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# Explicit representations of 3-dimensional Sklyanin algebras associated to a point of order 2

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The representation theory of a 3-dimensional Sklyanin algebra  $S$  depends on its (noncommutative projective algebro-) geometric data: an elliptic curve  $E$  in  $\mathbb{P}^2$ , and an automorphism  $\sigma$  of  $E$  given by translation by a point. Indeed, by a result of Artin, Tate, and van den Bergh, we have that  $S$  is module-finite over its center if and only if  $\sigma$  has finite order. In this case, all irreducible representations of  $S$  are finite-dimensional and of at most dimension  $|\sigma|$ .

In this work, we provide an algorithm in Maple to directly compute all irreducible representations of  $S$  associated to  $\sigma$  of order 2, up to equivalence. Using this algorithm, we compute and list these representations. To illustrate how the algorithm developed in this paper can be applied to other algebras, we use it to recover well-known results about irreducible representations of the skew polynomial ring  $\mathbb{C}_{-1}[x, y]$ .

## 1. Introduction

We work over the ground field  $\mathbb{C}$ . The motivation of this work is to study, up to equivalence, irreducible finite-dimensional representations (irreps) of *Sklyanin algebras*  $S$  of global dimension 3 (Definition 1.2). Past work on this problem includes results on bounds on the dimension of irreps of  $S$  [Walton 2012], and on a geometric parametrization of (trace-preserving) irreps of  $S$  [De Laet and Le Bruyn 2015]. The focus of this paper is to determine, for a class of Sklyanin algebras, all *explicit* irreps up to equivalence. Namely, we compute *irreducible matrix solutions* to the defining equations of  $S$ , up to an action of a general linear group. A geometric parametrization of the set of irreps of  $S$  is also presented, as this is the typical approach to understanding aspects of Sklyanin algebras.

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**Remark 1.1.** We directly compute the irreps via a Maple algorithm. A more conceptual technique, using noncommutative projective algebraic geometry (and Clifford theory for these particular Sklyanin algebras), can be used to solve this problem. We nevertheless hold to the computational approach because it can be adapted (much more easily in some cases) to other algebras; for further discussion of the complexity of this approach, see Remarks 1.10 and 1.11.

To begin, let us define the algebra under investigation.

**Definition 1.2** [Artin et al. 1990]. The 3-dimensional Sklyanin algebra  $S := S(a, b, c)$  over  $\mathbb{C}$  is generated by three noncommuting variables  $x, y, z$  subject to the relations

$$ayz + bzy + cx^2 = azx + bxz + cy^2 = axy + byx + cz^2 = 0. \quad (1.3)$$

Here,  $[a : b : c] \in \mathbb{P}_{\mathbb{C}}^2$ , with  $abc \neq 0$  and  $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$ .

This algebra is rather resistant to noncommutative Gröbner basis methods; that is, it is difficult to write down a  $\mathbb{C}$ -vector space basis of  $S$  (consisting of monomials in  $x, y, z$ ). See, for instance, [Bellamy et al. 2016, Exercise 1.7]. (The reader may also be interested in [Iyudu and Shkarin 2017].) In fact, it is common practice to consider the geometric data of  $S$  in the context of noncommutative projective algebraic geometry [Artin et al. 1990; Bellamy et al. 2016; Stafford and Van den Bergh 2001] to analyze its ring-theoretic behavior. By [Artin et al. 1990, Equations 1.6 and 1.7], the geometric data of  $S(a, b, c)$  consists of an elliptic curve  $E := E_{a,b,c} \subset \mathbb{P}_{\mathbb{C}}^2$  defined the equation

$$E_{a,b,c} : (a^3 + b^3 + c^3)(uvw) - (abc)(u^3 + v^3 + w^3) = 0, \quad (1.4)$$

and an automorphism of this elliptic curve  $\sigma := \sigma_{a,b,c}$  given by

$$\sigma_{a,b,c}([u : v : w]) = [acv^2 - b^2uw : bcu^2 - a^2vw : abw^2 - c^2uv]. \quad (1.5)$$

Here, the automorphism is given by translation of the point  $[a : b : c] \in E_{a,b,c}$ , where  $[1 : -1 : 0]$  is the origin of  $E_{a,b,c}$ . The *order* of  $\sigma$ , denoted by  $|\sigma|$ , is the smallest  $n \in \mathbb{N}$  such that  $\sigma^n = \text{id}_E$ . If no such  $n$  exists, then  $|\sigma| = \infty$ . Consider the following terminology.

**Definition 1.6.** We say that a Sklyanin algebra  $S(a, b, c)$  is *associated to a point*  $([a : b : c] \in E_{a,b,c})$  of *order*  $n$  if the automorphism  $\sigma_{a,b,c}$  has order  $n$ .

The role of this geometric data for our work will be explained towards the end of this section.

Now let us recall some basic representation theory terminology. Take  $n$  to be a positive integer. An  $n$ -dimensional representation of  $S := S(a, b, c)$  is an algebra homomorphism  $\psi : S \rightarrow \text{End}(V)$ , where  $V$  is a  $\mathbb{C}$ -vector space of dimension  $n$ . Since

$\text{End}(V)$  is isomorphic to  $\text{Mat}_n(\mathbb{C})$ , there is a one-to-one correspondence between the  $n$ -dimensional representations of  $S(a, b, c)$  and the  $n \times n$  matrix solutions  $(X, Y, Z)$  to the system of equations (1.3). Here,  $X = \psi(x)$ ,  $Y = \psi(y)$ , and  $Z = \psi(z)$ .

Next, we discuss irreducibility. Given a representation  $\psi : S \rightarrow \text{End}(V)$ , a subspace  $W$  of  $V$  is called  $S$ -stable if  $\psi(s)(w) \in W$  for all  $s \in S$ ,  $w \in W$ . Such a subspace  $W$  yields a *subrepresentation* of  $S$ , given as  $\psi' : S \rightarrow \text{End}(W)$ . We say that  $\psi$  is *irreducible* if the only  $S$ -stable subspaces of  $V$  are  $\{0\}$  and itself, that is, if there are no proper subrepresentations  $\psi'$  of  $\psi$ . Similarly, there is a notion of irreducibility for a matrix solution  $(X, Y, Z)$  to (1.3); see Lemma 2.1.

Now we recall when two representations/matrix solutions of  $S$  are equivalent. We say that  $n$ -dimensional representations  $\psi, \phi : S \rightarrow \text{End}(V)$  are *equivalent* if there exists a matrix  $Q \in \text{GL}_n(\mathbb{C})$  such that  $\psi(s) = Q\phi(s)Q^{-1}$  for all  $s \in S$ . Likewise, two matrix solutions  $(X_0, Y_0, Z_0)$  and  $(X_1, Y_1, Z_1)$  to (1.3) are *equivalent* if there exists  $Q \in \text{GL}_n(\mathbb{C})$  such that  $Q^{-1}X_0Q = X_1$ ,  $Q^{-1}Y_0Q = Y_1$ , and  $Q^{-1}Z_0Q = Z_1$ . Note that two equivalent representations/matrix solutions are either both irreducible or both reducible.

As the reader can imagine, studying explicit finite-dimensional representations of the algebras  $S(a, b, c)$  is difficult computationally. Now by [Walton 2012, Theorem 1.3], we only have nontrivial finite-dimensional representations of  $S$  when the automorphism  $\sigma$  of (1.5) has finite order. So, we refine our goal: we study the irreps of  $S(a, b, c)$  associated to a point  $[a : b : c] \in E_{a,b,c}$  of order 2. Note that the order-1 case is precisely the case when  $S$  is commutative (Lemma 2.4).

**Lemma 1.7** (Lemma 2.5). *A Sklyanin algebra  $S(a, b, c)$  is associated to a point  $[a : b : c] \in E_{a,b,c}$  of order 2 if and only if  $a = b$ .*

In this case, we assume that  $a = b = 1$  by rescaling. Therefore, our goal is to study the representation theory of the 3-dimensional Sklyanin algebra  $S(1, 1, c)$ , where by Definition 1.2,  $c \neq 0$ ,  $c^3 \neq 1, -8$ . By Lemma 2.6, all 1-dimensional irreps of  $S(1, 1, c)$  are trivial, and all irreps of  $S(1, 1, c)$  are finite-dimensional, of at most dimension 2. Thus, we only need to compute the irreps of dimension 2; we achieve this as follows.

**Theorem 1.8.** *The nontrivial explicit irreps (or matrix solutions) of the 3-dimensional Sklyanin algebra  $S(1, 1, c)$  are of dimension 2. They are classified up to equivalence; the representatives of equivalence classes of irreps of  $S(1, 1, c)$  are provided in (5.1)–(5.2) and (6.1)–(6.5) in Sections 5 and 6, respectively.*

In Section 2, we provide background material and some preliminary results. In Section 3, we give an outline (Steps 0–2, 3a, 3b) of our algorithm to prove Theorem 1.8. The algorithm then begins in Section 4, where we determine all of the 2-dimensional representations of  $S(1, 1, c)$ , and exclude “families” of reducible representations; this is Steps 0–2 of the algorithm. In Sections 5 and 6, we determine

representatives of equivalence classes of 2-dimensional irreps of  $S(1, 1, c)$ ; this is Steps 3a and 3b of the algorithm.

The study of the irreps of  $S(1, 1, c)$  ends in Section 7, where for completion, we discuss a geometric parametrization of equivalence classes of irreps of  $S(1, 1, c)$ ; e.g., we illustrate the *Azumaya locus* of  $S(1, 1, c)$  over the center of  $S(1, 1, c)$ . Namely we have the result below.

**Theorem 1.9** (Theorem 7.1). *The set of equivalence classes of irreps of  $S(1, 1, c)$  is in bijective correspondence with the points of the 3-dimensional affine variety:*

$$X_c := \mathbb{V}(g^2 - c^2(u_1^3 + u_2^3 + u_3^3) - (c^3 - 4)u_1u_2u_3) \subseteq \mathbb{C}_{\{u_1, u_2, u_3, g\}}^4.$$

*In particular,  $X_c \setminus \{0\}$  is the Azumaya locus of  $S$  over its center (i.e., points of  $X_c \setminus \{0\}$  correspond to 2-dimensional irreps of  $S$ ), and the origin of  $X_c$  corresponds to the trivial representation of  $S$ .*

**Remark 1.10.** We would like to point out that one can adjust our algorithm to prove Theorem 1.8 to examine equivalence classes of irreps of other algebras with generators and relations, especially those that are module-finite over their center. Although, the run-time and complexity of the output of the algorithm is in direct correlation with the number of generators and relations of the algebra, along with the algebra's *polynomial identity degree* (PI degree), if applicable.

We illustrate the remark above in Section 8, where we tailor our algorithm to examine irreps of the skew polynomial ring

$$\mathbb{C}_{-1}[x, y] := \mathbb{C}\langle x, y \rangle / (xy + yx).$$

Like  $S(1, 1, c)$ , it is well known that all irreps of  $\mathbb{C}_{-1}[x, y]$  are finite-dimensional, of dimension at most 2 (Lemma 8.1(c)). See Proposition 8.3 and Corollary 8.5 for the results on the representation theory of  $\mathbb{C}_{-1}[x, y]$ .

**Remark 1.11.** Part of the novelty of this work is that we obtain noncommutative algebraic/representation-theoretic results with Maple, which is a computer algebra system that is used typically for commutative computations. We hope that in the future the task of determining equivalence classes of irreps of noncommutative algebras (presented by generators and relations) can be achieved easily using a computer algebra system that handles noncommutative Gröbner bases, such as GAP [Cohen and Knopper 2016].

**Remark 1.12.** Unless stated otherwise, computational results in this work are performed with the computer algebra system Maple (version 16). All code (including comments) is available on the authors' professional websites, and in the preprint version of this work available on the ArXiv: <http://arxiv.org/abs/1512.09167>.

### 2. Preliminaries

We begin with a result on the irreducibility of a representation/matrix solution of a Sklyanin algebra  $S = S(a, b, c)$ . This result is well known, and we will use it often without mention.

**Lemma 2.1.** *Let  $\psi : S \rightarrow \text{End}(V)$  be an  $n$ -dimensional representation of  $S$ , with corresponding matrix solution  $(X, Y, Z)$  to the system of equations (1.3). Then, the following are equivalent:*

- (a)  $\psi$  is irreducible.
- (b) The corresponding  $S$ -module  $V$  (where  $S$  acts on  $V$  via  $\psi$ ) is simple.
- (c)  $\psi$  is surjective.
- (d)  $\psi(S)$  generates  $\text{End}(V) \cong \text{Mat}_n(\mathbb{C})$  as a  $\mathbb{C}$ -algebra.
- (e) Every matrix in  $\text{Mat}_n(\mathbb{C})$  can be expressed as a noncommutative polynomial in  $(X, Y, Z)$  over  $\mathbb{C}$ . □

If any of the above conditions hold, we say that the matrix solution  $(X, Y, Z)$  is irreducible.

On the other hand, we can determine when a matrix solution of  $S$  is reducible by using Lemma 2.1.

**Corollary 2.2.** *An  $n \times n$  matrix solution  $(X, Y, Z)$  to (1.3) (corresponding to a representation  $\psi$  of  $S$ ) is reducible if and only if there exists a subspace  $W$  of  $V$  of dimension  $m < n$  with  $X \cdot w, Y \cdot w, Z \cdot w \in W$  for all  $w \in W$ . Here, we embed  $W$  into  $V$  so that  $\cdot$  is given by matrix multiplication. □*

If  $S$  is a Sklyanin algebra associated to a point of infinite order, then by [Walton 2012, Theorem 1.3(i)], we have that all finite-dimensional irreps of  $S$  are trivial. On the other hand, Sklyanin algebras associated to points of finite order have an interesting representation theory, due to the following result.

**Proposition 2.3.** *Let  $S(a, b, c)$  be a Sklyanin algebra associated to a point of finite order. Then, all irreducible representations of  $S(a, b, c)$  are finite-dimensional, of at most dimension  $|\sigma_{a,b,c}|$ .*

*Proof.* In this case, we have that the Sklyanin algebra  $S(a, b, c)$  is module-finite over its center by [Artin et al. 1991, Theorem 7.1]. Further,  $S(a, b, c)$  has PI degree  $|\sigma_{a,b,c}|$  by [Walton 2012, Proposition 1.6]. Hence, the irreducible representations of  $S(a, b, c)$  are all finite-dimensional by [McConnell and Robson 2001, Theorem 13.10.3(a)], of dimension at most  $|\sigma_{a,b,c}|$  by [Brown and Goodearl 1997, Proposition 3.1]. □

Now we analyze parameters  $(a, b, c) \in \mathbb{C}^3$  so that the automorphism  $\sigma_{a,b,c}$  from (1.5) has finite order. Recall that two projective points  $[m_1 : m_2 : m_3], [n_1 : n_2 : n_3] \in \mathbb{P}^2_{\mathbb{C}}$

are equal if and only if  $m_1n_2 - m_2n_1 = m_1n_3 - m_3n_1 = m_2n_3 - m_3n_2 = 0$  if and only if  $n_i = \lambda m_i$  for all  $i = 1, 2, 3$ , for some nonzero  $\lambda \in \mathbb{C}$ . Omitting the conditions on parameters  $a, b, c$  for now, it is worth noting the following the result.

**Lemma 2.4.** *The automorphism  $\sigma_{a,b,c}$  from (1.5) has order 1 if and only if  $a = 1, b = -1, c = 0$ . In this case,  $S(1, -1, 0)$  is the commutative polynomial ring  $\mathbb{C}[x, y, z]$ .*

*Proof.* If  $\sigma$  has order 1, then we obtain  $[acv^2 - b^2uw : bcu^2 - a^2vw : abw^2 - c^2uv] = [u : v : w]$ . Therefore,  $bcu^2w - (a^2 + ab)vw^2 + c^2uv^2 = 0$ , which (by taking the coefficient of  $uv^2$ ) implies  $c = 0$ . Without loss of generality, take  $a = 1$ . Now,  $[-b^2uw : -vw : bw^2] = [u : v : w]$ , and we must have that  $b = -1$  since  $-vw^2 = bvw^2$ . Therefore, the forward direction holds. For the converse, note that  $\sigma_{1,-1,0}([u : v : w]) = [-uw : -vw : -w^2] = [u : v : w]$ , so  $\sigma_{1,-1,0}$  has order 1. The last statement is clear. □

Consider the following preliminary results about Sklyanin algebras associated to a point of order 2.

**Lemma 2.5.** *Take  $S = S(a, b, c)$  to be a 3-dimensional Sklyanin algebra associated to the automorphism  $\sigma_{a,b,c}$  of (1.5). Then,  $|\sigma_{a,b,c}| = 2$  if and only if  $a = b$ .*

*Proof.* Without loss of generality, take  $a = 1$ . The code for this result (see Remark 1.12) implies that  $b = 1$  and there are no conditions on  $c$  (other than those in Definition 1.2).

The converse is clear by the computation above, but we can verify this directly. If  $a = b = 1$ , then  $\sigma_{1,1,c}([u : v : w]) = [cv^2 - uw : cu^2 - vw : w^2 - c^2uv]$ . So,

$$\begin{aligned} \sigma_{1,1,c}^2([u : v : w]) &= [u(c^3u^3 + c^3v^3 + w^3 - 3c^2uvw) : v(c^3u^3 + c^3v^3 + w^3 - 3c^2uvw) \\ &\quad : w(c^3u^3 + c^3v^3 + w^3 - 3c^2uvw)] \\ &= [u : v : w], \end{aligned}$$

as desired. □

Hence, to work with Sklyanin algebras  $S(a, b, c)$  associated to a point of order 2, we take  $a = b = 1$ .

**Lemma 2.6.** *We have the following statements for the Sklyanin algebra  $S(1, 1, c)$ .*

- (a) *The only 1-dimensional representation of  $S(1, 1, c)$  is the trivial representation.*
- (b) *All irreducible representations of  $S(1, 1, c)$  are finite-dimensional, of at most dimension equal to 2.*

*Proof.* (a) One can compute this directly, or by using a short routine; see Remark 1.12.

(b) This follows from Proposition 2.3 and Lemma 2.5. □

### 3. Methodology and terminology

In this section, we provide an outline of the algorithm used to prove Theorem 1.8; see Sections 4–6 for the full details. The goal is to obtain *irreducible representative families* of  $S(1, 1, c)$  as defined below.

**Definition 3.1.** We say that a set of matrix solutions of the defining equations of  $S(1, 1, c)$  (or of equations (1.3) with  $a = b = 1$ ) is a *representative family of matrix solutions*, if no two members within the set are equivalent. Further, we call this set an *irreducible representative family* if all of its members are irreducible matrix solutions of  $S(1, 1, c)$ .

Note that we aim to have the parameter  $c$  of  $S(1, 1, c)$  free. So due to Maple’s default alpha ordering, we refer to  $c$  as  $zc$  in the code below.

First, we make the following simplification.

**Step 0:** assume the matrix  $X$  is in Jordan form. Due to Lemma 2.6 we know that all nontrivial irreps of  $S(1, 1, c)$  are of dimension 2. Hence, we only study  $2 \times 2$  matrix solutions  $(X, Y, Z)$  of (1.3) with  $(a, b, c) = (1, 1, c)$ . Initially, the entries of  $X, Y, Z$  are  $x_\ell, y_\ell, z_\ell$  for  $\ell = 1, 2, 3, 4$ . We further simplify the problem by assuming that  $X$  is in Jordan form. This simplification is made because we wish to classify the irreps up to equivalence, and equivalence is determined by simultaneous conjugation by an invertible matrix. So, we take  $X$  to be either a single  $2 \times 2$  Jordan block or diagonal so that we have 3 or 2 less unknowns, respectively. We consider these cases separately.

**Step 1:** find all families of matrix solutions. Now, we solve (1.3) with  $(a, b, c) = (1, 1, c)$  for  $2 \times 2$  matrices  $(X, Y, Z)$ . The output consists of 2-dimensional (*matrix solution*) families of  $S(1, 1, c)$ . The solutions are grouped according to the default behavior of Maple. We refer to these groups as Families.

**Step 2:** eliminate reducible matrix solutions. We run this step now to cut down on the run-time of the algorithm and the complexity of its output. Given a family of matrix solutions, we use Corollary 2.2 to determine if all members of this family are reducible. Namely, we let  $w = \langle p, q \rangle$  be a basis of a 1-dimensional subspace  $W$  of  $\mathbb{C}^2$ . Note that if  $p = p_1 + p_2i$  and  $q = q_1 + q_2i$  for  $i := \sqrt{-1}$  and  $p_1, p_2, q_1, q_2 \in \mathbb{R}$ , then  $(p, q) \neq (0, 0)$  precisely when  $p\bar{p} + q\bar{q} \neq 0$ . We examine when  $W$  is stable under the action of  $S(1, 1, c)$ ; namely, we need each of  $Xw, Yw, Zw$  to be a scalar multiple of  $w$ . So, we solve for  $p, q$  subject to the conditions

- $W$  is not the zero subspace  $\dots p \cdot \text{conjugate}(p) + q \cdot \text{conjugate}(q) > 0$ ,
- $XW \subset W \dots p \cdot Xw[2][1] - q \cdot Xw[1][1] = 0$ ,
- $YW \subset W \dots p \cdot Yw[2][1] - q \cdot Yw[1][1] = 0$ ,
- $ZW \subset W \dots p \cdot Zw[2][1] - q \cdot Zw[1][1] = 0$ ,
- conditions on  $c$ .

If there is a solution, then this implies that all members of the specified family are reducible. We remove such families from further computations by forming a list `NonRedFams` consisting of families for which there is no  $p, q$  satisfying the conditions above.

*Steps 3a and 3b are independent of each other, and either can be run after Step 2.*

**Step 3a:** account for equivalence between families. For the remaining families of matrix solutions, we determine conditions when members of one family `NonRedFams[i]` is equivalent to members of another family `NonRedFams[j]`. These conditions are collected in the list `BetweenFams`.

We do so as follows. First, we force variables of `NonRedFams[i]` to be in terms of  $u_\ell, v_\ell, w_\ell$  instead of  $x_\ell, y_\ell, z_\ell$  for  $\ell = 1, 2, 3, 4$ ; this is executed with

```
eval(NonRedFams[...], ChangeVars).
```

Next, we conjugate the relabeled matrices simultaneously by a  $2 \times 2$  matrix  $Q$  to form `Xconj`, `Yconj`, `Zconj`. Then, we solve for variables  $u_\ell, v_\ell, w_\ell, x_\ell, y_\ell, z_\ell$  subject to the conditions

- `Xconj` is equal to the  $X$ -matrix  $X_j$  of `NonRedFams[j]` ..... `Equiv1=0`,
- `Yconj` is equal to the  $Y$ -matrix  $Y_j$  of `NonRedFams[j]` ..... `Equiv2=0`,
- `Zconj` is equal to the  $Z$ -matrix  $Z_j$  of `NonRedFams[j]` ..... `Equiv3=0`,
- conditions on  $c$  and invertibility of  $Q$ .

The output is `[i, j, {conditions on  $u_\ell, v_\ell, w_\ell, x_\ell, y_\ell, z_\ell$ }]`, which we interpret as follows.

**Interpretation:** We can eliminate `NonRedFams[i]` from our consideration if all of its members are equivalent to members of `NonRedFams[j]` for some  $j \neq i$ . This occurs if we get an output

```
[i, j, ...{each of  $u_\ell, v_\ell, w_\ell$  is free }...] for  $i < j$ , or
[j, i, ...{each of  $x_\ell, y_\ell, z_\ell$  is free }...] for  $j < i$ .
```

We obtain that `NonRedFams[i]` forms a representative family if we get output

```
[i, i, ...{restrictions on  $u_\ell, v_\ell, w_\ell, x_\ell, y_\ell, z_\ell$  }...]
```

under one of the following conditions:

- (i) each of  $x_\ell, y_\ell, z_\ell$  is free and (ii) each of  $u_\ell, v_\ell, w_\ell$  is free, or depends only on  $x_\ell, y_\ell, z_\ell$ ; or
- (i) each of  $u_\ell, v_\ell, w_\ell$  is free and (ii) each of  $x_\ell, y_\ell, z_\ell$  is free, or depends only on  $u_\ell, v_\ell, w_\ell$ .

In either case above, we set the free variables in (ii) equal to 1 to obtain representative families. Otherwise, a careful examination is needed.

Conditions  $u_\ell, v_\ell, w_\ell, x_\ell, y_\ell, z_\ell$  may depend on entries of the matrix  $Q$ . In this case, we can conclude that such variables are free as long as this does not violate invertibility of  $Q$ .

**Step 3b:** check for full irreducibility conditions. Here, we run the same code as in Step 2 except that we solve for  $p, q$  along with all variables  $x_\ell, y_\ell, z_\ell$ . The conditions are collected in a list called `IrConditions`. If the output for `NonRedFams[i]` is `[i]` (or empty), then all members of `NonRedFams[i]` are irreducible.

#### 4. Families of nonreducible representations of $S(1, 1, c)$

Here, we execute Steps 0–2 of the algorithm discussed in the previous section. Namely, we find all 2-dimensional representations of  $S(1, 1, c)$  by determining  $2 \times 2$  matrix solutions  $(X, Y, Z)$  to (1.3) with  $a = b = 1$ . Here,  $X$  is in Jordan form, either one Jordan block or two Jordan blocks (diagonal). Moreover, we eliminate the families of solutions for which all of its members are reducible. See Remark 1.12 and we obtain the results below.

We start with the output of Steps 0–2 for `NonRedFams` when  $X$  is one Jordan block. For all matrix solutions, we have

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.1}$$

The rows below show the matrices  $Y, Z$  for the five matrix solutions:

$$\begin{aligned} & \begin{pmatrix} -y_4 & (y_4^2 + (y_4^4 - 8y_4z_4^3)^{1/2}) / (2cz_4^2) \\ -cz_4^2 & y_4 \end{pmatrix}, \quad \begin{pmatrix} -z_4 & 0 \\ \frac{1}{2}c(y_4^2 + (y_4^4 - 8y_4z_4^3)^{1/2}) - cy_4^2 & z_4 \end{pmatrix}; \\ & \begin{pmatrix} -y_4 & -(y_4^2 + (y_4^4 - 8y_4z_4^3)^{1/2}) / (2cz_4^2) \\ -cz_4^2 & y_4 \end{pmatrix}, \quad \begin{pmatrix} -z_4 & 0 \\ -\frac{1}{2}c(-y_4^2 + (y_4^4 - 8y_4z_4^3)^{1/2}) - cy_4^2 & z_4 \end{pmatrix}; \\ & \begin{pmatrix} \alpha / (cz_3) & -(cz_2^2z_3 - cz_2z_4^2 + 2\alpha z_4 / (cz_3)) / z_3 \\ -cz_2z_3 - cz_4^2 & -\alpha / (cz_3) \end{pmatrix}, \quad \begin{pmatrix} -z_4 & z_2 \\ z_3 & z_4 \end{pmatrix}; \\ & \begin{pmatrix} -\beta / (cz_3) & -(cz_2^2z_3 - cz_2z_4^2 + 2\beta z_4 / (cz_3)) / z_3 \\ -cz_2z_3 - cz_4^2 & \beta / (cz_3) \end{pmatrix}, \quad \begin{pmatrix} -z_4 & z_2 \\ z_3 & z_4 \end{pmatrix}; \\ & \begin{pmatrix} -y_4 & y_4^2 / (cz_4^2) \\ -cz_4^2 & y_4 \end{pmatrix}, \quad \begin{pmatrix} -z_4 & 2y_4 / (cz_4) \\ 0 & z_4 \end{pmatrix}; \end{aligned} \tag{4.2}$$

where,

$$\begin{aligned} \alpha &= c^2z_2z_3z_4 + c^2z_4^3 + (3c^4z_2^2z_3^2z_4^2 + 3c^4z_2z_3z_4^4 + c^4z_4^6 - cz_3^3 + c^4z_3^3z_2^3)^{1/2}, \\ \beta &= -c^2z_2z_3z_4 - c^2z_4^3 + (3c^4z_2^2z_3^2z_4^2 + 3c^4z_2z_3z_4^4 + c^4z_4^6 - cz_3^3 + c^4z_3^3z_2^3)^{1/2}. \end{aligned} \tag{4.3}$$

When  $X$  is two Jordan blocks, `NonRedFams` gives six matrix solutions:

$$\begin{aligned}
X &= \begin{pmatrix} cz_4^2/(2y_4) & 0 \\ 0 & -cz_4^2/(2y_4) \end{pmatrix}, & Y &= \begin{pmatrix} -y_4 & -(y_4^3 - z_4^3)/(y_4 y_3) \\ y_3 & y_4 \end{pmatrix}, \\
Z &= \begin{pmatrix} -z_4 & -z_4(8y_4^3 + c^3 z_4^3)/(4y_4^2 y_3) \\ 0 & z_4 \end{pmatrix}; \\
\\
X &= \begin{pmatrix} -x_4 & 0 \\ 0 & x_4 \end{pmatrix}, & Y &= \begin{pmatrix} 0 & 0 \\ y_3 & 0 \end{pmatrix}, & Z &= \begin{pmatrix} 0 & -cx_4^2/y_3 \\ 0 & 0 \end{pmatrix}; \\
\\
X &= \begin{pmatrix} cy_4^2/(2z_4) & 0 \\ 0 & -cy_4^2/(2z_4) \end{pmatrix}, & Y &= \begin{pmatrix} -y_4 & -y_4(8z_4^3 + c^3 y_4^3)/(4z_3 z_4^2) \\ 0 & y_4 \end{pmatrix}, \\
Z &= \begin{pmatrix} -z_4 & (y_4^3 - z_4^3)/(z_4 z_3) \\ z_3 & z_4 \end{pmatrix}; \\
\\
X &= \begin{pmatrix} -x_4 & 0 \\ 0 & x_4 \end{pmatrix}, & Y &= \begin{pmatrix} 0 & -cx_4^2/z_3 \\ 0 & 0 \end{pmatrix}, & Z &= \begin{pmatrix} 0 & 0 \\ z_3 & 0 \end{pmatrix}; \\
\\
X &= \begin{pmatrix} \gamma/(c^2 y_3 z_3) & 0 \\ 0 & -\gamma/(c^2 y_3 z_3) \end{pmatrix}, & Y &= \begin{pmatrix} -y_4 & -(2z_4 \gamma/(c^2 y_3 z_3) + cy_4^2)/(cy_3) \\ y_3 & y_4 \end{pmatrix}, \\
Z &= \begin{pmatrix} -z_4 & -(2\gamma y_4/(c^2 y_3 z_3) + cz_4^2)/(cz_3) \\ z_3 & z_4 \end{pmatrix}; \\
\\
X &= \begin{pmatrix} -\delta/(c^2 y_3 z_3) & 0 \\ 0 & \delta/(c^2 y_3 z_3) \end{pmatrix}, & Y &= \begin{pmatrix} -y_4 & -(2z_4 \delta/(c^2 y_3 z_3) + cy_4^2)/(cy_3) \\ y_3 & y_4 \end{pmatrix}, \\
Z &= \begin{pmatrix} -z_4 & -(2\delta y_4/(c^2 y_3 z_3) + cz_4^2)/(cz_3) \\ z_3 & z_4 \end{pmatrix};
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
\gamma &= -z_3^2 z_4 - y_3^2 y_4 + (z_3^4 z_4^2 + 2z_3^2 z_4 y_3^2 y_4 + y_3^4 y_4^2 + c^3 y_3 z_3^3 y_4^2 + c^3 y_3^3 z_3 z_4^2 - 2c^3 y_3^2 z_3 y_4 z_4)^{1/2}, \\
\delta &= z_3^2 z_4 + y_3^2 y_4 + (z_3^4 z_4^2 + 2z_3^2 z_4 y_3^2 y_4 + y_3^4 y_4^2 + c^3 y_3 z_3^3 y_4^2 + c^3 y_3^3 z_3 z_4^2 - 2c^3 y_3^2 z_3 y_4 z_4)^{1/2}.
\end{aligned} \tag{4.5}$$

### 5. Equivalence and irreducibility: one-Jordan-block case

We wish to classify the matrix solutions from Steps 0–2 (in the previous section) up to equivalence and extract the irreducible equivalence classes. So, we would like to know under what conditions is a matrix solution equivalent to a member of the same/different solution family. We then specify conditions for which the representative of an equivalence class of matrix solutions is irreducible. This achieved with Steps 3a and 3b, respectively, as described in Section 3. In this section, we continue the algorithm of Section 4 in the case when  $X$  is one Jordan block; see Remark 1.12.

The output of Steps 0–3a can be viewed by entering the following:

```
for i from 1 to nops(BetweenFams) do      print(BetweenFams[i]):      end do:
```

For interpretation, consider the snippets of output

```
[1, 2, {q1 = q1, q2 = q2, q3 = 0, q4 = q1, v4 = v4, w4 = w4,
      y4 =  $\frac{-4 w_4^3 + v_4^3 - v_4^{1/2} \%1}{v_4^2 - \%1}$ , z4 = -w4, zc =  $-\frac{4 q_1 w_4}{q_2 (v_4^2 - \%1^{1/2})}$ },
      \%1 :=  $v_4^4 - 8 w_4^3 v_4$ }]
```

```
[1, 5, {q1 = q1, q2 = q2, q3 = 0, q4 = q1, v4 = 0, w4 = w4, y4 =  $\frac{z_c q_2 w_4^2}{q_1}$ ,
      z4 = w4, zc = zc}]
```

In the first snippet, one sees that with a choice of  $q_1$  and  $q_2$ , the parameter  $c$  can be considered free without violating the invertibility of  $Q$ . We can also conclude that any member of  $\text{NonRedFams}[1]$  is equivalent to a member of  $\text{NonRedFams}[2]$ , except when  $v_4^2 - (v_4^4 - 8w_4^3v_4)^{1/2} = 0$ , or equivalently when  $v_4$  or  $w_4 = 0$ . From the second snippet of output, we see that any member of  $\text{NonRedFams}[1]$  is equivalent to a member of  $\text{NonRedFams}[5]$  when  $v_4 = 0$ . Moreover by (4.1)–(4.3), we have that in  $\text{NonRedFams}[1]$   $w_4$  (identified with  $z_4$ ) cannot be 0. So, we exclude  $\text{NonRedFams}[1]$  from further computation.

Now consider another two snippets of output:

```
[2, 4, {q1 = q1, q2 = q2, q3 = 0, q4 = q1, v4 = v4, w4 = w4, z2 = z2,
      z3 =  $\frac{(2 \text{RootOf}(\_Z^2 + 1 + \_Z) w_4 q_2 - q_1 z_2) q_1}{q_2^2}$ ,
      z4 =  $-\frac{\text{RootOf}(\_Z^2 + 1 + \_Z) w_4 q_2 - q_1 z_2}{q_2}$ ,
      zc =  $-\frac{2 (2 \text{RootOf}(\_Z^2 + 1 + \_Z) w_4 q_2 - q_1 z_2) q_1}{(v_4^2 + (v_4^4 - 8 w_4^3 v_4)^{1/2}) q_2^2}$ }]
```

```
[2, 5, {q1 = q1, q2 = q2, q3 = 0, q4 = q1, v4 = 0, w4 = w4, y4 =  $\frac{z_c q_2 w_4^2}{q_1}$ ,
      z4 = w4, zc = zc}]
```

Through a choice of  $q_1$  and  $q_2$ , we consider  $c$  to be free in  $[2, 4, \dots]$ . We conclude that any member of  $\text{NonRedFams}[2]$  is equivalent to a member of

NonRedFams [4] for all values of  $v_4$  and  $w_4$  except when  $v_4^2 + (v_4^4 - 8w_4^3 v_4)^{1/2} = 0$ , or equivalently when  $v_4$  or  $w_4 = 0$ . From the second snippet of output, we see that if  $v_4 = 0$ , any member of NonRedFams [2] is equivalent to a member of NonRedFams [5]. From (4.1)–(4.3), we see that  $w_4$  (identified with  $z_4$ ) in NonRedFams [2] cannot be 0. So, we exclude NonRedFams [2] from further computation.

Now take into account the following snippets of output:

$$[3, 4, \{q_1 = q_1, q_2 = q_2, q_3 = 0, q_4 = q_1, w_2 = w_2, w_3 = w_3, w_4 = w_4, z_2 = \frac{2 q_1 w_4 q_2 + w_2 q_1^2 - w_3 q_2^2}{q_1^2}, z_3 = w_3, z_4 = \frac{-w_3 q_2 + q_1 w_4}{q_1}, z_c = z_c\}]$$

[4, 5]

This implies NonRedFams [3] is equivalent to NonRedFams [4]. So, we exclude NonRedFams [3] from further computation. Further, no member of NonRedFams [4] is equivalent to a member of NonRedFams [5].

Finally, we determine when the remaining families are representative families. Consider

$$[4, 4, \{q_1 = q_1, q_2 = q_2, q_3 = 0, q_4 = q_1, w_2 = w_2, w_3 = w_3, w_4 = w_4, z_2 = \frac{2 q_1 w_4 q_2 + w_2 q_1^2 - w_3 q_2^2}{q_1^2}, z_3 = w_3, z_4 = \frac{-w_3 q_2 + q_1 w_4}{q_1}, z_c = z_c\}]$$

$$[5, 5, \{q_1 = q_1, q_2 = -\frac{q_1 (-y_4 + v_4)}{z_c w_4}, q_3 = 0, q_4 = q_1, v_4 = v_4, w_4 = w_4, y_4 = y_4, z_4 = w_4, z_c = z_c\}]$$

We get that a member of NonRedFams [5] is equivalent to another member of this family for any value of  $y_4$ . Without loss of generality, set  $y_4 = 1$ . So, NonRedFams [5] is a representative family with  $y_4 = 1$ .

In NonRedFams [4], we obtain any value for  $z_4$ , say  $a$ , by setting

$$q_2 = (w_4 - a)q_1/w_3.$$

(Note that by (4.1)–(4.3),  $z_3$ , identified by  $w_3$ , is not equal to 0.) This choice of  $q_2$  does not violate the invertibility of  $Q$ . Further, it is easy to check that in this case,  $z_2 = w_2$ . Thus, without loss of generality, set  $z_4 = 1$ . So, NonRedFams [4] is a representative family with  $z_4 = 1$ .

Given the results above, we only need to execute Step 3b for NonRedFams [4] and NonRedFams [5], but we complete this for the whole list NonRedFams as follows:

```

IrConditions:=[]:
for i from 1 to nops(NonRedFams) do
Xw:=Multiply(NonRedFams[i][1][1],w): Yw:=Multiply(NonRedFams[i][1][2],w):
Zw:=Multiply(NonRedFams[i][1][3],w):
Ir:=solve([p*conjugate(p)+q*conjugate(q)<>0,
           p*Xw[2][1]-q*Xw[1][1],p*Yw[2][1]-q*Yw[1][1],p*Zw[2][1]-q*Zw[1][1],
           zc<>0,zc^3<>1,zc^3<>-8]):
IrConditions:=[op(IrConditions),[i,Ir]]:
end do:

```

To see the output, enter

```
for i from 1 to nops(IrConditions) do print(IrConditions[i]): end do:
```

One gets that, for each  $i$ , all members of NonRedFams [i] are irreducible matrix solutions of  $S(1, 1, c)$ .

Now by entering

```
eval(NonRedFams[4],[z4=1]): eval(NonRedFams[5],[y4=1]):
```

one obtains the representatives of equivalence classes of irreducible matrix solutions  $(X, Y, Z)$  of equations (1.3), where  $X$  is assumed to be one Jordan block. The output is as follows:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} -\beta/(cz_3) & -(cz_2^2 z_3 - cz_2 + 2\beta/(cz_3))/z_3 \\ -cz_2 z_3 - c & \beta/(cz_3) \end{pmatrix}, \quad Z = \begin{pmatrix} -1 & z_2 \\ z_3 & 1 \end{pmatrix}; \quad (5.1)$$

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} -1 & 1/(cz_4^2) \\ -cz_4^2 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} -z_4 & 2/(cz_4) \\ 0 & z_4 \end{pmatrix};$$

where

$$\beta = -cz_2 z_3 - c^2 + (3c^4 z_2^2 z_3^2 + 3c^4 z_2 z_3 + c^4 - cz_3^3 + c^4 z_3^3 z_3^2)^{1/2}. \quad (5.2)$$

## 6. Equivalence and irreducibility: two-Jordan-block case

As in the one-Jordan-block case, we wish to classify the matrix solutions from Steps 0–2 (in Section 4) up to equivalence and extract the irreducible equivalence classes. So, we would like to know under what conditions a matrix solution is equivalent to a member of the same/different solution family. We then specify conditions for which the representative of an equivalence class of matrix solutions is irreducible. This achieved with Steps 3a and 3b, respectively, as described in Section 3. In this section, we continue the algorithm of Section 4 in the case when  $X$  is two Jordan blocks.

To execute Step 3a, as described in Section 3, enter the code for Step 3a used in Section 5 (see Remark 1.12). (The memory and time for this operation was

27068.0 MB and 523.78 seconds, respectively.) The output of Steps 0–3a can be viewed by entering the following:

```
for i from 1 to nops(BetweenFams) do      print(BetweenFams[i]):      end do:
```

Consider the following snippet of output:

$$[1, 1, \{q_1 = \frac{v_3 q_4}{y_3}, q_2 = 0, q_3 = 0, q_4 = q_4, v_3 = v_3, v_4 = y_4, w_4 = z_4,$$

$$y_3 = y_3, y_4 = y_4, z_4 = z_4, z_c = z_c\}]$$

Note that  $y_3 \neq 0$  in `NonRedFams[1]` by (4.4)–(4.5). So, `NonRedFams[1]` is a representative family with  $y_3$  (identified with  $v_3$ ) equal to 1 without loss of generality.

Now take

$$[2, 2, \{q_1 = \frac{v_3 q_4}{y_3}, q_2 = 0, q_3 = 0, q_4 = q_4, u_4 = x_4, v_3 = v_3, x_4 = x_4,$$

$$y_3 = y_3, z_c = z_c\}]$$

Note that  $y_3 \neq 0$  in `NonRedFams[2]` by (4.4)–(4.5). So, `NonRedFams[2]` is a representative family with  $y_3$  (identified with  $v_3$ ) equal to 1 without loss of generality.

Consider the output

$$[3, 3, \{q_1 = \frac{w_3 q_4}{z_3}, q_2 = 0, q_3 = 0, q_4 = q_4, v_4 = y_4, w_3 = w_3, w_4 = z_4,$$

$$y_4 = y_4, z_3 = z_3, z_4 = z_4, z_c = z_c\}]$$

Note that  $z_3 \neq 0$  in `NonRedFams[3]` by (4.4)–(4.5). So, `NonRedFams[3]` is a representative family with  $z_3$  (identified with  $w_3$ ) equal to 1 without loss of generality.

Next, consider the snippet of output below:

$$[2, 4, \{q_1 = 0, q_2 = -\frac{z_c x_4^2 q_3}{z_3 v_3}, q_3 = q_3, q_4 = 0, u_4 = -x_4, v_3 = v_3,$$

$$x_4 = x_4, z_3 = z_3, z_c = z_c\}]$$

By (4.4)–(4.5), we have that  $z_3 \neq 0$  for `NonRedFams[4]`. So by the output above, we get that any member of `NonRedFams[4]` is equivalent to a member `NonRedFams[2]`. We exclude `NonRedFams[4]` from further computation.

Consider the output:

$$[5, 5, \{q_1 = \frac{v_3 q_4}{y_3}, q_2 = 0, q_3 = 0, q_4 = q_4, v_3 = v_3, v_4 = y_4, w_3 = \frac{z_3 v_3}{y_3},$$

$$w_4 = z_4, y_3 = y_3, y_4 = y_4, z_3 = z_3, z_4 = z_4, z_c = z_c\}]$$

We have that  $y_3 \neq 0$  in `NonRedFams [5]` by (4.4)–(4.5). Without loss of generality, we can take  $y_3$  (identified with  $v_3$ ) to be 1. In this case,  $w_3 = z_3$ . So, `NonRedFams [5]` is a representative family with  $y_3 = 1$ .

Now let us take

$$[5, 6, \{q_1 = \frac{v_3 q_4}{y_3}, q_2 = 0, q_3 = 0, q_4 = q_4, v_3 = v_3, v_4 = y_4, w_3 = \frac{z_3 v_3}{y_3}, w_4 = z_4, y_3 = y_3, y_4 = y_4, z_3 = z_3, z_4 = z_4, z_c = z_c\}]$$

Note that by (4.4)–(4.5), we have  $y_3 \neq 0$  for `NonRedFams [6]`. So by the output above, we get that any member of `NonRedFams [6]` is equivalent to a member `NonRedFams [5]`. We exclude `NonRedFams [6]` from further computation.

We still need to analyze the equivalence between members of `NonRedFams [1]`, `NonRedFams [2]`, `NonRedFams [3]`, and `NonRedFams [5]`. In this case, the output is easier to interpret if we run Step 3b before Step 3a again.

Given the results above, we only need to execute Step 3b for `NonRedFams [1]`, `NonRedFams [2]`, `NonRedFams [3]`, and `NonRedFams [5]`, but we complete this for the whole list `NonRedFams` by entering the code for Step 3b (see Remark 1.12). Consider the snippets

$$[1, \{p = -\frac{y_4 q}{y_3}, q = q, y_3 = y_3, y_4 = y_4, z_4 = 0, z_c = z_c\}]$$

$$[2, \{p = 0, q = q, x_4 = 0, y_3 = y_3, z_c = z_c\}]$$

$$[3, \{p = -\frac{z_4 q}{z_3}, q = q, y_4 = 0, z_3 = z_3, z_4 = z_4, z_c = z_c\}]$$

$$[5, \{p = 0, q = q, y_3 = 0, y_4 = \text{RootOf}(\_Z^2 + 1 + \_Z) z_4, z_3 = z_3, z_4 = z_4, z_c = z_c\},$$

$$\{p = -\frac{z_4 q}{z_3}, q = q, y_3 = y_3, y_4 = \frac{z_4 y_3}{z_3}, z_3 = z_3, z_4 = z_4, z_c = z_c\}]$$

We obtain that

- members of `NonRedFams [1]`, `NonRedFams [2]`, and `NonRedFams [3]` are irreducible precisely when  $z_4 \neq 0$ ,  $x_4 \neq 0$ , and  $y_4 \neq 0$ , respectively, and
- members of `NonRedFams [5]` are irreducible precisely when  $\{y_3 \neq 0, y_4 \neq e^{\pm 2\pi i/3} z_4\}$  or  $\{y_3 z_4 \neq y_4 z_3\}$ .

We execute Step 3a again for the families highlighted above; we refer to the code in Remark 1.12. From the output, we obtain that  $z_4 = 0$  in `NewNonRedFams [1]` precisely when any member of `NewNonRedFams [1]` is equivalent to a member of `NewNonRedFams [2]`. On the other hand, we have that  $x_4 = 0$  in `NewNonRedFams [2]` precisely when any member of `NewNonRedFams [2]` is equivalent to a member of

$\text{NewNonRedFams [1]}$ . However, we know members of  $\text{NewNonRedFams [1]}$  and  $\text{NewNonRedFams [2]}$  are reducible when  $z_4 = 0$  and  $x_4 = 0$ , respectively.

Now by a choice of  $q_2, q_3$ , we can consider  $c$  to be free in  $[1, 3, \dots]$ . So, we get that  $z_4 = \zeta y_4$  for  $\zeta^3 = 1$  in  $\text{NewNonRedFams [3]}$  precisely when any member of  $\text{NewNonRedFams [3]}$  is equivalent to a member of  $\text{NewNonRedFams [1]}$ .

Putting this together we conclude that:

- $\text{NewNonRedFams [1]} = \text{eval}(\text{NonRedFams [1]}, [y_3=1])$  is an irreducible representative family when  $z_4 \neq 0$ ;
- $\text{NewNonRedFams [2]} = \text{eval}(\text{NonRedFams [2]}, [y_3=1])$  is an irreducible representative family when  $x_4 \neq 0$ ;
- $\text{NewNonRedFams [3]} = \text{eval}(\text{NonRedFams [3]}, [z_3=1])$  is an irreducible representative family when  $y_4 \neq 0$ , and there is no overlap with  $\text{NewNonRedFams [1]}$  when  $z_4 \neq \zeta y_4$  for  $\zeta^3 = 1$ ;
- $\text{NewNonRedFams [4]} = \text{eval}(\text{NonRedFams [5]}, [y_3=1])$  is an irreducible representative family when  $y_4 \neq e^{\pm 2\pi i/3} z_4$  and  $z_4 \neq y_4 z_3$ .

We obtain the following representatives of equivalence classes of irreducible matrix solutions  $(X, Y, Z)$  of equations (1.3), where  $X$  is assumed to be two Jordan blocks:

$$\begin{aligned}
 X &= \begin{pmatrix} cz_4^2/(2y_4) & 0 \\ 0 & -cz_4^2/(2y_4) \end{pmatrix}, & Y &= \begin{pmatrix} -y_4 & -(y_4^3 - z_4^3)/y_4 \\ 1 & y_4 \end{pmatrix}, \\
 Z &= \begin{pmatrix} -z_4 & -z_4(8y_4^3 + c^3z_4^3)/(4y_4^2) \\ 0 & z_4 \end{pmatrix}
 \end{aligned} \tag{6.1}$$

for  $z_4 \neq 0$ ,

$$X = \begin{pmatrix} -x_4 & 0 \\ 0 & x_4 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & -c^2x_4^2 \\ 0 & 0 \end{pmatrix} \tag{6.2}$$

for  $x_4 \neq 0$ ,

$$\begin{aligned}
 X &= \begin{pmatrix} cy_4^2/(2z_4) & 0 \\ 0 & -cy_4^2/(2z_4) \end{pmatrix}, & Y &= \begin{pmatrix} -y_4 & -y_4(8z_4^3 + c^3y_4^3)/(4z_4^2) \\ 0 & y_4 \end{pmatrix}, \\
 Z &= \begin{pmatrix} -z_4 & (y_4^3 - z_4^3)/z_4 \\ 1 & z_4 \end{pmatrix}
 \end{aligned} \tag{6.3}$$

for  $y_4 \neq 0$ ,  $z_4 \neq \zeta y_4$ ,  $\zeta^3 = 1$ , and

$$\begin{aligned}
 X &= \begin{pmatrix} \gamma/(c^2z_3) & 0 \\ 0 & -\gamma/(c^2z_3) \end{pmatrix}, & Y &= \begin{pmatrix} -y_4 & -(2z_4\gamma/(c^2z_3) + cy_4^2)/c \\ 1 & y_4 \end{pmatrix}, \\
 Z &= \begin{pmatrix} -z_4 & -(-2\gamma y_4/(c^2z_3) + cz_4^2)/cz_3 \\ z_3 & z_4 \end{pmatrix}
 \end{aligned} \tag{6.4}$$

for  $y_4 \neq e^{\pm 2\pi i/3} z_4$ ,  $z_4 \neq y_4 z_3$ , where

$$\gamma = -z_3^2 z_4 - y_4 + (z_3^4 z_4^2 + 2z_3^2 z_4 y_4 + y_4^2 + c^3 z_3^3 y_4^2 + c^3 z_3 z_4^2 - 2c^3 z_3^2 y_4 z_4)^{1/2}. \tag{6.5}$$

**7. Geometric parametrization of irreducible representations of  $S(1, 1, c)$**

Since the Sklyanin algebra  $S = S(1, 1, c)$  is module finite over its center, we can use the center  $Z$  of  $S$  to provide a geometric parametrization of the set of equivalence classes of irreducible representations of  $S$ . (Recall by Definition 1.2,  $c \neq 0$ ,  $c^3 \neq 1, -8$ .) Namely, we depict the *Azumaya locus* of  $S(1, 1, c)$  over its center [Brown and Goodearl 2002, III.1.7]. We refer the reader to [Smith et al. 2000] for an introduction to affine varieties.

**Theorem 7.1.** *Let  $Z$  be the center of the Sklyanin algebra  $S = S(1, 1, c)$ .*

(a) *We have that  $Z$  is generated by  $u_1 = x^2$ ,  $u_2 = y^2$ ,  $u_3 = z^2$ ,*

$$g = cy^3 + yxz - xyz - cx^3,$$

*subject to the degree-6 relation*

$$F := g^2 - c^2(u_1^3 + u_2^3 + u_3^3) - (c^3 - 4)u_1 u_2 u_3 = 0.$$

(b) *The set of equivalence classes of irreducible representations of  $S$  is in bijective correspondence with the set of maximal ideals of the center  $Z$  of  $S$ . Here, a representative  $\psi$  of an equivalence class of an irrep of  $S$  corresponds to  $(\ker \psi) \cap Z$ , a maximal ideal of  $Z$ .*

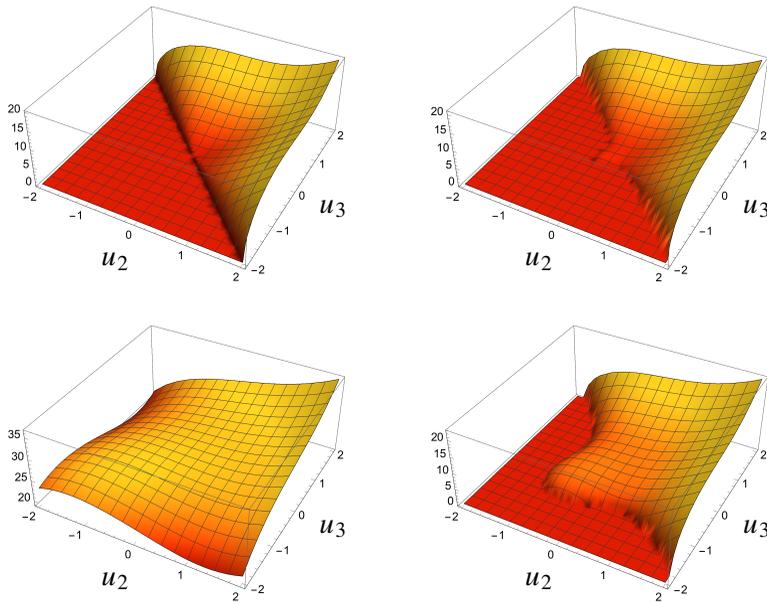
(b) *The geometric parametrization of the set of equivalence classes of irreducible representations of  $S$  is the 3-dimensional affine variety (3-fold)*

$$X_c := \mathbb{V}(F) \in \mathbb{C}_{\{u_1, u_2, u_3, g\}}^4.$$

*In particular,  $X_c \setminus \{0\}$  is the Azumaya locus of  $S$  over  $Z$ . Indeed, points of  $X_c \setminus \{0\}$  (the smooth locus of  $X_c$ ) correspond to irreducible 2-dimensional representations of  $S$ , and the origin of  $X_c$  corresponds to the trivial representation of  $S$ .*

Taking a value of  $c$ , say 5, we can visualize the 3-fold  $X_c$  by taking 2-dimensional slices at various values of  $u_1$ . See Figure 1 below.

*Proof of Theorem 7.1.* (a) We have that  $Z$  is generated by three algebraically independent elements  $u_1, u_2, u_3$  of degree 2 and one element  $g$  of degree 3, subject to one relation  $F$  of degree 6, by [Smith and Tate 1994, Theorems 3.7, 4.6, and 4.7]. Now part (a) follows by direct computation in the algebra  $S(1, 1, c)$ . One can do this by hand, but we execute this with the computer algebra software GAP using the GBNP package for noncommutative Gröbner bases [Cohen and Knopper 2016]. We check that  $u_1, u_2, u_3, g$  commute with each of  $x_1 := x$ ,  $y_1 := y$ ,  $z_1 := z$ ; the



**Figure 1.** Real part of  $\sqrt{25(u_1^3 + u_2^3 + u_3^3) + 21u_1u_2u_3}$  at  $u_1 = 0, 0.3, 1, 3$  (clockwise from the top left).

code for this is publicly available, see Remark 1.12. To view  $g$ , for instance, enter `PrintNP(g)`; The output of the last twelve lines are all 0. Thus,  $u_1, u_2, u_3, g$  are all central elements of  $S(1, 1, 5)$ . One can replace  $c = 5$  with various values of  $c \neq 0, 1, -8$ , and this yields the same output.

Now to see that  $F$  is the relation of  $Z$ , more care is needed. Enter

```
PrintNP(MulQA(g,g,GB));
PrintNP(MulQA(u1,MulQA(u1,u1,GB),GB));
PrintNP(MulQA(u2,MulQA(u2,u2,GB),GB));
PrintNP(MulQA(u3,MulQA(u3,u3,GB),GB));
PrintNP(MulQA(u1,MulQA(u2,u3,GB),GB));
```

and compare terms to derive the coefficients of  $F$  as claimed.

(b) The arguments below are standard in ring theory and in representation theory, but we provide details for the reader's convenience. Recall from Lemma 2.6 that all nontrivial irreducible representations of  $S$  are of dimension 2. Let  $\max\text{Spec}(A)$  denote the set of maximal ideals of an algebra  $A$ . Moreover, a *primitive* ideal of  $A$  is an ideal that arises as the kernel of an irreducible representation of  $A$ ; denote the set of such ideals by  $\text{prim}(A)$ . Take  $[\text{Irrep}(A)]$  to be the set of equivalence classes of irreducible representations of  $A$ .

Since  $S$  is PI, we see that there is a bijective correspondence between  $[\text{Irrep}(S)]$  and  $\text{prim}(S)$  as follows. Equivalent representations of  $S$  have the same kernel, so

we get a surjective map  $\phi: [\text{Irrep}(S)] \rightarrow \text{prim}(S)$ , given by  $\psi \mapsto \ker \psi$ . On the other hand, take  $P \in \text{prim}(S)$ , that is, the kernel of an irreducible representation  $\psi$  of  $S$ . Then,  $\psi$  is also an irreducible representation of  $S/P$ . Now  $S/P \cong \text{Mat}_t(\mathbb{C})$  for  $t = 1$  or  $2$  by [Brown and Goodearl 2002, Theorem I.13.5(1)], and all irreducible representations of matrix algebras are equivalent to the identity representation by the Skolem–Noether theorem. So,  $P \in \text{prim}(S)$  has a unique preimage  $\phi^{-1}(P)$  in  $[\text{Irrep}(S)]$ .

Moreover, we see that there is a bijective correspondence between  $[\text{Irrep}(S)]$  and  $\text{maxSpec}(S)$  as follows. Maximal ideals are primitive. On the other hand, take  $P$  to be a nonzero primitive ideal of  $S$ . Again, by [Brown and Goodearl 2002, Theorem I.13.5(1)],  $S/P$  is isomorphic to a matrix ring, which is simple. Thus,  $P$  is a maximal ideal of  $S$ . So it suffices to show that the ideals of  $\text{maxSpec}(S)$  and of  $\text{maxSpec}(Z)$  are in bijective correspondence.

Consider the map

$$\eta : \text{maxSpec}(S) \rightarrow \text{maxSpec}(Z), \quad M \mapsto M \cap Z.$$

The map  $\eta$  is well-defined and surjective by [Brown and Goodearl 2002, Proposition III.1.1(5)]. Now by Lemma 2.6, the trivial representation of  $S$  corresponds to the augmentation (maximal) ideal  $S_+ := (x, y, z)$  of  $S$ , and the set of equivalence classes of nontrivial irreducible representations of  $S$  corresponds to the maximal ideals  $M$  of  $S$  not equal to  $S_+$ . Thus,  $\eta(S_+) = Z_+$ , and it suffices to show that the ideals of  $\text{maxSpec}(S) \setminus S_+$  and of  $\text{maxSpec}(Z) \setminus Z_+$  are in bijective correspondence.

Take  $\text{Az}(S)$  to be the set of maximal ideals  $\mathfrak{m}$  of  $Z$  so that (i)  $\mathfrak{m} = M \cap Z$  for  $M \in \text{maxSpec}(S)$ , and (ii)  $M$  is the kernel of a 2-dimensional irreducible representation of  $S$ . Namely,  $\text{Az}(S)$  is the *Azumaya locus* of  $S$  over  $Z$ . Consider the map

$$\rho : \text{Az}(S) \rightarrow \text{maxSpec}(S), \quad \mathfrak{m} \mapsto \mathfrak{m}S.$$

We get that  $\eta\rho(\mathfrak{m}) = \eta(\mathfrak{m}S) = (\mathfrak{m}S) \cap Z = \mathfrak{m}$ ; the last equality holds by [Brown and Goodearl 2002, Theorem III.1.6(3)]. So,  $\eta$  is bijective on  $\rho(\text{Az}(S))$ . Since  $\text{Az}(S) = \text{maxSpec}(Z) \setminus Z_+$  by Lemma 2.6, and since  $\rho$  is injective, we conclude that  $\eta$  is bijective on  $\text{maxSpec}(S) \setminus S_+$ , as desired.

(c) To see that the claim follows from parts (a) and (b), we have to show that the smooth locus of  $X_c$  consists of all nonzero points. This is achieved by using [Smith et al. 2000, Theorem 6.2]; namely, we verify that the common zero set of the vanishing of all partial derivatives of  $F$  is the origin of  $X_c$ :

```
F:=g^2-c^2*(u1^3+u2^3+u3^3) - (c^3-4)*u1*u2*u3;
solve([diff(F,g),diff(F,u1),diff(F,u2),diff(F,u3)], [g,u1,u2,u3]);
> [[g = 0, u1 = 0, u2 = 0, u3 = 0]]
```

This completes the proof. □

**Remark 7.2.** One may push the result above further and study the *moduli space* (or *GIT quotient*) that parametrizes the set of equivalence classes of irreducible representations of  $S$ . But this is not the focus of this work here. On the other hand, if one wants to understand irreducible representations of  $S$  topologically, then one could consider the *Jacobson topology* (or *hull-kernel topology*) on the set  $\text{prim}(S)$ .

**Remark 7.3.** The code available via Remark 1.12 verifies that the irreps produced in (5.1)–(5.2) and (6.1)–(6.5) indeed correspond to points on  $X_c$ . One must first run the algorithm in the previous sections: Sections 4 and 5 for the one-Jordan-block case, and Sections 4 and 6 for the two-Jordan-block case.

By evaluating `simplify(U1);`, `simplify(U2);`, `simplify(U3);`, and `simplify(G);` for each of the six irreducible representative families above, we obtain the corresponding points on the 3-fold  $X_c = \mathbb{V}(F) \subset \mathbb{C}_{\{u_1, u_2, u_3, g\}}^4$ .

**8. Irreducible representations of  $\mathbb{C}_{-1}[x, y] := \mathbb{C}\langle x, y \rangle / (xy + yx)$**

The purpose of this section is to illustrate our algorithm of Sections 3-6 (Steps 0–2, 3a, 3b) by replacing the Sklyanin algebra  $S(1, 1, c)$  with a class of algebras that are much better understood. Here, we study irreducible representations of the skew polynomial ring

$$\mathbb{C}_{-1}[x, y] := \mathbb{C}\langle x, y \rangle / (xy + yx),$$

up to equivalence; these results are well known. At the end of the section, we provide a geometric parametrization of these irreps, akin to Theorem 7.1 for  $S(1, 1, c)$ . Now we remind the reader of a few preliminary results.

**Lemma 8.1.** (a) *The 1-dimensional irreps of  $\mathbb{C}_{-1}[x, y]$  are, up to equivalence, of the form*

$$\begin{aligned} \rho_\alpha : \mathbb{C}_{-1}[x, y] &\rightarrow \mathbb{C}, & x &\mapsto \alpha, & y &\mapsto 0 & \text{for } \alpha \in \mathbb{C}, \\ \rho_\beta : \mathbb{C}_{-1}[x, y] &\rightarrow \mathbb{C}, & x &\mapsto 0, & y &\mapsto \beta & \text{for } \beta \in \mathbb{C}. \end{aligned} \tag{8.2}$$

(b) *All irreducible representations of  $\mathbb{C}_{-1}[x, y]$  are finite-dimensional, of at most dimension 2.*

*Proof.* (a) This follows by an easy computation.

(b) By [Brown and Goodearl 1997, Proposition 3.1; 2002, Example I.14.3(1)], an irrep of  $\mathbb{C}_{-1}[x, y]$  is of at most dimension 2. □

With the lemma above, we see that to classify irreps of  $\mathbb{C}_{-1}[x, y]$ , we just need to compute the 2-dimensional irreps

$$\psi : \mathbb{C}_{-1}[x, y] \rightarrow \text{Mat}_2(\mathbb{C}), \quad x \mapsto X, \quad y \mapsto Y,$$

up to equivalence.

Without loss of generality, we can assume that  $X$  is in Jordan form; that is, either one Jordan block or diagonal. Now the code for this part (available publicly, see Remark 1.12) was adapted from Sections 3–6 by removing all lines and conditions involving the generator  $z$ , and by changing the defining relations of the algebra.

We obtain the result below.

**Proposition 8.3.** *All irreducible representations  $\phi$  of  $\mathbb{C}_{-1}[x, y]$  are of dimensions 1 or 2. In dimension 1, irreps are of the form (8.2). In dimension 2, all irreps, up to equivalence, take the form*

$$\psi_{\alpha,\beta} : \mathbb{C}_{-1}[x, y] \longrightarrow \text{Mat}_2(\mathbb{C}), \quad x \mapsto \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix} \quad (8.4)$$

for  $\alpha, \beta \in \mathbb{C}$  with  $\alpha\beta \neq 0$ .

*Proof.* The first two statements follow from Lemma 8.1. To get the last statement, we run the adapted algorithm above. We only obtain reducible representations in the one-Jordan-block case; just enter `NonRedFams`; and `IrConditions`; to see this.

On the other hand, in the two-Jordan-block case, we first print off `NonRedFams` (we’ve converted the output to standard format for readability):

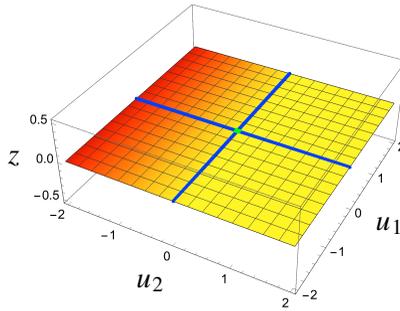
$$\begin{aligned} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}; & \begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & y_4 \end{bmatrix}; \\ & \begin{bmatrix} 0 & 0 \\ 0 & x_4 \end{bmatrix}, \begin{bmatrix} y_1 & 0 \\ 0 & 0 \end{bmatrix}; & \begin{bmatrix} x_1 & 0 \\ 0 & x_4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \\ & \begin{bmatrix} -x_4 & 0 \\ 0 & x_4 \end{bmatrix}, \begin{bmatrix} 0 & y_2 \\ y_3 & 0 \end{bmatrix}. \end{aligned}$$

Consider the following snippets of output from `BetweenFams`:

```
[2, 3, {q1 = 0, q2 = q2, q3 = q3, q4 = 0, u1 = x4, v4 = y1, x4 = x4,
y1 = y1}]
[5, 5, {q1 = 0, q2 = q2, q3 = q3, q4 = 0, u4 = -x4, v2 =  $\frac{y3 q2}{q3}$ , v3 =  $\frac{y2 q3}{q2}$ ,
x4 = x4, y2 = y2, y3 = y3}]
```

So, any member of `NonRedFams [3]` is equivalent to a member of `NonRedFams [2]`, and therefore `NonRedFams [3]` is removed from our consideration.

Moreover, `NonRedFams [5]` forms an equivalence family as  $x_4, y_2, y_3$  are free. Take into consideration the output from `IrConditions`:



**Figure 2.** Affine 2-space parametrizing irreps of  $\mathbb{C}_{-1}[x, y]$ ; axes parametrize 1-dimensional irreps.

$$[1, \{p = p, q = q, y_1 = \frac{y_3^2 p^2 + p y_4^2 q - y_2^2 q}{p q}, y_2 = y_2, y_3 = y_3, y_4 = y_4\}]$$

$$[2, \{p = p, q = 0, x_1 = x_1, y_4 = y_4\}]$$

$$[4, \{p = p, q = 0, x_1 = x_1, x_4 = x_4\}]$$

$$[5, \{p = p, q = 0, x_4 = x_4, y_2 = y_2, y_3 = 0\},$$

$$\{p = 0, q = q, x_4 = x_4, y_2 = 0, y_3 = y_3\},$$

$$\{p = p, q = q, x_4 = 0, y_2 = \frac{y_3^2 p}{2q}, y_3 = y_3\}]$$

Now, we can conclude that `NonRedFams [1]`, `NonRedFams [2]`, `NonRedFams [4]` consist of reducible representations, so these families are eliminated from our consideration. Further, `NonRedFams [5]` forms an irreducible representative family with  $y_2 = 1$ ; we can see this by adapting and running the algorithm for Step 3b in Section 6 in this case. □

The geometric parametrization of the equivalence classes of irreducible representations of  $\mathbb{C}_{-1}[x, y]$  is given as follows; see also Figure 2.

**Corollary 8.5.** *We have the following statements.*

(a) *We have that the center  $Z$  of  $\mathbb{C}_{-1}[x, y]$  is the commutative polynomial ring generated by  $u_1 := x^2$  and  $u_2 := y^2$ .*

(b) *The set of equivalence classes of irreducible representations of  $S$  are in bijective correspondence with the set of maximal ideals of  $\mathbb{C}[x^2, y^2]$ .*

(c) *The geometric parametrization of the set of equivalence classes of irreducible representations of  $\mathbb{C}_{-1}[x, y]$  is the 2-dimensional affine space  $\mathbb{C}_{\{u_1, u_2\}}^2$ . In particular*

- *points of  $\mathbb{C}^2 \setminus \mathbb{V}(u_1 u_2)$  correspond to irreducible 2-dimensional representations of  $\mathbb{C}_{-1}[x, y]$ ,*
- *points on the axes  $\mathbb{V}(u_1 u_2)$  not equal to the origin correspond to nontrivial 1-dimensional representations of  $\mathbb{C}_{-1}[x, y]$ , and*
- *the origin corresponds to the trivial representation of  $\mathbb{C}_{-1}[x, y]$ .*

*Proof.* (a) The algebra  $\mathbb{C}_{-1}[x, y]$  has a  $\mathbb{C}$ -vector space basis given by  $\{x^i y^j \mid i, j \in \mathbb{N}\}$ . Since  $(x^i y^j)x = (-1)^j x^{i+1} y^j = x^{i+1} y^j$  and  $y(x^i y^j) = (-1)^i x^i y^{j+1} = x^i y^{j+1}$  implies that  $i, j$  are even, the result is clear.

(b) This follows by the proof of Theorem 7.1(b).

(c) The first statement follows, as  $\text{Spec}(Z) = \mathbb{C}_{\{u_1, u_2\}}^2$ . Now the remaining statements hold by (8.4) and (8.2), where  $u_1 = \alpha^2$  and  $u_2 = \beta$ .  $\square$

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### References

- [Artin et al. 1990] M. Artin, J. Tate, and M. Van den Bergh, "Some algebras associated to automorphisms of elliptic curves", pp. 33–85 in *The Grothendieck Festschrift, I*, edited by P. Cartier et al., Progr. Math. **86**, Birkhäuser, Boston, 1990. MR Zbl
- [Artin et al. 1991] M. Artin, J. Tate, and M. Van den Bergh, "Modules over regular algebras of dimension 3", *Invent. Math.* **106**:2 (1991), 335–388. MR Zbl
- [Bellamy et al. 2016] G. Bellamy, D. Rogalski, T. Schedler, J. T. Stafford, and M. Wemyss, *Noncommutative algebraic geometry*, Mathematical Sciences Research Institute Publications **64**, Cambridge Univ. Press, 2016. MR Zbl
- [Brown and Goodearl 1997] K. A. Brown and K. R. Goodearl, "Homological aspects of Noetherian PI Hopf algebras of irreducible modules and maximal dimension", *J. Algebra* **198**:1 (1997), 240–265. MR Zbl
- [Brown and Goodearl 2002] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Birkhäuser, Basel, 2002. MR Zbl

- [Cohen and Knopper 2016] A. M. Cohen and J. W. Knopper, “GAP package GBNP: computing Gröbner bases of noncommutative polynomials”, software package, 2016, available at <http://www.gap-system.org/Packages/gbnp.html>.
- [De Laet and Le Bruyn 2015] K. De Laet and L. Le Bruyn, “The geometry of representations of 3-dimensional Sklyanin algebras”, *Algebr. Represent. Theory* **18**:3 (2015), 761–776. MR Zbl
- [Iyudu and Shkarin 2017] N. Iyudu and S. Shkarin, “Three dimensional Sklyanin algebras and Gröbner bases”, *J. Algebra* **470** (2017), 379–419. MR Zbl
- [McConnell and Robson 2001] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, revised ed., Graduate Studies in Mathematics **30**, American Mathematical Society, Providence, RI, 2001. MR Zbl
- [Smith and Tate 1994] S. P. Smith and J. Tate, “The center of the 3-dimensional and 4-dimensional Sklyanin algebras”, *K-Theory* **8**:1 (1994), 19–63. MR Zbl
- [Smith et al. 2000] K. E. Smith, L. Kahanpää, P. Kekäläinen, and W. Traves, *An invitation to algebraic geometry*, Springer, New York, 2000. MR Zbl
- [Stafford and Van den Bergh 2001] J. T. Stafford and M. Van den Bergh, “Noncommutative curves and noncommutative surfaces”, *Bull. Amer. Math. Soc. (N.S.)* **38**:2 (2001), 171–216. MR Zbl
- [Walton 2012] C. Walton, “Representation theory of three-dimensional Sklyanin algebras”, *Nuclear Phys. B* **860**:1 (2012), 167–185. MR Zbl

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