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Levi Angel and Matt Davis

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Properties of sets of nontransitive dice with few sides

Levi Angel and Matt Davis

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We define and investigate several properties that sets of nontransitive dice might have. We prove several implications between these properties, which hold in general or for dice with few sides. We also investigate some algorithms for creating sets of 3-sided dice that realize certain tournaments.

1. Nontransitive dice

Consider a set of three 3-sided dice, A, B, and C, numbered in the following way:

A	9	5	1
B	8	4	3
C	7	6	2

(1)

In this example, if we rolled each die one time, die A would beat die B $\frac{5}{9}$ of the time, die B would beat die C $\frac{5}{9}$ of the time, and die C would beat die A $\frac{5}{9}$ of the time. We say that die A "beats" or "wins against" die B if the probability that A rolls higher than B is greater than $\frac{1}{2}$. (Of course, in this case we could also say that B loses against A.) We use the notation $\succeq$ for the relation "beats", so that in this example $A \succeq B$, $B \succeq C$, and $C \succeq A$. This is an example of *nontransitivity*, since the relation $\succeq$ on $\{A, B, C\}$ is nontransitive. The study of such sets of dice dates back to [Steinhaus and Trybuła 1959; Trybuła 1961], although [Gardner 1970] was highly influential in raising interest in them. Numerous examples of nontransitive dice have since been constructed. This paper will examine a number of questions related to the construction of such sets of dice, particularly focusing on those with a small number of sides.

In what follows, we will always have a set of n k-sided dice, with the faces of each die labeled with a number from $\{1, 2, \ldots, kn\}$. We will assume each number from this set is used exactly once.

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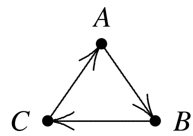

Figure 1. A nontransitive tournament on three vertices.

The relation $\succeq$ on a set of dice can be visualized as a directed graph. A *tournament* on n vertices is a directed realization of the complete graph K_n. In other words, it is a directed graph on the vertices $\{1, 2, \dots, n\}$ where for any pair of vertices i and j, either there is an edge from i to j or from j to i, but not both. We can interpret this as a definition of a relation on a set of dice — we say that a set of dice $\{X_1, X_2, \dots, X_n\}$ *realizes* a tournament T if $X_i \succeq X_j$ if and only if there is an edge from i to j in T. So the set of dice given in (1) realizes the tournament in Figure 1.

Previous work has shown that for any tournament T, it is possible to construct a set of dice that realizes this tournament. See [Angel and Davis 2017; Schaefer 2017; Bednay and Bozóki 2013] for some examples of such algorithms.

2. Properties of sets of dice

There are a number of properties that a set of dice might have that we will work with. By abuse of notation, for a given die X, we will use the same letter to represent the random variable giving the value when the die is rolled.

Uniform. We say a set of dice is *uniform* if there is a constant p so that, whenever $X \succeq Y$, we have $P(X>Y)=p$. Note that this is similar to the notion of *balanced dice* in [Schaefer and Schweig 2017], but uniformity is slightly stronger for nontransitive dice. The set of dice in (1) is uniform since $P(A>B) = P(B>C) = P(C>A) = \frac{5}{9}$.

Columned. We say a set of n dice with k sides is *columned* if the j-th smallest side on each die is chosen from the numbers $(j-1)n+1, \dots, jn$. That is, the smallest side from every die contains a number 1 through n, the second-smallest side of every die contains a number from $n+1$ through $2n$ and so on until the largest side of every die contains a number from $(k-1)n+1$ through kn. Put another way, the sides of each die are a transversal of the collection $\{\{1, 2, \dots, n\}, \{n+1, n+2, \dots, 2n\}, \dots, \{(k-1)n+1, \dots, kn\}\}$. The set of dice in (1) is columned since each die contains one number from $\{1, 2, 3\}$, one from $\{4, 5, 6\}$, and one from $\{7, 8, 9\}$.

Regular. If we have an odd number n of dice, we say the set of dice is *regular* if each die wins against $\frac{1}{2}(n-1)$ of the dice and loses against $\frac{1}{2}(n-1)$ dice. A set of dice is regular exactly if the tournament it realizes is a regular graph. The set of dice in (1) is regular since each die beats exactly one other die.

For our final property, we need one other notion. Given a die X in a set of dice, the *total number of face wins* for X is the number of ordered pairs (a, b) where a is a number on die X, b is a number on a different die in the set, and $a > b$. We similarly define the number of face wins for a die X over a die Y to be the number of ordered pairs (a, b) where a is on X, b is on Y, and $a > b$. In the example in (1), A has five face wins over B, corresponding to the pairs $(9, 8)$, $(9, 4)$, $(9, 3)$, $(5, 4)$, $(5, 3)$. Also, A has four face wins over C for a total of nine face wins.

This notion counts the total number of ways for die X to beat another die when X is rolled against another die. In other words, if we have a set S of k-sided dice containing X, and we sum $P(X>Y)$ for all $Y \neq X$ in S, the (unreduced) result will be a fraction with k^2 in the denominator. The total number of face wins is the numerator of that fraction. (Notice that in (1), $P(A>B) = \frac{5}{9}$ and $P(A>C) = \frac{4}{9}$, corresponding to its face wins.)

Equitable. We say a set of dice is *equitable* if each die has the same total number of face wins. The set of dice in (1) is equitable since each die has exactly nine total face wins.

We observe that, for a die X in a set of n k-sided dice,

$$\sum_{j \text{ is a face of } X} j = \text{total number of face wins for } X + \binom{k+1}{2}. \tag{2}$$

To see this, note that since our dice are numbered from 1 to nk, a face labeled j will be at least as large as the j faces labeled $1, 2, \ldots, j$. However, when counting total face wins, we do not count the wins a die's face would earn over faces on the same die (including the tie against itself). There are always exactly $1+2+\cdots+k = \binom{k+1}{2}$ of these, accounting for the extra term in (2). Thus, for a set of n k-sided dice that use the numbers $1, 2, \ldots, kn$ once each, equitability is equivalent to the condition that the total of the faces of each die is the same. This means that equitability is not always possible for a given number of sides and number of dice — specifically, an even number of dice each with an odd number of sides cannot be equitable.

We also explain here one way of thinking about sets of dice that is sometimes useful, which we call the *face rankings* of a die. For an ordered list of k-sided dice $X_1, X_2, \ldots, X_n$, we can associate to each die a list of numbers that encodes the number of face wins for each die over the next die in the list (or for X_n over X_1), one face at a time. Specifically, for each die, we give a list of k numbers. The first number corresponds to the highest face on the die and tells us how many faces of the *next* die it is higher than, i.e., how many face wins the given die has as a result of that face. The second number similarly corresponds to the second-highest face of the die in the same way, etc. So in the example in (1), the corresponding list would be

A	320
B	311
C	221

since, for example, die A's highest face beats all of B's faces, its middle face beats two of B's faces, and its lowest face beats none of B's faces. Notice that these lists give a number of relations between the faces, which in this case (but not in all cases) are enough to reconstruct the entire set of dice. We can see A's highest face is larger than B's highest, which is larger than C's highest two faces, which are larger than A's second-highest face, etc.

For example, the set of face rankings

A	320
B	221
C	221
D	311

would describe the set of dice

A	11	8	1
B	9	7	4/5
C	10	6	3
D	12	4/5	2

(3)

where the two spaces marked 4/5 contain the faces 4 and 5 in some order. These faces are not uniquely determined by the face rankings.

With the notion of face wins and (2), we can establish some general implications between the properties described above.

Theorem 1. *Given a regular tournament on an odd number n of dice, any uniform set of dice that realize that tournament is equitable.*

Proof. Assume the dice have k sides, and that if $X \succeq Y$, then $P(X{>}Y) = j/k^2$. Then a die X wins against $\frac{1}{2}(n-1)$ dice with j face wins each and loses against $\frac{1}{2}(n-1)$ dice with $k^2 - j$ face wins each. Thus the total number of face wins for X must be $\frac{1}{2}k^2(n-1)$, and so the set is equitable. $\square$

Theorem 2. *A uniform equitable set of an odd number n of k-sided dice must be regular.*

Proof. Note that a uniform equitable set of dice cannot be transitive, since the die that beats all others would have a greater total number of face wins than the other dice. Assume that if $X \succeq Y$, then $P(X{>}Y) = j/k^2$. Given a die X, there are $k^2(n-1)$ pairs consisting of a face of X and a face of another die. By equitability, the face from X is the higher value in exactly half of those pairs. So X has a total number of face wins equal to $\frac{1}{2}k^2(n-1)$. However, adding up the total number of face wins by comparing X to each other die means we must write $\frac{1}{2}k^2(n-1)$ as a sum of $n-1$ numbers, each of which is either j or k^2-j. This is only possible with exactly $\frac{1}{2}(n-1)$ of each. □

3. Implications between properties for 3-sided dice

For sets of small dice, there are some additional implications between these properties. In what follows, we will use the following theorem; see [Savage 1994] or [Trybuła 1961].

Theorem 3. *Suppose the numbers* $1, 2, \ldots, kn$ *are arranged on a set of three* k*-sided dice, labeled* A_1, A_2, A_3. *Then at least one of the probabilities* $P(A_1>A_2)$, $P(A_2>A_3)$ *is less than* $\frac{1}{2}(\sqrt{5}-1)$.

For a set of three dice, at least one of the given probabilities must be less than or equal to $\frac{5}{9}$, since $\frac{6}{9} > \frac{1}{2}(\sqrt{5}-1)$. However, for a set of four dice, it is possible to arrange the dice in a cycle so each one beats the next with probability $\frac{2}{3}$. The dice described by Gardner [1970], now known as Efron dice for their discoverer, are an example of such dice.

This theorem also inspires the following theorem, which is particular to sets of 3-sided dice of any size.

Theorem 4. *Suppose the numbers* $1, 2, \ldots, kn$ *are arranged on a set of* k 3*-sided dice, labeled* A_1, A_2, $A_3, \ldots, A_k$. *Then at least one of the probabilities* $P(A_1>A_2)$, $P(A_2>A_3), \ldots, P(A_k>A_1)$ *is less than* $\frac{2}{3}$.

Proof. Assume that the dice are numbered so that $A_1 \succeq A_2, \ldots, A_{k-1} \succeq A_k$, $A_k \succeq A_1$. If no such k-cycle can be formed, then one of the given probabilities is in fact less than $\frac{1}{2}$. Also, assume each winning probability is at least $\frac{2}{3}$. This means each die has at least six face wins over the next die in the cycle. For 3-sided dice, this implies the middle face of a die is larger than at least two faces on the next die. (The only possible lists of face rankings with six face wins are 330 or 222.) Thus each middle face of a die is greater than the middle face of the next die in the cycle. But, this implies (by going all the way around the cycle) that each middle face is larger than itself, a contradiction. □

Theorem 5. *A nontransitive uniform set of* 3*-sided dice is columned.*

Proof. In the case that any die X has two numbers from $1, \ldots, n$ or $2n+1, \ldots, 3n$, there would be another die Y that had no numbers from that set. This would lead to one of X or Y beating the other with probability at least $\frac{2}{3}$. Unless the set of dice is transitive, this would force a cycle of at least three dice, each of which beats the next with probability at least $\frac{2}{3}$ (by uniformity), contradicting Theorem 4. □

Of course, one can easily make a transitive uniform set of 3-sided dice that is not columned merely by making each die in the list strictly better than the next.

Theorem 6. *A set of an odd number of equitable columned* 3*-sided dice must be uniform.*

Proof. Recall that an odd number of dice are necessary in this case for equitability to be possible. By the columned property, every die must have at least three face wins against any other die, since the largest number on each die is guaranteed to be higher than the smaller two numbers on the other dice, etc. Thus for any dice X and Y (assuming without loss of generality that $X \succeq Y$), we have $P(X{>}Y) \leq \frac{2}{3}$. However, if $P(X{>}Y) = \frac{2}{3}$, this would imply that X's largest face is greater than Y's largest face, X's second-largest face is greater than Y's, etc. This contradicts equitability. Thus if $X \succeq Y$, then $P(X{>}Y) = \frac{5}{9}$. □

Theorem 7. *A set of an odd number of regular, columned* 3*-sided dice has to be uniform.*

Proof. Since the set of dice is columned, the only way for a die X to have six face wins over a die Y is if X's largest face is greater than Y's largest face, X's second-largest face is greater than Y's, etc. This, however, implies that X beats every die that Y beats, as well as Y, so the set could not be regular. So the only possible numbers of face wins for one die over another are 5 and 4. Thus if $X \succeq Y$, $P(X{>}Y) = \frac{5}{9}$, the definition of uniformity. □

Theorem 8. *A regular equitable set of an odd number n of* 3*-sided dice must be uniform.*

Proof. By regularity, each die wins against exactly $\frac{1}{2}(n-1)$ other dice. By equitability, the total number of face wins for any die is $\frac{9}{2}(n-1)$. So the average number of face wins for any die against another is $\frac{9}{2}$. Thus if a die X has six face wins against a die Y, then X must have three face wins or fewer against some other die Z, or else its average number of face wins would be greater than $\frac{9}{2}$. Similarly, Z would have only three face wins against some die, and this will eventually create a cycle of 3-sided dice where each die beats the next with probability $\frac{2}{3}$. This contradicts Theorem 4. Thus, if $X \succeq Y$, then $P(X{>}Y) \neq \frac{2}{3}$, and so $P(X{>}Y) = \frac{5}{9}$. □

The previous few theorems, along with Theorems 1 and 2, imply the following corollary.

Corollary 9. *If a set of an odd number of* 3*-sided dice has any three of the properties equitable, columned, uniform, and regular, then it must have the fourth property. If the set of dice has two of these properties, at least one of which is regular or equitable, then it has all four properties.*

Note that it is possible for a set of dice to be only uniform and columned.

Example 10.

A	9	6	1
B	8	5	2
C	7	4	3

In this case A beats both B and C $\frac{5}{9}$ of the time and B beats C $\frac{5}{9}$ of the time, making this set of dice uniform. While this set of dice is columned, C doesn't win against any die, and A's face sum is 16 whereas C's is 14, so the example is not equitable or regular.

The theorems at the end of Section 2 suggest that generally, uniformity is the strongest condition, but the others become slightly more powerful with small dice. Generally, any one of the properties can exist alone, although a set of 3-sided dice which is uniform but not columned must be transitive.

Example 11.

A	15	5	4
B	12	11	1
C	14	7	3
D	13	9	2
E	10	8	6

This set of dice is equitable (since each die has 18 face wins) but has none of the other properties.

Example 12.

A	15	7	2
B	14	5	4
C	13	11	1
D	12	9	3
E	10	8	6

This set of dice is regular (since each die beats 2 other dice) but has none of the other properties.

Example 13.

A	14	7	1
B	11	8	2
C	15	9	3
D	12	10	4
E	13	6	5

This set of dice is columned (since each die contains one face each from the sets $\{1, 2, 3, 4, 5\}$, $\{6, 7, 8, 9, 10\}$, and $\{11, 12, 13, 14, 15\}$) but has none of the other properties.

Note that for an even number of 3-sided dice, regularity and equitability are impossible. However, we can replace these notions with weak versions. For a set of an even number n of k-sided dice (where k is odd), we say the set is *weakly regular* if every die beats either $\frac{n}{2}$ or $\frac{n}{2}-1$ other dice. We say the set is *weakly equitable* if the number of face wins for each die is within $\frac{1}{2}$ of the average number of face wins. For our dice, this is equivalent to the sum of the labels on each die being either $\lceil \frac{1}{2}k(kn+1) \rceil$ or $\lfloor \frac{1}{2}k(kn+1) \rfloor$.

Some of the theorems above generalize to the weaker versions, with the same proof. Theorems 1, 2, and 6 directly generalize to the weaker notions of regularity and equitability. However, Theorems 7 and 8 do not generalize.

First, a weakly regular columned set of dice need not be uniform.

Example 14.

A	10	8	3
B	9	7	2
C	12	6	1
D	11	5	4

Here, A and D beat two dice each, while B and C each beat one die, so these dice are weakly regular. However, $P(A>B) = \frac{2}{3}$, while the other winning probabilities are $\frac{5}{9}$, so the dice are not uniform.

Also, a weakly regular and weakly equitable set of dice need not be uniform.

Example 15.

A	8	7	5
B	12	4	3
C	11	6	2
D	10	9	1

The face sums of these dice are all 19 or 20, so they are weakly equitable, and each die beats 1 or 2 other dice, so they are weakly regular. However, $P(A>B) = \frac{2}{3}$ and $P(D>A) = \frac{2}{3}$, while the other winning probabilities are $\frac{5}{9}$, so these dice are not uniform.

Returning to our strong versions of the properties, we note that the statement that any three of these properties implies the fourth is specific to 3-sided dice. For sets of dice with more sides, it is possible to create sets of dice which have three of these properties, but not the fourth, if the missing property is either columned or uniform.

Example 16.

A	15	13	7	3	2
B	14	12	9	4	1
C	11	10	8	6	5

This is an example of three 5-sided dice which are equitable, regular, and uniform, but not columned. Note also that Algorithm 4.2 in [Schaefer and Schweig 2017] gives a way of constructing more examples which are equitable, regular, and uniform, but not columned.

The following theorem gives a large class of counterexamples.

Theorem 17. *For an odd number $n > 3$, there exists a set of n n-sided dice which is regular, equitable and columned but not uniform. In fact, each die beats the dice that it wins against with a different probability. The set of winning probabilities for a given die are the $\frac{1}{2}(n-1)$ possible winning probabilities closest to $\frac{1}{2}$.*

Proof. We begin by constructing the dice so that each die beats the next one in the list (cyclically) with $\frac{1}{2}n(n+1)-1$ face wins. To do so, take the numbers $1, n+1, 2n+1, \dots, n^2-n+1$ (all congruent to 1 mod n) and place them on different dice. This can be done arbitrarily, but we assume without loss of generality that they are placed as shown here:

A	1					
B		$n{+}1$				
C			$2n{+}1$			
D				$3n{+}1$		
E					$4n{+}1$	
$\vdots$						$\ddots$

Then, place the number that is congruent to 2 mod n in each column above the number congruent to 1 mod n, cycling around to the bottom row when necessary.

A	1	$n{+}2$				
B		$n{+}1$	$2n{+}2$			
C			$2n{+}1$	$3n{+}2$		
D				$3n{+}1$	$4n{+}2$	
E					$4n{+}1$	$\ddots$
$\vdots$						$\ddots$

Then we repeat the process, placing the number congruent to 3 mod n in each column above the number congruent to 2 mod n, etc. This process creates a columned set of dice, shown here for $n = 5$.

A	1	7	13	19	25
B	5	6	12	18	24
C	4	10	11	17	23
D	3	9	15	16	22
E	2	8	14	20	21

Now, each die will contain exactly one number from each congruence class mod n, so the total on the die will be $\frac{1}{2}n(n^2+1)$. By construction, a die X earns $\frac{1}{2}n(n+1)-1$ face wins over the die Y after it, since each face of X is larger than the corresponding face of Y except the face of X congruent to 1 mod n. But X earns $\frac{1}{2}n(n+1)-2$ face wins over the die Z after Y, since each face of X is greater than the corresponding face of Z except the faces of X congruent to 1 or 2 mod n. This pattern repeats, and X earns one fewer face win against every successive die after it in the list. Thus X wins against exactly $\frac{1}{2}(n-1)$ other dice, but with different winning probabilities. □

Note that using face rankings, it possible to show that the construction above is the only way to create a columned n-cycle of n-sided dice where each die has $\frac{1}{2}n(n+1)-1$ face wins against the next one in the cycle. For a columned set of dice, there are exactly n ways for one die to have $\frac{1}{2}n(n+1)-1$ face wins over another. If die X beats die Y with exactly $\frac{1}{2}n(n+1)$ face wins, each face of X would be greater than the face of Y in the same column, so for X to get one fewer face win, exactly one of its faces must be smaller than the corresponding face of Y. But, no such list of face rankings can repeat in a set of n n-sided dice, or else we would be missing one such pattern, which would create a cycle within a single column, which is impossible.

This section gives a relatively complete picture of the possibilities for 3-sided dice. We attempt to generalize to sets of 4-sided dice with some success.

4. Implications between properties for 4-sided dice

Note that for 4-sided dice, it is no longer necessarily the case that a uniform set of nontransitive dice must be columned.

Example 18.

A	16	8	6	3
B	15	13	5	2
C	14	11	9	1
D	12	10	7	4

This set of dice is not columned since D has no face from the set {13, 14, 15, 16}. However, every winning probability is $\frac{9}{16}$, and the set contains the cycle $A \succeq B$, $B \succeq C$, $C \succeq A$.

In fact, we have the following:

Theorem 19. *A uniform columned set of three* 4*-sided dice is transitive.*

Proof. Assume that $A \succeq B$, $B \succeq C$, and $C \succeq A$. Then by uniformity and Theorem 3, $P(A>B) = P(B>C) = P(C>A) = \frac{9}{16}$. Thus, since the dice are columned, the only face rankings that are possible are 4320, 4311, 4221, or 3321. But, choosing any three of those will give us one column where each face ranking has the same number, implying that each face in that column would have to be larger than the corresponding face on the next die, even cyclically, which is impossible. □

Corollary 20. *A uniform columned set of* 4*-sided dice is transitive.*

Proof. Given a uniform columned set of 4-sided dice, if it is not transitive, then it contains some 3-cycle. Call the dice in that cycle A, B, and C. Then we can convert A, B, and C into a set of columned dice labeled by $1, \dots, 12$ by "compressing" the numbers in each column. So the smallest number on each die is changed to 1, 2, or 3, but keeping the numbers in the same relative order as in the original set of dice. Repeating this process for each column gives us a uniform columned set of three 4-sided dice, which must be transitive, a contradiction. □

This theorem gives us two more corollaries.

Corollary 21. *A set of columned equitable* 4*-sided dice must contain some evenly matched dice — dice with equal probability of beating each other.*

Proof. By the columned property, every die must have at least six face wins against any other die, since the largest number on each die is guaranteed to be higher than the smaller three numbers on the other dice, etc. Thus for any dice X and Y (assuming $X \succeq Y$ without loss of generality), $P(X>Y) \leq \frac{10}{16}$. However, if $P(X>Y) = \frac{10}{16}$, this would imply that X's largest face is greater than Y's largest face, X's second-largest face is greater than Y's, etc. This contradicts equitability. Thus for every pair of dice X and Y where $X \succeq Y$, we have $P(X>Y) = \frac{9}{16}$. Thus, if there were no evenly matched dice, the set of dice would be uniform, a contradiction. □

Corollary 22. *There are no sets of regular columned* 4*-sided dice.*

Proof. For a set of an odd number of 4-sided dice that is regular and columned, if $P(X>Y) = \frac{10}{16}$, then each face of X is higher than the corresponding face of Y. Thus X beats any die that Y beats, contradicting the regularity assumption. So if $X > Y$, then $P(X>Y) = \frac{9}{16}$. Then, since regularity implies that there are no evenly matched dice, the set of dice must be uniform, a contradiction. □

Note that Theorem 19 and Corollary 20 can in some sense theoretically be generalized to larger sizes of dice. However, the theorem is not as powerful, since it applies only to uniform sets of dice where the winning probability is $\big(\frac{1}{2}n(n+1)-1\big)/n^2$. So, for example, a columned set of 5-sided dice with uniform winning probability $\frac{14}{25}$ is impossible (see the note after Theorem 17), but if we want probability $\frac{13}{25}$, such a set of dice is possible:

A	21	20	12	7	5
B	22	19	11	9	4
C	23	18	15	6	3
D	24	17	14	8	2
E	25	16	13	10	1

But, if we ignore the columned property for the moment, we could ask whether any two of the other properties will imply the last remaining one for a set of 4-sided dice. Two of these implications are special cases of the theorems of Section 2. However, the question of whether a regular and equitable set of 4-sided dice must be uniform is unclear. To this point, we have yet to even find an example of a regular and equitable set of 4-sided dice to test the implication. We suspect that an equitable set of 4-sided dice must include at least two evenly matched dice somewhere, but have not been able to prove this conjecture. (This would be a strengthening of Corollary 21.)

We note here for completeness that our theorems on 4-sided dice are based on the fact that relatively few winning probabilities are possible. For larger sizes of dice, there are multiple possible winning probabilities, and generalizations of Theorem 19 (and its implications) tend not to hold.

The following example gives a set of three 6-sided dice which are columned, uniform, equitable and regular, showing that for sets of dice with a larger even number of sides, the columned property can coexist with the others.

Example 23.

A	18	14	12	7	4	2
B	17	13	11	9	6	1
C	16	15	10	8	5	3

Here, each die beats one other, with probability $\frac{19}{36}$, so the dice are regular and uniform. Moreover, the face sums all equal 57, so the dice are equitable.

However, uniformity does not imply columned for larger sets of dice. The following set of dice is adapted from [Savage 1994]. It is regular, uniform, and equitable, but not columned.

Example 24.

A	18	10	9	8	7	5
B	17	16	15	4	3	2
C	14	13	12	11	6	1

Note that each die still beats one other, and all winning probabilities are still $\frac{19}{36}$, so regularity and uniformity still hold. Also, each die has face sum 57, so the dice are still equitable.

So generally, for larger dice, the columned property is independent of the others.

5. Some tournaments achievable on 3-sided dice

One area of interest in the study of nontransitive dice is finding sets of dice with relatively few sides that realize a given tournament; see [Bozóki 2014] for one example. Given our focus on properties of sets of 3-sided dice, it is interesting to investigate which tournaments are actually realizable on 3-sided dice.

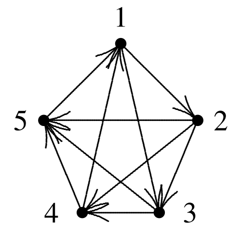

Figure 2. The 2-almost transitive tournament on five vertices.

First, for $1 < j < n$ we define the j-almost transitive tournament on n dice to be the tournament on n dice $X_1, \ldots, X_n$, where $X_i \succeq X_k$ if $i < k$, or if $k = 1$ and $i \in \{n - j + 1, n - j + 2, \ldots, n\}$. Intuitively, this tournament is almost transitive since each die except X_1 beats the dice after it in the list. However, the last j dice beat X_1. Figure 2 shows the 2-almost transitive tournament on five vertices.

Theorem 25. *Given integers $j < n$, there exists a columned set of n 3-sided nontransitive dice which realize the j-almost transitive tournament.*

Proof. We construct a table as follows. The third column contains $3n$ through $2n + 1$, in order, from top to bottom. In the second column, die X_{n-j+1}, which is the lowest-numbered die which beats X_1, has face $2n$. The remaining numbers are added downward from it in order, wrapping around to the top after placing $2n - j + 1$ on die X_n. In the first column, X_1 receives the face 1, then dice X_{n-j+1} through X_n contain the numbers 2 through $j + 1$, in order, and dice X_2 through X_{n-j} receive the numbers $j + 2$ through n, in order.

For example, the 2-almost transitive tournament on seven dice is realized by

X_1	1	12	21
X_2	4	11	20
X_3	5	10	19
X_4	6	9	18
X_5	7	8	17
X_6	2	14	16
X_7	3	13	15

Then one can check easily that the dice from X_2 through X_{n-j} defeat each other transitively since the last two columns are in descending order. However, X_1 loses to X_{n-j+1} through X_n since its smallest two faces are smaller. But dice X_2 through X_{n-j} all beat X_{n-j+1} through X_n because of the first and third columns. Lastly, the dice X_{n-j+1} through X_n beat each other transitively because of the second and third columns. □

For a tournament T, let T' denote the *opposite tournament*, the tournament on the same set of dice with all edges reversed. Note that if we have a set of n k-sided dice labeled with the numbers 1 through nk that realizes a tournament T, we can replace each face label j with the label $nk+1-j$ to get a set of dice that realize T'. In the case of the j-almost transitive tournament T, we can see that T' is the tournament where $X_i \succeq X_k$ if $i > k$, or if $i = 1$ and $k \in \{n-j+1, n-j+2, \ldots, n\}$. We call this the j-upsetter tournament on n vertices, since it can be obtained from a transitive tournament by making the "last-place" die beat the j dice that won against the most dice in the transitive tournament. Thus as a corollary of this theorem, it is always possible to construct a set of 3-sided dice that realize the j-upsetter tournament on n vertices.

We also define the cyclic tournament on $2n+1$ vertices to be the regular tournament on the dice $X_1, \ldots, X_{2n+1}$ where each die beats the next n dice in the list, wrapping around to the beginning as necessary. (This name is given to it because it can be constructed from the data of a cyclic group.)

Theorem 26. *The cyclic tournament on* $2n+1$ *vertices is realizable with* 3*-sided dice.*

Proof. We construct a table as follows. In the first column, we add the numbers in the order $2, 4, 6, \ldots, 2n, 1, 3, \ldots, 2n+1$, i.e., counting by twos mod $2n+1$. In the second column, place $4n+2$ on the same die as the entry 1 and add the remaining numbers in the second column in order downward from that die. Then in the third column we add the numbers $6n+3$ through $4n+3$ starting with $6n+3$ on the first die and moving downward in order.

For example, the cyclic tournament on seven dice is realized by

X_1	2	10	21
X_2	4	9	20
X_3	6	8	19
X_4	1	14	18
X_5	3	13	17
X_6	5	12	16
X_7	7	11	15

To see that this realizes the cyclic tournament, notice that X_{n+1} has the entries $1, 4n+2, 5n+3$. So it loses to the n dice above it because of the first and third columns, but it beats the n dice below it because of the last two columns. Then for any die X_k where $k < n+1$, we can see that X_k will beat X_{k+1} through X_n because of the last two columns. It will beat X_{n+1} through X_{n+k} because of the first and last columns. However, X_{n+k+1}'s first column contains the entry that is one more than X_k's first column, so X_k loses to X_{n+k+1} through X_{2n+1}. The dice after X_{n+1} can be examined similarly. □

Another construction that is very helpful is the "blow-up" of a tournament. (The terminology is borrowed from a vaguely similar concept in algebraic geometry.) Say we have a tournament S on the vertices $Y_1, \ldots, Y_m$, and a tournament T on the vertices $X_1, \ldots, X_n$. We can form a new tournament U on the vertices $Y_1, \ldots, Y_m, X_2, X_n$, where in U, the relation $\succeq$ between X_i and X_j or Y_i and Y_j is the same as in T or S respectively, and $Y_i \succeq X_j$ exactly if $X_1 \succeq X_j$ in T. Intuitively, the vertex X_1 in T has been "blown up" into an entire copy of S, which has the same relationship to the other X_j as X_1 did. We call U the blow-up of T at X_1 with by S.

Theorem 27. *If there is a columned set of k-sided dice that realize S and a set of k-sided dice that realize T, then there is a set of k-sided dice that realize the blowup of T at any vertex X by S.*

Proof. Let X be the die representing the vertex at which we blow up T, and assume it has faces $a_1, a_2, \ldots, a_k$, where $a_1 < a_2 < \cdots < a_k$. We choose a small $\epsilon > 0$. Then for each die Y_i in our realization of S, we replace its smallest label y_{i1} by $a_1 + y_{i1}\epsilon$, its second-lowest face y_{i2} by $a_2 + y_{i2}\epsilon$, etc. (This will of course create a set of dice labeled with numbers other than the usual integers, but we will adjust accordingly at the end of the algorithm.) We claim that the new dice Y_i that we have just constructed will have the same relationships to each other as the original dice realizing S. To see this, note that for faces in the same "column", we have $y_{ij} < y_{kj}$ if and only if $a_j + y_{ij}\epsilon < a_j + y_{kj}\epsilon$. For numbers in different columns, $y_{ij} < y_{km}$ whenever $j < m$. But since $a_j < a_m$ in this case, we will also have $a_j + y_{ij}\epsilon < a_m + y_{km}\epsilon$. Moreover, we can choose ϵ small enough that $a_j + y_{ij}\epsilon < a_j + 1$ always holds, so that every entry $a_j + y_{ij}\epsilon$ is in the same position as a_j relative to the faces of the other dice that realized T. That is, the new die Y_i will beat (or lose to) those other dice in the same way that X did. So, if we remove X from the set of dice and include the altered Y_i's, the resulting set will realize the blowup of T at X by S. And finally, we can alter the actual numbers on the resulting dice set by replacing the lowest number on all the faces by a 1, the second-lowest number by 2, etc., without changing the structure of $\succeq$. □

Example 28. As an example of this theorem, we can take both S and T to be the dice set of (1), and let X be die C. For clarity, we will call the dice in S lowercase a, b, and c. The algorithm (using $\epsilon = .1$) originally gives

A	1	5	9
B	3	4	8
a	2.1	6.5	7.9
b	2.3	6.4	7.8
c	2.2	6.6	7.7

Converting these to the numbers 1 through 15, we obtain

A	1	7	15
B	5	6	14
a	2	9	13
b	4	8	12
c	3	10	11

As another example, the algorithm above for constructing a set of 3-sided dice realizing the 1-almost transitive tournament implicitly makes use of the blow-up algorithm. If we start with the dice set of (1) as T, and S as the dice set

X_1	1	$2n$	$3n$
X_2	2	$2n-1$	$3n-1$
X_3	3	$2n-2$	$3n-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

realizing the transitive tournament, then performing the algorithm to blow up T at B by S will give the same construction of a dice set realizing the 1-almost transitive tournament as Theorem 25.

Notice that these theorems allow us to construct a wide range of 3-sided realizations of tournaments. But in general, not all tournaments are realizable with 3-sided dice. An exhaustive computer search found that all tournaments with up to seven vertices could be realized on 3-sided dice, but that approximately 95 tournaments on eight vertices (out of 6880) could not be realized on 3-sided dice. On nine vertices, there are even some regular tournaments that cannot be realized with 3-sided dice. The question of exactly which tournaments can be realized on 3-sided dice seems difficult but interesting.

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langel@muskingum.edu *Department of Mathematics and Computer Science, Muskingum University, New Concord, OH, United States*

mattd@muskingum.edu *Department of Mathematics and Computer Science, Muskingum University, New Concord, OH, United States*

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