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Let G be a group. For any $\mathbb{Z}G$ -module M and any integer $d > 0$, we define a function $\text{FV}_M^{d+1} : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ generalizing the notion of $(d+1)$ -dimensional filling function of a group. We prove that this function takes only finite values if M is of type FP_{d+1} and $d > 0$, and remark that the asymptotic growth class of this function is an invariant of M . In the particular case that G is a group of type FP_{d+1} , our main result implies that its $(d+1)$ -dimensional homological filling function takes only finite values.

1. Introduction

For a contractible cellular complex X and an integer $d > 0$, the homological filling function $\text{FV}_X^{d+1} : \mathbb{N} \rightarrow \mathbb{N}$ measures the difficulty of filling cellular d -cycles with $(d+1)$ -chains; a precise definition is below. They are higher-dimensional homological generalizations of isoperimetric functions. For a group G admitting a compact classifying space $K(G, 1)$ with universal cover X , the equivalence growth rate of the function FV_X^{d+1} provides an invariant of the group. The initial motivation of this work was to provide a direct argument that FV_X^{d+1} takes only finite values for such complex X , addressing what the authors perceived as a gap in the literature. In this article we provide a self-contained proof based on the algebraic approach to define the homological filling functions from [Hanlon and Martínez-Pedroza 2016], and on our way, we prove a more general result that defines a new collection of invariants for $\mathbb{Z}G$ -modules.

The topological perspective. We assume all spaces are combinatorial complexes and all maps are combinatorial; see for example [Bridson and Haefliger 1999, Part I, Chapter 8, Appendix]. A G -action on a complex X is *proper* if for all compact subcomplexes K of X the collection $\{g \in G \mid K \cap g(K) \neq \emptyset\}$ is finite. The G -action is *cocompact* if there is a compact subcomplex K of X such that the collection $\{gK \mid g \in G\}$ covers X . For a complex X , the cellular d -dimensional chain group

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$C_d(X, \mathbb{Z})$ is a free \mathbb{Z} -module with a natural ℓ_1 -norm induced by a basis formed by the collection of all d -dimensional cells of X , each cell with a chosen orientation from each pair of opposite orientations. This norm, denoted by $\|\cdot\|_1$, is the sum of the absolute value of the coefficients in the unique representation of the chain as a linear combination of elements of the basis. Let $Z_d(X, \mathbb{Z})$ denote the \mathbb{Z} -module of integral d -cycles, and $\partial_{d+1} : C_{d+1}(X, \mathbb{Z}) \rightarrow Z_d(X, \mathbb{Z})$ be the boundary map. The $(d+1)$ -dimensional filling function of X is the function $FV_X^{d+1} : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ defined as

$$FV_X^{d+1}(k) = \sup\{\|\gamma\|_{\partial} \mid \gamma \in Z_d(X, \mathbb{Z}), \|\gamma\|_1 \leq k\},$$

where

$$\|\gamma\|_{\partial} = \inf\{\|\mu\|_1 \mid \mu \in C_{d+1}(X, \mathbb{Z}), \partial(\mu) = \gamma\},$$

where the supremum and infimum of the empty set are defined as zero and ∞ respectively. In words, $FV_X^{d+1}(k)$ is the most efficient upper bound on the size of fillings by $(d+1)$ -chains of d -cycles of norm at most k . A complex X is d -acyclic if the reduced homology groups $\bar{H}_i(X, \mathbb{Z})$ are trivial for $0 \leq i \leq d$. As mentioned above, the initial motivation of this work was to provide a proof of Theorem 1.1, which the authors perceived as a gap in the literature. The main contribution of this note is a generalization to an algebraic framework of the following statement; see Theorem 1.3.

Theorem 1.1. *Let d be a positive integer and let G be a group acting properly and cocompactly by cellular automorphisms on a d -acyclic complex X . Then $FV_X^{d+1}(m)$ is finite for all $m \in \mathbb{N}$.*

Theorem 1.1 was known to hold in the following cases:

- For $d = 1$, it is a result of [Gersten 1999, Proposition 2.4].
- For $d \geq 1$ and under the extra assumption that G admits a combing, it follows from [Epstein et al. 1992, Theorem 10.3.6]; see also [Behrstock and Druţu 2015, Lemma 3.7].
- For $d \geq 3$, Hanlon and the second author observed in [Hanlon and Martínez-Pedroza 2016, Section 3.3] that Theorem 1.1 holds using results of Alonso, Pride and Wang [Alonso et al. 1999] in conjunction with an argument from Abrams, Brady, Dani and Young [Abrams et al. 2013]. The results in [Alonso et al. 1999] rely on nontrivial machinery from homotopy theory. The failure of the argument for $d = 2$ relies on an application of the Hurewicz theorem; for details see [Hanlon and Martínez-Pedroza 2016, Section 3.3].

Current results in the literature leave open the statement of Theorem 1.1 for the case $d = 2$. Our argument in this note proving Theorem 1.1 does not rely on previous results, it is valid for all $d > 0$, and it is elementary. The argument might be known to the experts, but to our knowledge does not appear in the literature, and

this note fills this gap. Let us sketch the argument from a topological perspective; for an algebraic proof see Section 2.

Sketch of the proof of Theorem 1.1, from a topological perspective. Consider the combinatorial path metric on the 1-skeleton of X , and for any d -cycle σ (which is a formal finite sum of d -cells) define its diameter $\text{diam}(\sigma)$ as the diameter of the set consisting of vertices (0-cells) which are in the closure of at least one d -cell defining σ . A d -cycle σ is called *connected* if the subcomplex of X formed by taking the closure of the union of d -cells defining σ is connected (and has no cut-points).

Let $m > 0$. Since G acts properly and cocompactly on X , there is an integer $C \geq 0$ that bounds the diameter of any d -cell of X , and hence for any connected d -cycle σ ,

$$\text{diam}(\sigma) \leq C \|\sigma\|_1.$$

From here, it follows that the induced G -action on the set of connected d -cycles with ℓ_1 -norm $\leq m$ has finitely many G -orbits. Since X is d -acyclic, $\|\sigma\|_\partial < \infty$ for each d -cycle σ . Therefore, there is an integer $M = M(m)$ such that

$$\sigma \text{ is connected and } \|\sigma\|_1 \leq m \implies \|\sigma\|_\partial \leq M.$$

Let σ be an arbitrary d -cycle with ℓ_1 -norm $\leq m$. Then one shows that σ can be decomposed as a sum of connected d -cycles $\sum_{i=1}^k \sigma_i$, where $k \leq \|\sigma\|_1 = \sum_{i=1}^k \|\sigma_i\|_1$. Hence

$$\|\sigma\|_\partial \leq \sum_{i=1}^k \|\sigma_i\|_\partial \leq k \cdot M \leq m \cdot M.$$

Therefore $\text{FV}_X^{d+1}(m) \leq m \cdot M < \infty$. □

Remark 1.2. Under the assumptions of Theorem 1.1, it is known that the growth rate of the function FV_X^{d+1} is a quasi-isometry invariant of the group G . This was first addressed by Fletcher [1998, Theorem 2.1] under the assumption that X is the universal cover of $K(G, 1)$. Young [2011, Lemma 1] provided a proof of the quasi-isometry invariance in the general context of Theorem 1.1. Notably, these works do not address that these functions are finite.

The algebraic perspective, and our main result. Our main result is an algebraic analog of Theorem 1.1. Recall that for a group G , a $\mathbb{Z}G$ -module M is of type FP_n if there exists a partial resolution of $\mathbb{Z}G$ -modules

$$P_n \xrightarrow{\varphi_n} P_{n-1} \xrightarrow{\varphi_{n-1}} \dots \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \rightarrow M \rightarrow 0$$

such that each P_i is a finitely generated projective $\mathbb{Z}G$ -module. For a $\mathbb{Z}G$ -module M of type FP_{d+1} we define the $(d+1)$ -filling function FV_M^{d+1} of M , see Definition 2.5, and prove the following result.

Recall that the *growth rate class* of a function $\mathbb{N} \rightarrow \mathbb{N}$ is defined as follows. Given two functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$, define the relation $f \preceq g$ if there is $C > 0$ such that $f(n) \leq Cg(Cn + C) + Cn + C$ for all $n \in \mathbb{N}$, and let $f \sim g$ if both $f \preceq g$ and $g \preceq f$. This yields an equivalence relation where the equivalence class of a function f is called the *growth rate class of f* .

Theorem 1.3. *Let M be a $\mathbb{Z}G$ -module of type FP_{d+1} :*

- (1) *For all positive integers d and k , we have $FV_M^{d+1}(k) < \infty$.*
- (2) *The growth rate of the function $FV_M^{d+1} : \mathbb{N} \rightarrow \mathbb{N}$ only depends on M .*

This result provides a new collection of invariants for $\mathbb{Z}G$ -modules that remains to be studied. The invariant is interesting even in the case that $M = \mathbb{Z}$ and G is suitable. In this case, the filling functions $FV_{\mathbb{Z}}^{d+1}$ correspond to the filling invariants of the group G , usually denoted by FV_G^{d+1} , in the context of Theorem 1.1 and Remark 1.2. There are computations by Young [2016] in the case that G corresponds to a discrete Heisenberg group answering a conjecture of Gromov [1993, Chapter 5], estimations in the case that G is the special linear group $SL(n, \mathbb{Z})$ by Epstein and Thurston [1992, Chapter 10], and general results in the case that G is a hyperbolic group by Gersten [1996] and Mineyev [2000], among others. In [Hanlon and Martínez-Pedroza 2016, Remark 3.4], it was observed that there was no proof in the literature that if G is of type FP_3 (i.e., \mathbb{Z} is of type FP_3 as a module over $\mathbb{Z}G$) then FV_G^3 is finite-valued; observe that this is a consequence of Theorem 1.3.

This note contains a proof of the first statement of Theorem 1.3. The proof of the second statement appears in [Hanlon and Martínez-Pedroza 2016, Theorem 3.5] for the case that $M = \mathbb{Z}$, but the argument works verbatim for the general case.

Organization. The rest of the paper is organized as follows: Section 2 contains some preliminary definitions including the definition of FV_M^{d+1} , the statement of the main technical result of the article, Proposition 2.4, and arguments implying Theorems 1.1 and 1.3. Section 3 is devoted to the proof of Proposition 2.4. Section 4 discusses some geometric examples illustrating some matters about Theorem 1.1.

2. Main technical result and proofs of the main theorems

Let G be a group and let S be a G -set. The set of all orbits of S under the G -action is denoted by S/G . The free abelian group $\mathbb{Z}[S]$ with S as a free generating set can be made into a $\mathbb{Z}G$ -module that we shall call the permutation module on S . The \mathbb{Z} -basis S induces a G -equivariant norm, called the ℓ_1 -norm, given by $\|\sum_{s \in S} n_s s\|_S = \sum_{s \in S} |n_s|$, where $n_s \in \mathbb{Z}$.

If the G -action on S is free, then $\mathbb{Z}[S]$ is a free module over $\mathbb{Z}S$. Conversely, if F is a free $\mathbb{Z}G$ -module with a chosen $\mathbb{Z}G$ -basis $\{\alpha_i \mid i \in I\}$, then F is isomorphic

to the permutation module $\mathbb{Z}[S]$, where $S = \{g\alpha_i \mid g \in G, i \in I\}$ with the natural G -action. In this case the $\mathbb{Z}G$ -basis $\{\alpha_i \mid i \in I\}$ of F induces an ℓ_1 -norm as before.

Definition 2.1 (Gersten’s filling norms). Let $\eta : F \rightarrow M$ be a surjective morphism of $\mathbb{Z}G$ -modules where F is finitely generated and free with a chosen finite $\mathbb{Z}G$ -basis, and the induced *filling norm* on M is defined by

$$\|m\|_\eta = \min\{\|x\|_F \mid x \in F, \eta(x) = m\},$$

where $\|\cdot\|_F$ denotes the induced ℓ_1 -norm on F .

Remark 2.2 (induced ℓ_1 -norms are filling norms). Let $\mathbb{Z}[S]$ be a permutation $\mathbb{Z}G$ -module such that G acts freely on S and the quotient S/G is finite. Then $\mathbb{Z}[S]$ is a finitely generated free $\mathbb{Z}G$ -module and the ℓ_1 -norm $\|\cdot\|_S$ is a filling norm. This statement holds without the assumption that G acts freely on S . Since we do not use this fact, we leave its verification to the reader.

Definition 2.3. Let $\rho : \mathbb{Z}[S] \rightarrow \mathbb{Z}[T]$ be a morphism of permutation $\mathbb{Z}G$ -modules such that the kernel $K = \ker \rho$ is finitely generated. Let $\|\cdot\|_K$ denote a filling norm on K and let $\|\cdot\|_S$ denote the ℓ_1 -norm on $\mathbb{Z}[S]$ induced by S . Define the function $FV_\rho : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ as

$$FV_\rho(n) = \sup\{\|x\|_K \mid x \in K, \|x\|_S \leq n\}.$$

Proposition 2.4. *Let $\rho : \mathbb{Z}[S] \rightarrow \mathbb{Z}[T]$ be a morphism. Suppose that S/G and T/G are finite, T has finite G -stabilizers for all $t \in T$, and $\ker \rho$ is finitely generated. Then $FV_\rho(n) < \infty$ for all $n \in \mathbb{N}$.*

In the rest of this section, we deduce Theorems 1.1 and 1.3 from Proposition 2.4.

Proof of Theorem 1.1. Let G be a group acting properly and compactly by cellular automorphisms on a d -connected complex X . The free abelian groups $C_d(X)$ and $C_{d+1}(X)$ are permutation $\mathbb{Z}G$ -modules over the G -sets of d -cells and $(d + 1)$ -cells of X , respectively. Observe that the definition of FV_X^{d+1} coincides with Definition 2.3 of FV_{∂_d} for the boundary map $C_d(X) \xrightarrow{\partial_d} C_{d-1}(X)$. The proof concludes by verifying the hypothesis of Proposition 2.4 for this morphism.

Since the G -action on X is cocompact, there are finitely many G -orbits of d -cells and $(d+1)$ -cells; in particular $C_{d+1}(X)$ is a finitely generated $\mathbb{Z}G$ -module. Since X is d -acyclic, the sequence

$$C_{d+1}(X) \xrightarrow{\partial_{d+1}} C_d(X) \xrightarrow{\partial_{d+1}} C_{d-1}(X)$$

is exact and hence $\ker(\partial_d)$ is a finitely generated $\mathbb{Z}G$ -module. Since the G -action is proper, the stabilizer of each d -cell of X is finite. □

Definition 2.5. Let M be a $\mathbb{Z}G$ -module of type FP_{d+1} . The $(d+1)$ -filling function of M is the function

$$FV_M^{d+1} : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$$

defined as follows. Let

$$P_{d+1} \xrightarrow{\varphi_{d+1}} P_d \xrightarrow{\varphi_d} \dots \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \rightarrow M \rightarrow 0$$

be a FP_{d+1} -resolution for M . Chose filling norms on P_{d+1} and P_d denoted by $\|\cdot\|_{P_{d+1}}$ and $\|\cdot\|_{P_d}$ respectively. Then

$$FV_M^{d+1}(k) = \sup\{\|x\|_{\varphi_{d+1}} \mid x \in \ker \varphi_d, \|x\|_{P_d} \leq k\},$$

where

$$\|x\|_{\varphi_{d+1}} = \min\{\|y\|_{P_{d+1}} \mid y \in P_{d+1}, \varphi_{d+1}(y) = x\}.$$

The proof of Theorem 1.3 uses the following lemma.

Lemma 2.6 [Brown 1982, Chapter VIII, Proposition 4.3]. *A $\mathbb{Z}G$ -module M is of type FP_d if and only if M admits a partial resolution of free finitely generated $\mathbb{Z}G$ -modules of the form*

$$F_{d+1} \rightarrow F_d \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0.$$

Proof of Theorem 1.3. Since M is of type FP_{d+1} , by Lemma 2.6, there exists a partial resolution of free and finitely generated $\mathbb{Z}G$ -modules

$$F_{d+1} \xrightarrow{\varphi_{d+1}} F_d \xrightarrow{\varphi_d} \dots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \rightarrow M \rightarrow 0$$

such that $\ker \varphi_n$ is finitely generated for n such that $d \geq n \geq 0$. Consider the finitely generated free modules F_d and F_{d-1} as permutation modules $\mathbb{Z}[S]$ and $\mathbb{Z}[T]$ respectively. Finite generation and freeness implies that we can assume that G acts freely and with finitely many orbits on both S and T . Since the induced ℓ_1 -norms on $\mathbb{Z}[S]$ and $\mathbb{Z}[T]$ are in particular filling norms, the definition of FV_M^{d+1} coincides with Definition 2.3 of $FV_{\varphi_{d+1}}$. Then the first statement of the theorem on the finiteness of FV_M^{d+1} follows by applying Proposition 2.4 to $FV_{\varphi_{d+1}}$.

The proof of the second statement that the growth rate of FV_M^{d+1} is independent of the choice of partial resolution and filling norms appears in [Hanlon and Martínez-Pedroza 2016, Theorem 3.5] for the case that $M = \mathbb{Z}$ and G is a group of type FP_{d+1} . The argument for arbitrary M follows verbatim by replacing each occurrence of \mathbb{Z} by M . Let us remark that the heart of the argument is the fact that any two projective resolutions of M are chain homotopy equivalent [Brown 1982, p. 24, Theorem 7]. □

3. Finiteness

This section contains the proof Proposition 2.4. Let S and T be G -sets. For $x \in \mathbb{Z}[S]$ with $x = \sum_{s \in S} n_s s$, we denote by $\langle x, s \rangle$ the integer n_s . For $x \in \mathbb{Z}[T]$ and $t \in T$ we define analogously $\langle x, t \rangle$.

Definition 3.1 (x is a part of y). Let $x, y \in \mathbb{Z}[S]$. We say x is a part of y , denoted by $x \leq_S y$, to mean that for each $s \in S$ if $\langle x, s \rangle > 0$ then $\langle x, s \rangle \leq \langle y, s \rangle$, and if $\langle x, s \rangle < 0$ then $\langle y, s \rangle \leq \langle x, s \rangle$. Note that this is equivalent to $\langle x, s \rangle \langle y, s \rangle \geq \langle x, s \rangle^2$ for all $s \in S$.

Definition 3.2 (S -intersect). For $x, y \in \mathbb{Z}[S]$, the S -intersection of x and y is defined as $x \cap_S y = \{s \in S \mid \langle x, s \rangle \langle y, s \rangle < 0\}$.

Remark 3.3. Let $x, y \in \mathbb{Z}[S]$. Then $\|x + y\|_S = \|x\|_S + \|y\|_S$ if and only if $x \cap_S y = \emptyset$. Indeed,

$$\|x + y\|_S = \sum_{s \in S} |\langle x, s \rangle + \langle y, s \rangle| \leq \sum_{s \in S} |\langle x, s \rangle| + \sum_{s \in S} |\langle y, s \rangle| = \|x\|_S + \|y\|_S$$

with equality if and only if $\langle x, s \rangle$ and $\langle y, s \rangle$ have the same sign for all $s \in S$.

Throughout the rest of this section, let

$$\mathcal{D}_1 = S \cup \{-s \mid s \in S\}.$$

Furthermore, let $\rho : \mathbb{Z}[S] \rightarrow \mathbb{Z}[T]$ denote a morphism of $\mathbb{Z}G$ -modules.

Definition 3.4 (ρ -intersect). A pair of elements $x, y \in \mathbb{Z}[S]$ have *nontrivial ρ -intersection*, denoted by $x \cap_\rho y \neq \emptyset$, if there exists $x_1, y_1 \in \mathcal{D}_1$ such that $\rho(x_1) \cap_T \rho(y_1) \neq \emptyset$ where $x_1 \leq_S x$ and $y_1 \leq_S y$.

Definition 3.5 (ρ -connected). For each integer $n \geq 1$, let \mathcal{D}_n be the collection of elements of $\mathbb{Z}[S]$ of the form $x = \sum_{i=1}^n x_i$, where each $x_i \in \mathcal{D}_1$ and for every $k < n$ the elements $\sum_{i=1}^k x_i$ and x_{k+1} have trivial S -intersection and nontrivial ρ -intersection. An element $x \in \mathbb{Z}[S]$ is *ρ -connected* if $x \in \mathcal{D}_n$ for some $n \geq 1$.

Remark 3.6. For $x \in \mathbb{Z}[S]$, we have $x \in \mathcal{D}_n$ if and only if x is ρ -connected and $\|x\| = n$.

Lemma 3.7. *If $0 \neq z \in \ker \rho$, then there exists x such that*

- (1) $x \leq_S z$, in particular, $\|z - x\|_S < \|z\|_S$,
- (2) $x \in \ker \rho$, and
- (3) x is ρ -connected.

Proof. Let $0 \neq z \in \ker \rho$ be an arbitrary element. Consider the set

$$\Omega = \{x \leq_S z \mid x \neq 0, x \text{ is } \rho\text{-connected}\};$$

this is a nonempty finite set partially ordered by \preceq_S . Let $x \in \Omega$ be a maximal element. We claim that $x \in \ker \rho$. Suppose that $x \notin \ker \rho$. We have $\rho(x)$ and $\rho(z - x)$ are nonzero and satisfy

$$\rho(x) + \rho(z - x) = 0.$$

Since $\rho(x) \neq 0$ there exists $t \in T$ such that $\langle \rho(x), t \rangle \neq 0$. Therefore

$$\langle \rho(z - x), t \rangle = -\langle \rho(x), t \rangle.$$

Since $\rho(z - x) \neq 0$, there exists $s \in S$ for which

$$\langle z - x, s \rangle \langle \rho(s), t \rangle \langle \rho(z - x), t \rangle > 0.$$

This implies

$$\langle z - x, s \rangle \langle \rho(s), t \rangle \langle \rho(x), t \rangle < 0.$$

Now define $\lambda = \langle z - x, s \rangle / |\langle z - x, s \rangle|$. We show $x + \lambda s$ is ρ -connected. First observe that $x \cap_S \lambda s = \emptyset$ since $x \preceq_S z$ and $\lambda s \preceq_S z$. Moreover, note that $x \cap_\rho \lambda s \neq \emptyset$ since

$$\langle \rho(x), t \rangle \langle \rho(\lambda s), t \rangle = \langle \rho(x), t \rangle \langle \rho(s), t \rangle \lambda < 0.$$

Therefore $x + \lambda s$ is ρ -connected and $x \not\preceq_S x + \lambda s \preceq_S z$. This contradicts the maximality of x and therefore $x \in \ker \rho$. \square

Proposition 3.8. *For all nonzero $z \in \ker \rho$, there exist ρ -connected elements $x_1, \dots, x_n \in \ker \rho$ such that*

- (1) $z = x_1 + \dots + x_n$,
- (2) $x_i \preceq_S z$ for each i .

Proof. Applying Lemma 3.7 to $z \in \ker \rho$, there exists a ρ -connected element $x_1 \in \ker \rho$ such that $x_1 \preceq_S z$. If $z - x_1 \neq 0$ then there exists a ρ -connected element $x_2 \in \ker \rho$ such that $x_2 \preceq_S z - x_1 \prec_S z$. If $z - x_1 - x_2 \neq 0$ then there exists a ρ -connected element $x_3 \in \ker \rho$ such that $x_3 \preceq_S z - x_1 - x_2 \prec z - x_1 \prec z$. This process must terminate for some positive integer n since

$$\|z - x_1 - \dots - x_k\| > \|z - x_1 - \dots - x_k - x_{k+1}\| \geq 0$$

if $z - x_1 - \dots - x_k \neq 0$. Hence we obtain ρ -connected elements $x_1, \dots, x_n \in \ker \rho$ such that $x_i \preceq_S z$ for each i , and $z = x_1 + \dots + x_n$. \square

Remark 3.9. For $x, y \in \mathbb{Z}[S]$, the relations $x \preceq_S y$, $x \cap_S y \neq \emptyset$, and $x \cap_\rho y \neq \emptyset$ are preserved by the G -action on $\mathbb{Z}[S]$. Thus, if $x \in \mathcal{D}_n$ and $g \in G$ then $gx \in \mathcal{D}_n$. It follows that \mathcal{D}_n is a G -set.

Proposition 3.10. *Suppose that S and T have finitely many G -orbits and each element of T has finite G -stabilizer. Then for every $n \geq 1$, the set \mathcal{D}_n has finitely many G -orbits.*

Before the proof of the Proposition 3.10, we introduce the following lemmas.

Lemma 3.11. *Suppose S has finitely many G -orbits, and each element of T has finite G -stabilizer. Then for every $t \in T$, the set $S(t) = \{s \in S \mid \langle \rho(s), t \rangle \neq 0\}$ is finite.*

Proof. For any $t \in T$, $s \in S$, and $g \in G$, we have $\langle \rho(gs), gt \rangle = \langle \rho(s), t \rangle$. For each $s \in S$, let $T(s) = \{t \in T \mid \langle \rho(s), t \rangle \neq 0\}$. As ρ is a morphism, $T(s)$ is a finite set for all $s \in S$. Now, fix $t \in T$ and let s_1, \dots, s_m be representatives of G -orbits of S . Then

$$S(t) = \bigcup_{i=1}^m \{gs_i \mid g \in G, \langle \rho(s_i), g^{-1}t \rangle \neq 0\} = \bigcup_{i=1}^m \bigcup_{r \in T(s_i)} \{gs_i \mid g \in G, g^{-1}t = r\}.$$

Observe that the set $\{g \in G \mid g^{-1}t = r\}$ is in one-to-one correspondence with $G_t = \{g \in G \mid gt = t\}$. By assumption, G_t is finite and thus for each $i \in \{1, \dots, m\}$ and $r \in T(s_i)$ the set $\{gs_i \mid g \in G, g^{-1}t = r\}$ is finite. Therefore, the set $S(t)$ is finite. □

Lemma 3.12. *Suppose S has finitely many G -orbits and that T has finite G -stabilizers for each $t \in T$. Then for all $n \in \mathbb{Z}_+$ and for all $y \in \mathcal{D}_n$ the set $\{x \in \mathcal{D}_1 \mid x \cap_\rho y \neq \emptyset\}$ is finite.*

Proof. For $y \in \mathbb{Z}[S]$ denote by $\mathcal{D}_1(y)$ the set $\{x \in \mathcal{D}_1 \mid x \cap_\rho y \neq \emptyset\}$. Let $y \in \mathcal{D}_n$. By definition, $y = \sum_{i=1}^n x_i$, where each $x_i \in \mathcal{D}_1$ and for each $k < n$, the elements $\sum_{i=1}^k x_i$ and x_{k+1} have trivial S -intersection and nontrivial ρ -intersection. It follows from the definition of ρ -intersect that

$$\mathcal{D}_1(y) = \{x \in \mathcal{D}_1 \mid x \cap_\rho y \neq \emptyset\} = \bigcup_{i=1}^n \{x \in \mathcal{D}_1 \mid x \cap_\rho x_i \neq \emptyset\} = \bigcup_{i=1}^n \mathcal{D}_1(x_i).$$

Therefore, to conclude it is enough to show that $\mathcal{D}_1(s)$ is finite for every $s \in \mathcal{D}_1$.

Let $s \in \mathcal{D}_1$. Observe that

$$\mathcal{D}_1(s) = \bigcup_{t \in T} \{x \in \mathcal{D}_1 \mid \langle \rho(x), t \rangle \langle \rho(s), t \rangle < 0\} \subset \bigcup_{t \in T} \{x \in \mathcal{D}_1 \mid \langle \rho(x), t \rangle \langle \rho(s), t \rangle \neq 0\}.$$

It is immediate that $\{t \in T \mid \langle \rho(s), t \rangle \neq 0\}$ is finite. Hence the union on the right is over a collection with finitely many nonempty sets. By Lemma 3.11, for any $t \in T$ the set $\{x \in \mathcal{D}_1 \mid \langle \rho(x), t \rangle \neq 0\}$ is finite, and hence $\{x \in \mathcal{D}_1 \mid \langle \rho(x), t \rangle \langle \rho(s), t \rangle \neq 0\}$ is finite. Therefore the expression on the right is the union of a finite collection of finite sets, and we conclude that $\mathcal{D}_1(s)$ is finite. □

Proof of Proposition 3.10. We prove by induction on n . For $n = 1$ the result follows from the assumption that S has finitely many G -orbits and the definition of \mathcal{D}_1 .

Suppose \mathcal{D}_n has finitely many G -orbits with representatives y_1, \dots, y_ℓ . For each $1 \leq k \leq \ell$, let A_k be the collection of elements A_k of \mathcal{D}_1 such that

$$y_k \cap_S z = \emptyset \quad \text{and} \quad y_k \cap_\rho z \neq \emptyset.$$

By Lemma 3.12, the collection A_k is finite. The proof concludes with the verification of the following claim.

Claim. *The set*

$$\{y_k + z \mid 1 \leq k \leq \ell \text{ and } z \in A_k\}$$

is a collection of representatives of G -orbits of \mathcal{D}_{n+1} .

Let $x \in \mathcal{D}_{n+1}$. Then $x = \sum_{i=1}^{n+1} x_i$, where each $x_i \in \mathcal{D}_1$ and for every $k < n$ the elements $\sum_{i=1}^k x_i$ and x_{k+1} have trivial S -intersection and nontrivial ρ -intersection. By definition, $\sum_{i=1}^n x_i$ is in \mathcal{D}_n . Hence $\sum_{i=1}^n x_i = gy_j$ for some $g \in G$ and some $1 \leq j \leq \ell$. It follows that $x = gy_j + x_{n+1}$ and therefore $g^{-1}x = y_j + g^{-1}x_{n+1}$. By Remark 3.9, we have that $z = g^{-1}x_{n+1}$ is an element of A_j . Therefore $x = gy_j + gz = g(y_j + z)$. This proves the claim. \square

Proof of Proposition 2.4. Let K denote $\ker \rho$, and let $\|\cdot\|_K$ denote a chosen filling norm on K . By Proposition 3.10, for each positive integer n , the G -set $\bigcup_{i=1}^n \mathcal{D}_i$ has finitely many G -orbits. Therefore, for each $n \in \mathbb{Z}_+$ there is an integer B_n such that for every $x \in \bigcup_{i=1}^n \mathcal{D}_i$, we have $\|x\|_K \leq B_n$.

Let $0 \neq z \in K$ such that $\|z\|_S \leq n$. By Proposition 3.8, there exist ρ -connected elements $x_1, \dots, x_m \in K$ such that $m \leq n$, $z = x_1 + \dots + x_m$, and $x_i < z$, $i = 1, \dots, m$. By Remark 3.6, each $x_i \in \mathcal{D}_n$. Therefore, by the triangle inequality,

$$\|z\|_K \leq \sum_{i=1}^m \|x_i\|_K \leq m \cdot B_n \leq n \cdot B_n.$$

This shows that $FV_\rho(n) \leq n \cdot B_n < \infty$. \square

Remark 3.13. Observe that Proposition 2.4 can be generalized as follows. Consider the sequence of modules $\ker \rho \rightarrow \mathbb{Z}[S] \xrightarrow{\rho} \mathbb{Z}[T]$, where $|S/G|, |T/G| < \infty$ and T has finite G -stabilizers for all $t \in T$. Let $\|\cdot\|_K$ be a G -invariant norm on K ; then for all $n \in \mathbb{N}$,

$$\sup\{\|x\|_K \mid x \in K, \|x\|_S \leq n\} < \infty.$$

In particular, K being finitely generated induces a filling norm which is G -invariant.

4. Examples

A graph Γ is called *fine* if for every edge e and each integer $n > 0$, the number of circuits of length at most n which contain e is finite. By a circuit we mean a

closed edge path that does not pass through a vertex more than once. The length of a circuit is defined as the number of edges.

Theorem 4.1 [Martínez-Pedroza 2016, Theorem 1.3]. *Let X be a cocompact G -cell complex with finite stabilizers of 1-cells. The following two statements are equivalent:*

- (1) X has fine 1-skeleton and the homology group $H_1(X, \mathbb{Z})$ is trivial.
- (2) $FV_X(k) < \infty$ for any integer k .

This result allows us to exhibit examples that contrast with Theorem 1.1 as follows:

- There is a group G acting cocompactly, not properly, and by cellular automorphisms on a simply connected complex X for which $FV_X^2(m)$ is finite for all $m \in \mathbb{N}$.

In particular, the converse of Theorem 1.1 does not hold.

- There is a group G acting cocompactly by cellular automorphisms on a simply connected complex X for which $FV_X^2(m)$ is infinite for some $m \in \mathbb{N}$. In particular, the properness assumption in Theorem 1.1 cannot be removed.

The two examples are based on the notion of coned-off Cayley complex. We use the version from [Groves and Manning 2008], which we briefly recall below; for another version see [Martínez-Pedroza 2017, Section 3].

Let G be a group and let P be a subgroup. The group G is *finitely generated relative to P* if there is a finite subset $S \subset G$ such that the natural map $F(S) * P \rightarrow G$ is surjective, where $F(S)$ denotes the free group on S , and $F(S) * P$ denotes the free product of $F(S)$ and P . In this case S is called a *finite relative generating set of G with respect to P* .

Suppose that S is a finite relative generating set of G with respect to P . Without loss of generality assume that S is closed under inverses. The *coned-off Cayley graph* $\widehat{\Gamma} = \widehat{\Gamma}(G, P, S)$ is the graph with vertex set consisting of all elements of G and all left cosets of P ; the edge set is the collection of pairs $(g, gs) \in G \times G$ for $g \in G$ and $s \in S$, and pairs (g, gP) for $g \in G$. Observe that the left action of G on itself extends to a left action on $\widehat{\Gamma}$. Vertices of $\widehat{\Gamma}$ of the form gP are called *cone-vertices*. Observe that the G -stabilizers of cone-vertices correspond to conjugates of P ; in particular, if P is infinite, the action is not proper. Moreover, the G -stabilizers of 1-cells of $\widehat{\Gamma}$ are trivial. It is well known that the assumption that S is a relative generating set implies that $\widehat{\Gamma}$ is path-connected as a combinatorial complex; in fact, this is an equivalence, as remarked in [Hruska 2010].

Under the assumptions, the group G is *finitely presented relative to P* if there is a finite subset $R \subset F(S) * P$ such that the kernel of the map $F(S) * P \rightarrow G$ is the

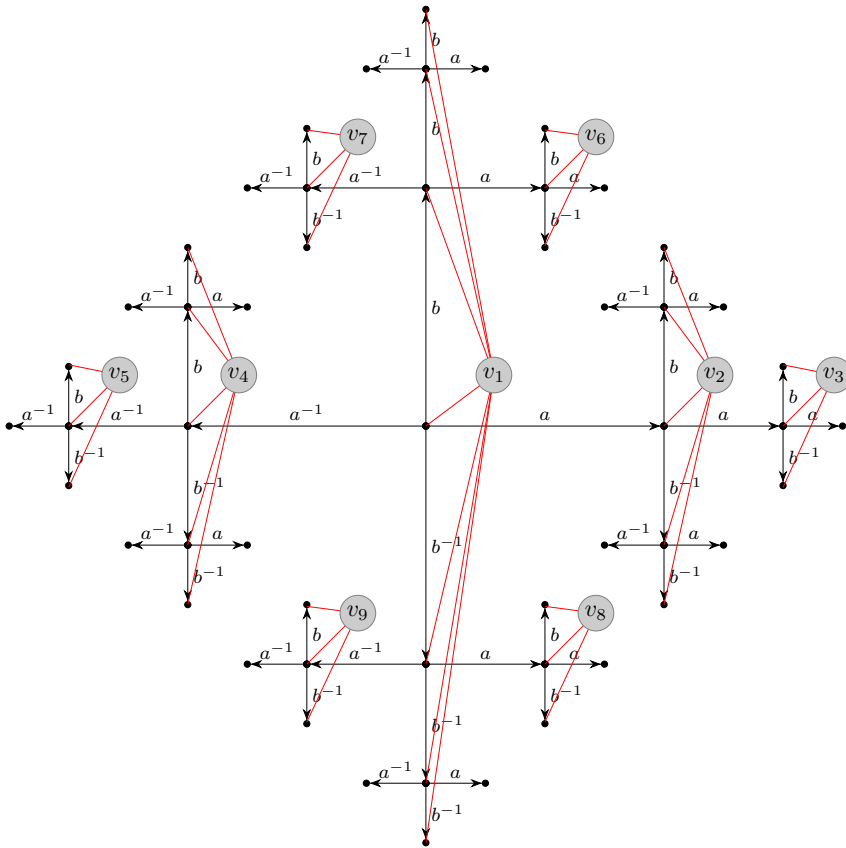


Figure 1. The coned-off Cayley graph $\widehat{\Gamma}(G, P, S)$, where G is the free group in two letters $S = \{a, b\}$ and P is the cyclic subgroup $\langle b \rangle$.

smallest normal subgroup containing R . In this case, we say that

$$\langle S, P | R \rangle \tag{1}$$

is a finite relative presentation of G with respect to P . It is an exercise to show that if G is finitely presented and P is finitely generated, then G is finitely presented relative to P . We refer the reader to [Osin 2006] for an exposition on finite relative presentations.

Assume that P is finitely generated, that (1) is a finite relative presentation of G with respect to P , and that $S \cap P$ is a generating set of P . The *coned-off Cayley complex* $\widehat{C} = \widehat{C}(G, P, S, R)$ is the 2-dimensional complex with 1-skeleton the coned-off Cayley graph $\widehat{\Gamma}(G, P, S)$ obtained by equivariantly attaching 2-cells as follows. For each word $r \in R$ correspond a loop in $\widehat{\Gamma}$. Attach a 2-cell with trivial stabilizer to each such loop, and extend in a manner equivariant under the G -action

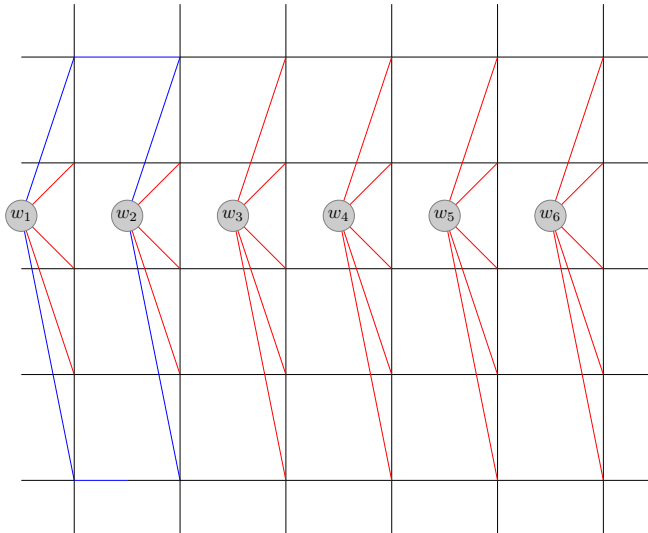


Figure 2. The coned-off Cayley graph $\widehat{\Gamma}(G, P, S)$, where G is the free abelian group in two letters $S = \{a, b\}$ and P is the cyclic subgroup $\langle b \rangle$.

on $\widehat{\Gamma}$. Similarly, for each $P \in \mathcal{P}$, for each generator in $s \in S \cap P$ and each $g \in G$, we have a corresponding loop in $\widehat{\Gamma}$ of length 3 passing through the vertices g, gs, gP . Attach a 2-cell with trivial stabilizer to each such loop, equivariantly under the G -action. The resulting G -complex \widehat{C} is simply connected [Groves and Manning 2008, Lemma 2.48], the G -action is cocompact by construction, and if P is infinite, the G -action is not proper. Now we consider the 2-dimensional filling function $FV_{\widehat{C}}^2$ of \widehat{C} .

Example 4.2. Let G be the free group of rank 2, let $S = \{a, b\}$ be a free generating set, and let P be the cyclic subgroup generated by b . It is an observation that the coned-off Cayley graph $\widehat{\Gamma}(G, P, S)$, see Figure 1, is a fine graph and hence Theorem 4.1 implies that $FV_{\widehat{C}}^2(m) < \infty$ for every $m \in \mathbb{N}$. Similar examples can be constructed by considering relatively hyperbolic groups.

Example 4.3. Let G be the free abelian group of rank 2, let $S = \{a, b\}$ be a generating set, and let P be the cyclic subgroup generated by b . The coned-off Cayley graph $\widehat{\Gamma}(G, P, S)$, see Figure 2, is not fine since there are infinitely many circuits of length 6 passing through the edge from b to P . By Theorem 4.1, we have that $FV_{\widehat{C}}^2(m) = \infty$ for some $m \in \mathbb{N}$. In fact, one can verify that $FV_{\widehat{C}}^2(6) = \infty$.

Remark 4.4. Theorem 1.1 does not hold for $d = 0$ in the natural setting of defining FV_X^1 by taking $Z_0(X, \mathbb{Z})$ to be the kernel of the augmentation map. Consider a finitely generated infinite group G acting properly and cocompactly on a connected

graph X ; for example, take X to be the Cayley graph of G with respect to a finite generating set. Then X is infinite, and the formal difference $\gamma = b - a$ between two distinct vertices a and b of X is a 0-cycle for which $|\gamma|_{\partial}$ can be made arbitrarily large by taking a and b sufficiently far apart; roughly speaking, a 1-chain μ such that $\partial\mu = b - a$ contains a combinatorial edge path from a to b and hence $\|\mu\|_1$ is at least the length of the shortest edge path from a to b . Hence $\text{FV}_X^1(2) = \infty$ in this case.

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
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Modeling of breast cancer through evolutionary game theory KE'YONA BARTON, CORBIN SMITH, JAN RYCHTÁŘ AND TSVETANKA SENDOVA	541
The isoperimetric problem in the plane with the sum of two Gaussian densities JOHN BERRY, MATTHEW DANNENBERG, JASON LIANG AND YINGYI ZENG	549
Finiteness of homological filling functions JOSHUA W. FLEMING AND EDUARDO MARTÍNEZ-PEDROZA	569
Explicit representations of 3-dimensional Sklyanin algebras associated to a point of order 2 DANIEL J. REICH AND CHELSEA WALTON	585
A classification of Klein links as torus links STEVEN BERES, VESTA COUFAL, KAIA HLAVACEK, M. KATE KEARNEY, RYAN LATTANZI, HAYLEY OLSON, JOEL PEREIRA AND BRYAN STRUB	609
Interpolation on Gauss hypergeometric functions with an application HINA MANOJ ARORA AND SWADESH KUMAR SAHOO	625
Properties of sets of nontransitive dice with few sides LEVI ANGEL AND MATT DAVIS	643
Numerical studies of serendipity and tensor product elements for eigenvalue problems ANDREW GILLETTE, CRAIG GROSS AND KEN PLACKOWSKI	661
Connectedness of two-sided group digraphs and graphs PATRECK CHIKWANDA, CATHY KRILOFF, YUN TECK LEE, TAYLOR SANDOW, GARRETT SMITH AND DMYTRO YEROSHKIN	679
Nonunique factorization over quotients of PIDs NICHOLAS R. BAETH, BRANDON J. BURNS, JOSHUA M. COVEY AND JAMES R. MIXCO	701
Locating trinomial zeros RUSSELL HOWELL AND DAVID KYLE	711

