p \in \mathcal{P}$ lie within the unit disk. But we can say more; there is a *desert*, the open disk $\bigl\{z : z \in T_\alpha \text{ with } 0 < \alpha < \frac{1}{2}\bigr\}$, in which critical points of p cannot occur. This desert corresponds to the white disk in Figure 1.

Theorem 6. *No polynomial $p \in \mathcal{P}$ has a critical point strictly inside $T_{1/2}$.*

Proof. Let $c_1, c_2 \neq 1$ be nontrivial critical points of $p(z) = (z-1)(z-r_1)(z-r_2)^2$ with $c_1 \in T_\alpha$ and $c_2 \in T_\beta$. As the trivial critical point lies on T_2, Theorem 1 gives

$$\frac{1}{2} + \frac{1}{\alpha} + \frac{1}{\beta} = 3. \tag{2}$$

Suppose for the sake of contradiction that $\alpha < \frac{1}{2}$. Then

$$\frac{1}{\beta} < \frac{5}{2} - 2 = \frac{1}{2}$$

implies $\beta > 2$, which violates the Gauss–Lucas theorem. □

A similar analysis leads to the following theorem.

Theorem 7. *Let $c_1, c_2 \neq 1$ be nontrivial critical points of $p \in \mathcal{P}$. If c_1 lies on $T_{4/5}$ so does c_2. Otherwise, c_1 and c_2 lie on opposite sides of $T_{4/5}$.*

Proof. Let $c_1 \in T_\alpha$ and $c_2 \in T_\beta$. Then, (2) implies $1/\alpha + 1/\beta = \frac{5}{2}$. Therefore, $\alpha = \frac{4}{5}$ if and only if $\beta = \frac{4}{5}$ and $\alpha > \frac{4}{5}$ if and only if $\beta < \frac{4}{5}$. □

4. The second desert

Figure 1 suggests the existence of two desert regions in which critical points cannot occur. Methods from [Frayer et al. 2014] quickly identify the desert region $\{z : z \in T_\alpha \text{ with } 0 < \alpha < \frac{1}{2}\}$. See Theorem 6. Determining the second desert, the white region enclosed by the "bean"-shaped curve in Figure 1, requires a significant amount of analysis.

To begin this analysis we investigate the relationship between the roots and nontrivial critical points of a polynomial in $\mathcal{P}$. Given $p(z) = (z-1)(z-r_1)(z-r_2)^2$ with a nontrivial critical point at c, we have

$$0 = q'(c) = 4c^2 - (3r_1 + 2r_2 + 3)c + r_1 r_2 + 2r_1 + r_2.$$

Direct calculations give

$$r_1 = \frac{(1-2c)r_2 + 4c^2 - 3c}{-r_2 + 3c - 2} \quad \text{and} \quad r_2 = \frac{(2-3c)r_1 + 4c^2 - 3c}{-r_1 + 2c - 1}.$$

Definition 8. Given $c \in \mathbb{C}$, define

$$f_{1,c}(z) = \frac{(1-2c)z + 4c^2 - 3c}{-z + 3c - 2} \quad \text{and} \quad f_{2,c}(z) = \frac{(2-3c)z + 4c^2 - 3c}{-z + 2c - 1}$$

and let $S_1 = f_{1,c}(T_2)$ and $S_2 = f_{2,c}(T_2)$.

Observe that $f_{1,c}$ and $f_{2,c}$ are fractional linear transformations with $f_{1,c}(r_2) = r_1$ and $f_{2,c}(r_1) = r_2$. We have established the following theorem.

Theorem 9. *The polynomial* $p(z) = (z-1)(z-r_1)(z-r_2)^2 \in \mathcal{P}$ *has a nontrivial critical point at* $c \neq 1$ *if and only if* $f_{1,c}(r_2) = r_1$ *and* $f_{2,c}(r_1) = r_2$.

When $c = 1$,

$$f_{1,c}(z) = f_{2,c}(z) = \frac{-z+1}{-z+1} = 1.$$

If $c \neq 1$, then $f_{1,c}$ and $f_{2,c}$ are one-to-one with $(f_{1,c})^{-1} = f_{2,c}$. Furthermore, $f_{1,c}(r_2) = r_1 \in T_2$, so that $r_1 \in S_1 \cap T_2$, and $f_{2,c}(r_1) = r_2 \in T_2$, so that $r_2 \in S_2 \cap T_2$. We can use these facts to classify the polynomials in $\mathcal{P}$ having a critical point at $c \neq 1$ in the closed unit disk. We will show that $|S_1 \cap T_2| = |S_2 \cap T_2|$ (Lemma 10) and that the cardinality of $S_1 \cap T_2$ is the number of polynomials in $\mathcal{P}$ having a nontrivial critical point at c (Lemma 11).

As fractional linear transformations map circles and lines to circles and lines, S_1 is a circle or line. Therefore, $S_1 = T_2$ or $|S_1 \cap T_2| \leq 2$. We will show that $S_1 \neq T_2$. If $S_1 = T_2$, then $f_{1,c}(T_2) = T_2$. Since

$$f_{1,c}(z) = \frac{(1-2c)z + 4c^2 - 3c}{-z + 3c - 2},$$

Theorem 2 implies that $\overline{1-2c} = 2 - 3c$ and $\overline{4c^2 - 3c} = 1$. The second equation implies $4c^2 - 3c = 1$ and it follows that

$$0 = 4c^2 - 3c - 1 = (4c+1)(c-1)$$

so that $c = -\frac{1}{4}$ or $c = 1$. However, $c = -\frac{1}{4}$ does not satisfy the equation $\overline{1-2c} = 2 - 3c$, and when $c = 1$, we know $f_{1,1}(z) = 1$ does not satisfy the hypothesis of Theorem 2. Therefore, $S_1 \neq T_2$. Likewise, as $(f_{1,c})^{-1} = f_{2,c}$, there is no c for which $S_2 = T_2$.

Lemma 10. *If* $c \neq 1$, *then* $|S_1 \cap T_2| = |S_2 \cap T_2| \in \{0, 1, 2\}$.

Proof. Without loss of generality, suppose $|S_1 \cap T_2| = 1$ and $S_2 \cap T_2 = \{a, b\}$ with $a \neq b$. By definition of S_2, there exist $a_0, b_0 \in T_2$ with $f_{2,c}(a_0) = a$, $f_{2,c}(b_0) = b$ and $a_0 \neq b_0$. Hence, $f_{1,c}(f_{2,c}(a_0)) = f_{1,c}(a)$ and $f_{1,c}(f_{2,c}(b_0)) = f_{1,c}(b)$, which implies

$$f_{1,c}(a) = a_0 \quad \text{and} \quad f_{1,c}(b) = b_0$$

so that $|S_1 \cap T_2| > 1$; a contradiction. Therefore, $|S_1 \cap T_2| = |S_2 \cap T_2|$. □

The following lemma characterizes the three possible cardinalities of $S_1 \cap T_2$.

Lemma 11. *Suppose* $c \neq 1$.

(1) *If* S_1 *and* T_2 *are disjoint, then no* $p \in \mathcal{P}$ *has a critical point at* c.

(2) *If* S_1 *and* T_2 *are tangent, then* c *is the nontrivial critical point of exactly one* $p \in \mathcal{P}$.

(3) *If* S_1 *and* T_2 *intersect in two distinct points, then* c *is the nontrivial critical point of exactly two polynomials in* $\mathcal{P}$.

Proof. Suppose $c \neq 1$. If $S_1 \cap T_2 = \varnothing$, then no point in $\mathbb{C}$ is eligible to be r_1 or r_2 and it follows that no $p \in \mathcal{P}$ has a critical point at c. If $S_1 \cap T_2 = \{a\}$, it follows from Lemma 10 that $S_2 \cap T_2 = \{b\}$. By the definitions of S_1 and S_2, there exist $a_0, b_0 \in T_2$ with $f_{1,c}(a_0) = a$ and $f_{2,c}(b_0) = b$. As $(f_{1,c})^{-1} = f_{2,c}$, we have

$$a_0 = f_{2,c}(a) \quad \text{and} \quad b_0 = f_{1,c}(b).$$

Therefore $a_0 = b$ and $b_0 = a$. By Theorem 9, c is a nontrivial critical point of $p(z) = (z-1)(z-a)(z-b)^2$. Furthermore, as $r_1 \in S_1 \cap T_2 = \{a\}$ and $r_2 \in S_2 \cap T_2 = \{b\}$, no other $p \in \mathcal{P}$ has a nontrivial critical point at c.

If $S_1 \cap T_2 = \{a, b\}$ with $a \neq b$, it follows from Lemma 10 that $S_2 \cap T_2 = \{d, e\}$ with $d \neq e$. By the definition of S_1, there exist $a_0, b_0 \in T_2$ with $f_{1,c}(a_0) = a$, $f_{1,c}(b_0) = b$ and $a_0 \neq b_0$. Hence, $a_0 = f_{2,c}(a)$ and $b_0 = f_{2,c}(b)$ and it follows that $\{a_0, b_0\} = \{d, e\}$. Therefore, $f_{2,c}(a) = a_0$ and $f_{1,c}(a_0) = a$. Theorem 9 implies that c is a nontrivial critical point of $p_1(z) = (z-1)(z-a)(z-a_0)^2$. Likewise, $f_{2,c}(b) = b_0$ and $f_{1,c}(b_0) = b$ implies that c is also a nontrivial critical point of $p_2(z) = (z-1)(z-b)(z-b_0)^2$. Moreover, as $r_1 \in S_1 \cap T_2 = \{a, b\}$, we have exhausted the potential candidates for r_1 and no other $p \in \mathcal{P}$ has a nontrivial critical point at c. When $|S_1 \cap T_2| = 2$, there are exactly two polynomials in $\mathcal{P}$ with a nontrivial critical point at c. □

In light of Lemmas 10 and 11, S_1 alone is sufficient to characterize the nontrivial critical points of polynomials in $\mathcal{P}$.

4.1. ***Analyzing S_1.*** To determine the boundary equation of the second desert region, we need to further explore S_1. Let $1 \neq c \in \mathbb{C}$. Since

$$f_{1,c}(z) = \frac{(1-2c)z + 4c^2 - 3c}{-z + 3c - 2}$$

is a fractional linear transformation, S_1 will be a line when there exists $z \in T_2$ with $-z + 3c - 2 = 0$. This occurs when

$$|3c-2| = |z| = 1 \iff \left|c - \tfrac{2}{3}\right| = \tfrac{1}{3}.$$

Therefore, S_1 is a line whenever $c \in T_{2/3}$. We now investigate an example for future reference.

Example 12. Let $c \in T_{2/3}$. Then, S_1 is a line passing through $f_{1,c}(1) = \frac{1}{3}(4c-1)$ and $f_{1,c}(-1) = (4c^2 - c - 1)/(3c-1)$. Moreover,

$$f_{1,c}(1) - f_{1,c}(-1) = \frac{4-4c}{9c-3}. \tag{3}$$

Substituting $c = \frac{2}{3} + \frac{1}{3}e^{i\theta}$ into (3) and simplifying yields $\mathrm{Re}(f_c(1) - f_c(-1)) = 0$. When $c \in T_{2/3}$, we have S_1 is a vertical line through $f_{1,c}(1) = \frac{1}{3}(4c-1) \in T_{8/9}$.

For $c \notin T_{2/3}$, we will determine the center and radius of S_1. By definition, $z \in S_1$ if and only if there exists a $w \in T_2$ with $f_{1,c}(w) = z$. That is, $w = (f_{1,c})^{-1}(z) = f_{2,c}(z) \in T_2$, which is true if and only if $|f_{2,c}(z)| = 1$. Equivalently,

$$\left|\frac{(2-3c)(z)+4c^2-3c}{-z+2c-1}\right| = 1.$$

Therefore, $z \in S_1$ if and only if

$$|z-(2c-1)| = |2-3c|\left|z - \frac{3c-4c^2}{2-3c}\right|. \tag{4}$$

For $k \neq 1$, the solution set of

$$|z-u| = k|z-v|$$

is a circle with center C and radius R satisfying

$$C = v + \frac{v-u}{k^2-1} \quad \text{and} \quad R^2 = |C|^2 - \frac{k^2|v|^2-|u|^2}{k^2-1}.$$

Observe that when $k = |2-3c| = 1$,

$$\left|\tfrac{2}{3}-c\right| = \tfrac{1}{3} \iff c \in T_{2/3}$$

and by Example 12, S_1 is a line. When $c \in T_\alpha$ with $\alpha \neq \frac{2}{3}$, we have $k = |2-3c| \neq 1$ and routine calculations establish the following lemma.

Lemma 13. *Suppose $c \neq 1$ and $c \in T_\alpha$ with $\alpha \neq \frac{2}{3}$. Then, S_1 is a circle with center γ and radius r given by*

$$\gamma = \frac{4c-1}{3} + \frac{2\alpha}{9\alpha-6} \quad \textit{and} \quad r = \frac{2\alpha}{3|3\alpha-2|}.$$

We now study a special case.

Example 14. Suppose $c \in T_2$ with $c \neq 1$. Direct calculations give

$$f_{1,c}(c) = c, \quad f_{1,c}(1) = \frac{4c-1}{3} \quad \text{and} \quad f_{1,c}(-1) = \frac{4c^2-c-1}{3c-1},$$

so that

$$\left|f_{1,c}(z) - \tfrac{4}{3}c\right| = \tfrac{1}{3}$$

for $z \in \{c, \pm 1\}$. Therefore, for $c \in T_2$ with $c \neq 1$, we have S_1 is a circle with radius $\frac{1}{3}$ and center $\frac{4}{3}c$, which is externally tangent to T_2 at c. See Figure 2.

When $1 \neq c \in T_2$, it follows from Example 5 that the other nontrivial critical point, $c_2 = \frac{3}{4} + \frac{1}{4}c \in T_{1/2}$, lies on the line segment $\overline{1c}$. Similar calculations show that for $c_2 = \frac{3}{4} + \frac{1}{4}c$, we have S_1 is a circle with radius $\frac{1}{3}$ and center $\frac{2}{3}c$, which is internally tangent to T_2 at c. See Figure 2.

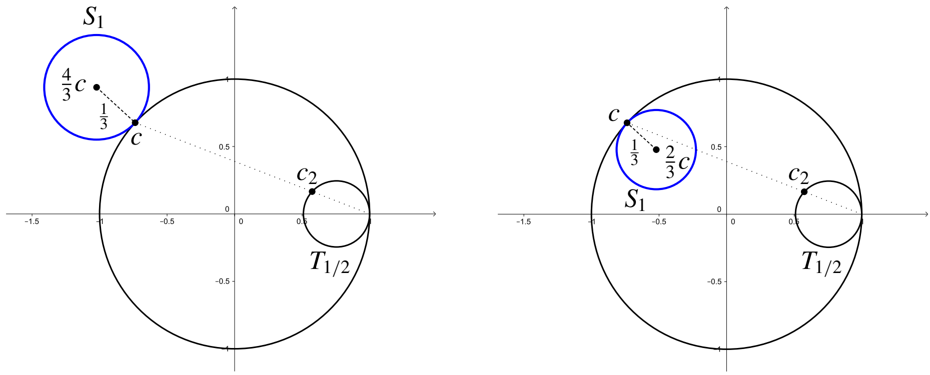

Figure 2. Left: for $c \in T_2$ with $c \neq 1$, the circle S_1 is externally tangent to T_2 at c. Right: for the corresponding nontrivial critical point, c_2, the circle S_1 is internally tangent to T_2 at c.

4.2. ***When is S_1 tangent to T_2?*** Let $1 \neq c \in \mathbb{C}$. When $S_1 \cap T_2 = \varnothing$, Lemma 11 implies that c is not the critical point of any $p \in \mathcal{P}$. To better understand this case, we will determine when $|S_1 \cap T_2| = 1$. That is, for what c in the unit disk will S_1 and T_2 be tangent? By Example 14, if $c \in T_{1/2}$, where $T_{1/2}$ is the boundary of the first desert region, then S_1 is internally tangent to T_2. Additionally, if $c \in T_\alpha$ with $\alpha < \frac{1}{2}$, it follows from Theorem 6 that S_1 and T_2 are disjoint.

For $1 \neq c \in T_\alpha$ with $\frac{1}{2} \leq \alpha \leq 2$, if S_1 is internally tangent to T_2, then

$$|\gamma| + r = 1. \tag{5}$$

See Figure 3. For $R = 2\alpha/(9\alpha - 6)$, the circle S_1 has center $\gamma = \frac{1}{3}(4c - 1) + R$ and radius $r = |R|$. Substituting into (5) and setting $c = x + iy$ gives

$$(4x - 1 + 3R)^2 + 16y^2 = 9(1 - |R|)^2. \tag{6}$$

Since R depends upon α, we denote (6) by D_α.

Since $r > 0$, (5) is satisfied if and only if S_1 is internally tangent to T_2 or $S_1 = T_2$. Recalling that there is no c for which $S_1 = T_2$, we obtain the following result.

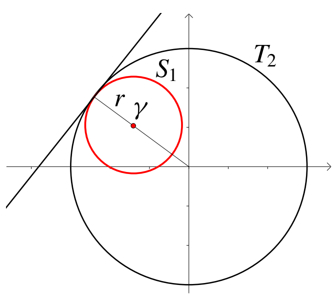

Figure 3. When $|\gamma| + r = 1$, the circle S_1 will be internally tangent to T_2.

Lemma 15. *Let $c \neq 1$ and $\frac{1}{2} \leq \alpha \leq 2$. Then, S_1 is internally tangent to T_2 if and only if $c \in T_\alpha \cap D_\alpha$.*

To apply Lemma 15 we need to determine when and where the circles T_α and D_α intersect, that is, the values of α for which $T_\alpha \cap D_\alpha \neq \varnothing$, and the corresponding points of intersection. Because of the $|R| = |2\alpha/(9\alpha - 6)|$ appearing in (6), we consider three cases:

(1) $\frac{1}{2} \leq \alpha < \frac{2}{3}$;

(2) $\alpha = \frac{2}{3}$;

(3) $\frac{2}{3} < \alpha \leq 2$.

In the first case, $|R| = -R$ and (6) becomes

$$\left(x - \left(1 - \frac{11\alpha - 6}{12\alpha - 8}\right)\right)^2 + y^2 = \left(\frac{11\alpha - 6}{12\alpha - 8}\right)^2.$$

For $\frac{1}{2} \leq \alpha < \frac{2}{3}$, circles T_α and D_α intersect if and only if $\alpha = \frac{1}{2}$. When $\alpha = \frac{1}{2}$, $T_{1/2} = D_{1/2}$ and by Lemma 15, when $c \in T_{1/2}$, we have S_1 is internally tangent to T_2. See Example 14.

In the second case, $\alpha = \frac{2}{3}$ and D_α is undefined. By Example 12, when $c \in T_{2/3}$, S_1 is a vertical line passing through $f_{1,c}(1) = \frac{1}{3}(4c - 1) \in T_{8/9}$ and S_1 is not tangent to T_2.

In the third case, $|R| = R$ and (6) becomes

$$\left(x - \left(-\frac{1}{2} + \frac{7\alpha - 6}{12\alpha - 8}\right)\right)^2 + y^2 = \left(\frac{7\alpha - 6}{12\alpha - 8}\right)^2.$$

For $\frac{2}{3} < \alpha \leq 2$, the circles T_α and D_α intersect if and only if $1 \leq \alpha \leq \frac{3}{2}$. To determine the values of c where S_1 is internally tangent to T_2, we need to find the intersection of the circles D_α and T_α. Upon simplification, these equations become

$$(4x - 1 + 3R)^2 + 16y^2 = 9(1 - R)^2,$$
$$\alpha(1 - x) - (1 - x)^2 = y^2.$$

By setting $R = 2\alpha/(9\alpha - 6)$ and using substitution, we eventually obtain

$$x = \frac{2(\alpha - 1)^2}{(2\alpha - 1)(\alpha - 2)} \quad \text{and} \quad y = \pm\frac{\alpha\sqrt{(3 - 2\alpha)(\alpha - 1)}}{(2\alpha - 1)(\alpha - 2)}. \tag{7}$$

As α varies from 1 to $\frac{3}{2}$, a parametric curve is formed. See Figure 4. For each value of c on the parametric curve, S_1 is internally tangent to T_2. Using resultant methods, see [Sederberg et al. 1984], the curve can be implicitized. Substituting $t = \alpha - 1$ into (7) implies $0 \leq t \leq \frac{1}{2}$ and

$$x = \frac{2t^2}{2t^2 - t - 1} \quad \text{and} \quad y^2 = \frac{-2t^4 - 3t^3 + t}{4t^4 - 4t^3 - 3t^2 + 2t + 1}, \tag{8}$$

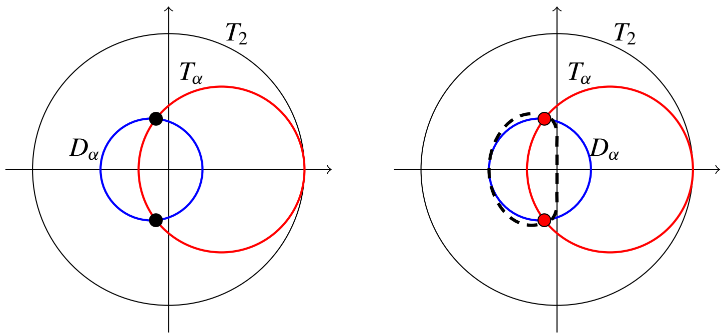

Figure 4. Left: when $1 \le \alpha \le \frac{3}{2}$, we have $|D_\alpha \cap T_\alpha| = 2$. Right: as α varies from 1 to $\frac{3}{2}$, parametric equations (7) trace the boundary of the second desert.

so that

$$f = (2x-2)t^2 + (-x)t + (-x) = 0,$$
$$g = (4y^2+2)t^4 + (-4y^2+3)t^3 + (-3y^2)t^2 + (2y^2-1)t + y^2 = 0.$$

The resultant of f and g with respect to t,

$$\operatorname{Res}(f, g; t) = \begin{vmatrix} 2x-2 & -x & -x & 0 & 0 & 0 \\ 0 & 2x-2 & -x & -x & 0 & 0 \\ 0 & 0 & 2x-2 & -x & -x & 0 \\ 0 & 0 & 0 & 2x-2 & -x & -x \\ 4y^2+2 & -4y^2+3 & -3y^2 & 2y^2-1 & y^2 & 0 \\ 0 & 4y^2+2 & -4y^2+3 & -3y^2 & 2y^2-1 & y^2 \end{vmatrix},$$

eliminates the variable t and is the implicit form of the curve. With the assistance of Mathematica, we find

$$\operatorname{Res}(f, g; t) = 2x^4 - 3x^3 + x + 4x^2y^2 - 3xy^2 + 2y^4$$

and the cartesian representation of (7) is

$$2x^4 - 3x^3 + x + 4x^2y^2 - 3xy^2 + 2y^4 = 0. \tag{9}$$

Equation (9) represents the boundary of the second desert region.

Theorem 16. *No polynomial in $\mathcal{P}$ has a critical point strictly inside $2x^4 - 3x^3 + x + 4x^2y^2 - 3xy^2 + 2y^4 = 0$.*

Proof. Let $c = x + iy \in T_\alpha$ with $\alpha \in \left[1, \frac{3}{2}\right]$. Then, c lies inside $2x^4 - 3x^3 + x + 4x^2y^2 - 3xy^2 + 2y^4 = 0$ whenever

$$(4x - 1 + 3R)^2 + 16y^2 < 9(1-R)^2 \quad \text{and} \quad x + iy \in T_\alpha.$$

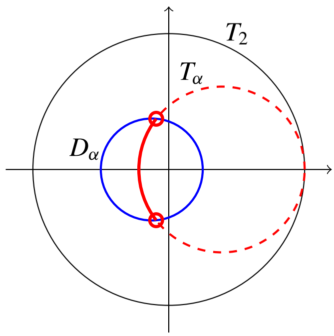

Figure 5. The bold semicircle lies strictly inside the circle $(4x-1+3R)^2+16y^2=9(1-R)^2$ and on T_α.

See Figure 5. Equivalently, (5) and (6) imply $|\gamma|+r<1$ and $c \in T_\alpha$. Therefore, S_1 and T_2 are disjoint. By Lemma 11, c is not the critical point of any $p \in \mathcal{P}$. □

The analysis of the circles D_α and T_α has established the following result.

Lemma 17. *The circle S_1 is internally tangent to T_2 if and only if $c=x+iy$ is on $T_{1/2}$ or $2x^4-3x^3+x+4x^2y^2-3xy^2+2y^4=0$.*

Furthermore, for $c \in T_\alpha$ with $\frac{1}{2} \le \alpha \le 2$, the circle S_1 will be externally tangent to T_2 if and only if $|\gamma|-r=1$. A similar, but less involved, analysis leads to the following result.

Lemma 18. *The circle S_1 is externally tangent to T_2 if and only if $c \in T_2$.*

5. Main result

We are now ready to characterize the critical points of a polynomial in $\mathcal{P}$. Let O represent the region strictly inside the closed unit disk and outside of $T_{1/2}$ and $2x^4-3x^3+x+4x^2y^2-3xy^2+2y^4=0$. That is, O is the gray shaded region in Figure 1. Denote the closure of O by $\overline{O}$.

Theorem 19. *Let $c \in \mathbb{C}$.*

(1) *The polynomial $p \in \mathcal{P}$ has a nontrivial critical point at $c=1$ if and only if $p(z)=(z-1)^2(z-r)^2$ or $p(z)=(z-1)^3(z-r)$ for some $r \in T_2$.*
(2) *If $c \notin \overline{O}$, there is no $p \in \mathcal{P}$ with a critical point at c.*
(3) *If $1 \neq c \in \overline{O}-O$, there is a unique $p \in \mathcal{P}$ with a nontrivial critical point at c.*
(4) *If $c \in O$, there are exactly two polynomials in $\mathcal{P}$ with a nontrivial critical point at c.*

Proof. A polynomial $p \in \mathcal{P}$ has a nontrivial critical point at $c=1$ if and only if p has a repeated root at 1, that is, $p(z)=(z-1)^2(z-r)^2$ or $p(z)=(z-1)^3(z-r)$ for some $r \in T_2$. See Example 4.

Let c lie strictly inside $T_{1/2}$, strictly inside $2x^4-3x^3+x+4x^2y^2-3xy^2+2y^4=0$, or strictly outside T_2. Then, it follows from Theorems 6, 16 and the Gauss–Lucas theorem respectively, that no $p \in \mathcal{P}$ has a critical point at c.

Let $c \neq 1$ lie on T_2, $T_{1/2}$, or $2x^4-3x^3+x+4x^2y^2-3xy^2+2y^4=0$. Lemmas 17 and 18 imply that S_1 is tangent to T_2. Therefore, by Lemma 11, there is exactly one $p \in \mathcal{P}$ with a nontrivial critical point at c.

Lastly, we need to show that for $c \in O$, we have $|S_1 \cap T_2| = 2$. This follows from a "root dragging" argument. Without loss of generality, suppose $S_1 \cap T_2 = \varnothing$ with S_1 contained inside of T_2. As we "drag" c to T_2 along a line segment contained in O, S_1 is continuously transformed into a circle externally tangent to T_2. By continuity, there exists a c_0 on the line segment with S_1 internally tangent to T_2. As c never crosses $T_{1/2}$ or $2x^4-3x^3+x+4x^2y^2-3xy^2+2y^4=0$, this contradicts Lemma 17. Therefore, $|S_1 \cap T_2| = 2$ and by Lemma 11 there are exactly two polynomials in $\mathcal{P}$ with a nontrivial critical point at c. □

This completes the characterization of critical points of polynomials in $\mathcal{P}$. Our results can be extended to polynomials of the form

$$p(z) = (z-1)^k (z-r_1)^m (z-r_2)^n$$

with $|r_1| = |r_2| = 1$ and $\{k, m, n\} \subseteq \mathbb{N}$. Similar to $\mathcal{P}$, when $m \neq n$, the unit disk contains two "desert" regions in which critical points cannot occur, and each c inside the unit disk and outside of the desert regions is the critical point of exactly two such polynomials. However, some questions remain unanswered. For example, if a polynomial has four or more distinct roots, all of which lie on the unit circle, how many desert regions will be in the unit disk?

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frayerc@uwplatt.edu *Department of Mathematics, University of Wisconsin, Platteville, WI, United States*

gauthierl@uwplatt.edu *University of Wisconsin, Platteville, WI, United States*

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