involve

a journal of mathematics

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msp

2018 vol. 11, no. 4

INVOLVE 11:4 (2018)
dx.doi.org/10.2140/involve.2018.11.711

Locating trinomial zeros

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(Communicated by Michael Dorff)

We derive formulas for the number of interior roots (i.e., zeros with modulus less than 1) and exterior roots (i.e., zeros with modulus greater than 1) for trinomials of the form $z^n + z^k - 1$, where $1 \le k \le n-1$. Combined with earlier work by Brilleslyper and Schaubroeck, who focus on unimodular roots (i.e., zeros that lie on the unit circle), we give a complete count of the location of zeros of these trinomials.

1. Introduction

The investigation of zeros of analytic functions has a long and rich history, with many important results focusing on specialized cases. Indeed, the study of zeros of trinomials dates to the 19th century, and a recent paper by Melman [2012] gives historical references in addition to providing information on the location of zeros. Even more recently, [Brilleslyper and Schaubroeck 2014], which won a Pólya award, investigated trinomials of the form

$$p(z) = z^n + z^k - 1 \quad (n \ge 2,\ 1 \le k \le n-1). \tag{1}$$

Their main result characterizes the *unimodular roots* (i.e., zeros that lie on the unit circle) of $p(z)$:

Theorem 1. *Let $p(z) = z^n + z^k - 1$ and let $g = \gcd(n, k)$. If 6 divides $n/g + k/g$, then p has exactly $2g$ unimodular roots, occurring in conjugate pairs z_m and $\bar{z}_m$, determined by $z_m = \exp[i(\pi/(3g) + 2\pi m/g)]$, where $0 \le m \le g-1$.*

In that paper, they called for the discovery of a formula (involving n and k) that would calculate the number of *interior roots* (i.e., zeros with modulus less than 1) of these trinomials. In [Brilleslyper and Schaubroeck $\ge$ 2018] they developed a conjecture,

$$\text{number of interior roots} = 2g\left\lfloor \frac{n+k-g}{6g} \right\rfloor + g,$$

and proved it for the special case when $k = 1$.

MSC2010: primary 30C15, 97I80; secondary 11-02.
Keywords: trinomials, complex analysis, Diophantine equations, zeros of functions.

Here we show that their conjecture is correct in general. Specifically we prove, for $1 \le k \le n-1$, the equivalent formula

$$\text{number of interior roots} = 2g\left\lceil \frac{n+k}{6g} \right\rceil - g. \tag{2}$$

Our proof proceeds in three steps.

First, we show that any interior root must lie in what we call an *interior region*, that any such region contains at most one root, and that the maximum number of these regions matches (2). Next, we show that a similar situation holds for *exterior roots* (i.e., zeros with modulus greater than 1) with respect to *exterior regions*, where the maximum number of these regions matches (3), given by

$$\text{number of exterior roots} = n - 2g\left\lfloor \frac{n+k}{6g} \right\rfloor - g. \tag{3}$$

Finally, we show that adding together the number of unimodular roots (if any), the maximum number of interior regions, and the maximum number of exterior regions results in n, the degree of the trinomial, so that these regions contain exactly one root.

We begin by analyzing where interior roots must be located. To do so, we generally follow the approach in [Brilleslyper and Schaubroeck $\geq$ 2018], but with some modifications. Throughout, the term *trinomial* and the notation $p(z)$ designate a function as defined in (1).

2. The location of interior roots

In what follows we suppose $p(z_0) = 0$ for some z_0 with $|z_0| < 1$.

2.1. ***Native zones for interior roots.*** The assumption that $p(z_0) = 0$ leads to the equation $z_0^k(z_0^{n-k}+1) = 1$. Using the additional assumption that $|z_0| < 1$ and taking the modulus of both sides reveal that $|z_0^k| < 1$ and $|z_0^{n-k}+1| > 1$. Thus, z_0^{n-k} must lie outside the circle $|z+1| = 1$, and z_0^k must lie inside the circle $|z| = 1$. But if $|z_0^k| < 1$, then $|z_0^{n-k}| < 1$ as well, so z_0^{n-k} must also lie inside the circle $|z| = 1$. The two circles intersect at points whose arguments are $\pm\frac{2}{3}\pi$, so $\operatorname{Arg}(z_0^{n-k}) \in \left(-\frac{2}{3}\pi, \frac{2}{3}\pi\right)$. It follows that the point z_0 itself must lie inside one of $n-k$ possible disjoint regions, which we dub *native zones*:

$$N_m = \left\{ re^{i\theta} : \theta \in \left(-\frac{2\pi}{3(n-k)} + m\frac{2\pi}{(n-k)}, \frac{2\pi}{3(n-k)} + m\frac{2\pi}{(n-k)} \right) \right\}, \tag{4}$$

where $0 < r < 1$ and $m \in \mathbb{Z}$.

Although there are only $n-k$ distinct native zones N_m, we allow the index m to range over the integers. Doing so will assist us later in counting the number of these zones satisfying certain restrictions.

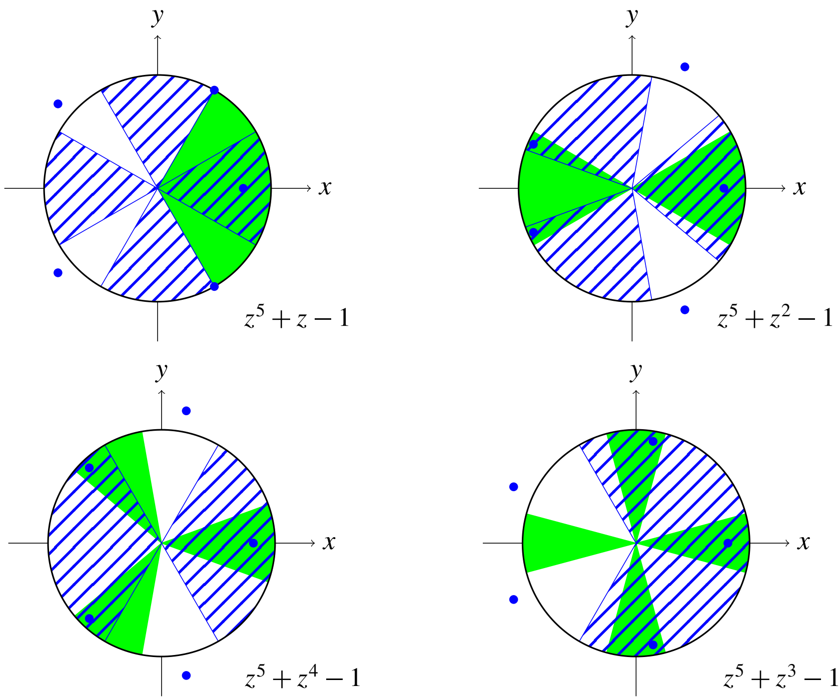

Figure 1. The unit disk with native zones (hatched), echo zones (shaded), and roots (large dots) for the trinomials $z^5 + z^k - 1$, where $1 \le k \le 4$.

2.2. ***Echo zones for interior roots.*** We can get further information on the location of z_0 by considering a related polynomial $q(z)$ defined by

$$q(z) = -z^n p(1/z) = z^n - z^{n-k} - 1.$$

A straightforward calculation reveals that $p(z) = 0$ if and only if $q(1/\bar{z}) = 0$.

Let $w_0 = 1/\bar{z}_0$, and note that $\operatorname{Arg}(z_0) = \operatorname{Arg}(w_0)$. Thus, z_0 and w_0 are echos of each other across the unit circle, and are zeros, respectively, of $p(z)$ and $q(z)$.

Write $q(w_0) = {w_0}^n - {w_0}^{n-k} - 1 = 0$ as $w_0^{n-k}({w_0}^k - 1) = 1$. Taking the modulus of both sides reveals that $|w_0^{n-k}| > 1$ (because $|z_0| < 1$) and $|{w_0}^k - 1| < 1$. Using an analysis similar to that which led to the definition of native zones enables us to conclude that $\operatorname{Arg}(w_0^k) = \operatorname{Arg}(z_0^k) \in \left(-\frac{1}{3}\pi, \frac{1}{3}\pi\right)$. It follows that the point z_0 itself must lie inside one of k possible disjoint regions E_j, which we call *echo zones*:

$$E_j = \left\{ re^{i\theta} : \theta \in \left(-\frac{\pi}{3k} + j\frac{2\pi}{k}, \frac{\pi}{3k} + j\frac{2\pi}{k} \right) \right\}, \tag{5}$$

where $0 < r < 1$ and $j \in \mathbb{Z}$.

As with the native zones, we allow the index j for the echo zones E_j to range over the integers.

2.3. ***Interior regions for interior roots.*** The preceding analysis shows that any interior root must lie in a nonempty intersection of a native zone and an echo zone, which we call an *interior region*. Figure 1 depicts this result for the trinomials $z^n + z^k - 1$, where $n = 5$ and $1 \le k \le 4$. Note that every interior root is in an interior region, and, in the case of Figure 1 (upper-left), there are $2g = 2$ unimodular roots as guaranteed by Theorem 1. Further, extending the radii of native and echo zones indicates that every exterior root is in neither a native nor an echo zone. The next section shows more precisely that these roots must be located in what we call *exterior regions*.

3. The location of exterior roots

Under the hypothesis that $p(z_0) = 0$, where $|z_0| > 1$, the same process for analyzing interior roots can be used to show that all exterior roots belong to an intersection of an *exterior native zone* and an *exterior echo zone*, defined respectively as

$$EN_m = \left\{re^{i\theta} : \theta \in \left(\frac{2\pi}{3(n-k)} + m\frac{2\pi}{(n-k)},\ \frac{4\pi}{3(n-k)} + m\frac{2\pi}{(n-k)}\right)\right\}, \tag{6}$$

where $1 < r < \infty$, $m \in \mathbb{Z}$; and

$$EE_j = \left\{re^{i\theta} : \theta \in \left(\frac{\pi}{3k} + j\frac{2\pi}{k},\ \frac{5\pi}{3k} + j\frac{2\pi}{k}\right)\right\}, \tag{7}$$

where $1 < r < \infty$, $j \in \mathbb{Z}$. As with the corresponding native and echo zones, we allow m and j to range over the integers.

We call any nonempty intersection of (6) and (7) an *exterior region*.

4. Upper bounds for roots

The last two sections collectively show that every interior root must belong to an interior region, and every exterior root must belong to an exterior region. In this section we establish that each such region contains at most one root.

In proving (2) for the case when $k = 1$, Brilleslyper and Schaubroeck demonstrated that exactly one root of $p(z)$ resides in each of the disjoint angular regions

$$R_a = \left\{re^{i\theta} : \theta \in \left(\frac{2a\pi}{n} - \frac{\pi}{2n},\ \frac{2a\pi}{n} + \frac{\pi}{2n}\right)\right\}, \tag{8}$$

where $0 < r < 2$ and $0 \le a \le n-1$.

They called these regions *Rouché sectors* [Brilleslyper and Schaubroeck $\ge$ 2018], an appropriate choice because their demonstration makes creative use of Rouché's theorem, which can be found in almost any standard text for a first course in complex analysis [Mathews and Howell 2012, pp. 340–341]. For completeness we state the theorem here.

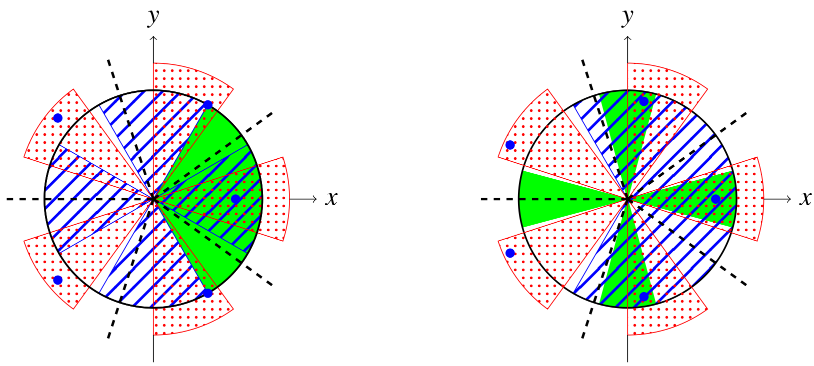

Figure 2. Native zones (hatched), echo zones (shaded), Rouché sectors (dotted), and roots (large dots) for $z^5 + z - 1$ (left) and $z^5 + z^4 - 1$ (right). The dashed lines are midway between the Rouché sectors.

Theorem 2 (Rouché's theorem). *Let Γ be a simple closed positively oriented contour in $\mathbb{C}$, and let f and g be analytic functions in a simply connected domain that contains Γ. If $|f(z) - g(z)| < |g(z)|$ for all $z \in \Gamma$, then f and g have the same number of zeros inside Γ.*

The demonstration that $p(z)$ has a zero (henceforth root) in any sector R_a comes from applying Rouché's theorem to the functions $f(z) = p(z)$ and $g(z) = z^n - 1$ evaluated on the boundary of the sector defined in (8). Each sector is centered around only one n-th root of unity, so $g(z)$ has exactly one root in each. Therefore, $p(z)$ has exactly one root in each Rouché sector.

Figure 2 illustrates this situation for the trinomials $z^5 + z - 1$ and $z^5 + z^4 - 1$, where all interior roots lie in the intersection of an interior region and a Rouché sector, and all exterior roots lie in the intersection of an exterior region and a Rouché sector. In each case the number of interior and exterior regions match, respectively, (2) and (3).

Now, if an interior region contained more than one root, then that region would have to intersect at least two Rouché sectors, and for some integer a contain one of the rays $\{z = re^{i\theta_a} : 0 < r < 1\}$, where $\theta_a = \pi/n + 2\pi a/n$, which is midway between the respective Rouché sectors (see Figure 2).

Suppose that some ray $z = re^{i\theta_a}$ were in an interior region. Then, for some integers m and j, we have $re^{i\theta_a} \in N_m$ and $re^{i\theta_a} \in E_j$ for $0 < r < 1$. According to the definitions of N_m and E_j, see (4) and (5), we thus get the inequalities

$$-\frac{2\pi}{3(n-k)} + m\frac{2\pi}{(n-k)} < \frac{\pi}{n} + a\frac{2\pi}{n} < \frac{2\pi}{3(n-k)} + m\frac{2\pi}{(n-k)},$$

that is,

$$-\frac{5}{3} < -\frac{k}{n} + 2a - a\frac{2k}{n} - 2m < -\frac{1}{3}, \tag{9}$$

if θ_a were in a native zone, and

$$-\frac{\pi}{3k} + j\frac{2\pi}{k} < \frac{\pi}{n} + a\frac{2\pi}{n} < \frac{\pi}{3k} + j\frac{2\pi}{k},$$

that is,

$$-\frac{1}{3} < \frac{k}{n} + a\frac{2k}{n} - 2j < \frac{1}{3}, \tag{10}$$

if θ_a were in an echo zone. Combining (9) and (10) gives

$$-2 < 2a - 2n - 2j < 0 \quad \text{or} \quad -1 < a - n - j < 0,$$

which is impossible because j, m, and a are integers.

By the same process we can determine that no ray $z = re^{i\theta_a}$ is in an exterior region, so that each exterior region has at most one root.

Thus, an upper bound for the number of interior and exterior roots is, respectively, the number of interior and exterior regions. The next few sections establish that the maximum number of these regions matches (2) and (3).

5. Counting interior regions

Each native and echo zone has the general form $\{re^{i\theta} : \alpha < \theta < \beta,\ 0 < r < 1\}$. To simplify language we will call the ray $z = re^{i\beta}$, where $0 < r < 1$, the *right border* of the given zone. (For exterior zones, of course, $1 < r < \infty$.) With this understanding, we proceed to count how many interior regions there are for a given trinomial $p(z)$, where a working assumption will be $\gcd(n, k) = 1$. In a subsequent section we show how to extend this assumption to the case when $\gcd(n, k) = g > 1$.

Recall that an interior region consists of a nonempty intersection $N_m \cap E_j$ of a native and echo zone. Figure 3 illustrates that there are three cases to consider for such an intersection: the right border of an echo zone belongs to a native zone (Figure 3, left), the right border of a native zone belongs to an echo zone (Figure 3, center) or their right borders coalign (Figure 3, right). Our task is to count the interior regions in each case.

Case 1: The right border of an echo zone belongs to a native zone (Figure 3, left).

Then, by (4) and (5), for some $j,\ m \in \mathbb{Z}$,

$$\begin{gathered} -\frac{2\pi}{3(n-k)} + m\frac{2\pi}{(n-k)} < \frac{\pi}{3k} + j\frac{2\pi}{k} < \frac{2\pi}{3(n-k)} + m\frac{2\pi}{(n-k)} \quad \text{or} \\ -\frac{n+k}{6} < j(n-k) - mk < \frac{3k-n}{6}. \end{gathered} \tag{11}$$

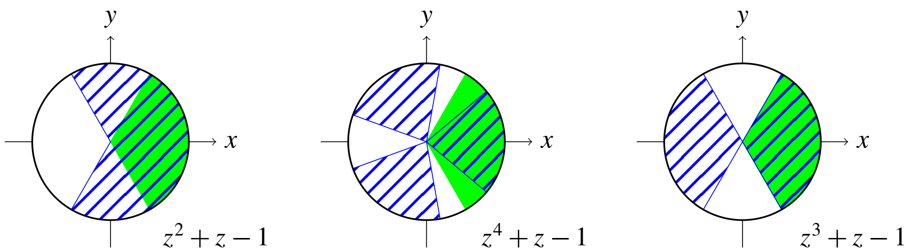

Figure 3. Trinomials illustrating that either the right border of an echo zone belongs to a native zone (left), the right border of a native zone belongs to an echo zone (center), or their right borders coalign (right).

To count the interior regions in this category, we first determine all values of m and j satisfying (11). By a standard result in number theory (see, for example, [Uspensky and Heaslet 1939, pp. 54–57]) we know that, because $\gcd(n-k, k) = 1$, the Diophantine equation $j(n-k) - mk = c$ has a solution j_c, m_c for any integer $c \in \left(-\frac{1}{6}(n+k), \frac{1}{6}(3k-n)\right)$. Furthermore, the set of all solutions is given by

$$j = j_c + kt \quad \text{and} \quad m = m_c + (n-k)t \quad \text{for} \quad t \in \mathbb{Z}. \tag{12}$$

According to (4) and (5), $E_{j_c} = E_{j_c+kt}$ and $N_{m_c} = N_{m_c+(n-k)t}$ for all $t \in \mathbb{Z}$. Hence, from solution set (12), we see that to every integer $c \in \left(-\frac{1}{6}(n+k), \frac{1}{6}(n-3k)\right)$ there corresponds exactly one interior region $N_{m_c} \cap E_{j_c}$. In other words, the maximum number of interior regions in this category — and thus the maximum number of interior roots — is the number of integers between $-\frac{1}{6}(n+k)$ and $\frac{1}{6}(3k-n)$. The number of integers in an open interval (a, b) for $a, b \in \mathbb{R}$ is the difference between the last integer and first integer plus 1, that is, $(\lceil b \rceil - 1) - (\lfloor a \rfloor + 1) + 1$. Combining that fact with the result that, for $x \in \mathbb{R}$, $\lfloor -x \rfloor = -\lceil x \rceil$, yields a formula for the number of integers in the interval $\left(-\frac{1}{6}(n+k), \frac{1}{6}(3k-n)\right)$, and thus the maximum number of interior regions for Case 1:

$$\left(\left\lceil \frac{3k-n}{6} \right\rceil - 1\right) - \left(\left\lfloor -\frac{n+k}{6} \right\rfloor + 1\right) + 1 = \left\lceil \frac{3k-n}{6} \right\rceil + \left\lceil \frac{n+k}{6} \right\rceil - 1. \tag{13}$$

Cases 2 *and* 3: The right border of a native zone belongs to an echo zone, or their right borders coalign (Figure 3, center and right, respectively).

Then, for some $j, m \in \mathbb{Z}$,

$$\begin{gathered} -\frac{\pi}{3k} + j\frac{2\pi}{k} < \frac{2\pi}{3(n-k)} + m\frac{2\pi}{(n-k)} \le \frac{\pi}{3k} + j\frac{2\pi}{k} \quad \text{or} \\ -\frac{n+k}{6} < mk - j(n-k) \le \frac{n-3k}{6}. \end{gathered} \tag{14}$$

By using (14) and the same analysis as in Case 1, we find that there is exactly one interior region for each integer in the interval $\left(-\frac{1}{6}(n+k), \frac{1}{6}(n-3k)\right]$. The last integer in this interval is $\left\lfloor \frac{1}{6}(n-3k) \right\rfloor$ and the first integer is $\left\lfloor -\frac{1}{6}(n+k) \right\rfloor + 1$. Therefore, the number of integers in the interval $\left(-\frac{1}{6}(n+k), \frac{1}{6}(n-3k)\right]$, and thus the maximum number of interior regions in this category, is

$$\left\lfloor \frac{n-3k}{6} \right\rfloor - \left(\left\lfloor -\frac{n+k}{6} \right\rfloor + 1\right) + 1 = -\left\lceil \frac{3k-n}{6} \right\rceil + \left\lceil \frac{n+k}{6} \right\rceil. \tag{15}$$

Combining the cases: Adding together (13) and (15) gives the desired formula for the maximum number of interior regions, and therefore the maximum number of interior roots when $\gcd(n, k) = 1$:

$$2\left\lceil \frac{n+k}{6} \right\rceil - 1. \tag{16}$$

6. Counting exterior regions

Again using the assumption that $\gcd(n, k) = 1$, we now obtain counts for exterior regions. As with interior regions, we have three cases to consider: the right border of an exterior echo zone (7) belongs to an exterior native zone (6), the right border of an exterior native zone belongs to an exterior echo zone, or their right borders coalign.

With the same techniques used in the previous section, we find that, in the first case, we must count the integers in the interval

$$\left(\frac{n+k}{6} - n + k, \frac{n+3k}{6} - n + k\right).$$

The identities $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ and $\lceil x+n \rceil = \lceil x \rceil + n$ (valid for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$) assist in obtaining the following count:

$$\left(\left\lceil \frac{n+3k}{6} - n + k \right\rceil - 1\right) - \left(\left\lfloor \frac{n+k}{6} - n + k \right\rfloor + 1\right) + 1 = \left\lceil \frac{n+3k}{6} \right\rceil - \left\lfloor \frac{n+k}{6} \right\rfloor - 1. \tag{17}$$

For the last two cases combined we must count the integers in the interval

$$\left(\frac{n+k}{6} - k, -\frac{n+3k}{6} + n - k\right].$$

Floor and ceiling function identities then assist in yielding the following amount:

$$\left\lfloor -\frac{n+3k}{6} + n - k \right\rfloor - \left(\left\lfloor \frac{n+k}{6} - k \right\rfloor + 1\right) + 1 = -\left\lceil \frac{n+3k}{6} \right\rceil + n - \left\lfloor \frac{n+k}{6} \right\rfloor. \tag{18}$$

Adding together the counts described in (17) and (18) reveals that the maximum number of exterior regions is

$$n - 2\left\lfloor \frac{n+k}{6} \right\rfloor - 1, \tag{19}$$

which is thus an upper bound for the maximum number of exterior roots when $\gcd(n,k)=1$.

7. Verifying the general formulas

For the interior roots of $p(z)=z^n+z^k-1$, where $\gcd(n,k)=g>1$, we appeal to the related polynomial $\tilde{p}(z)=z^{n/g}+z^{k/g}-1$. From [Brilleslyper and Schaubroeck 2014, Lemma 2], we know that the roots of $p(z)$ are in g-to-one correspondence with the roots of $\tilde{p}(z)$, and this correspondence does not disrupt the classification of roots into interior, unimodular, or exterior categories. Since $\gcd(n/g,k/g)=1$, we can use n/g and n/k, respectively, in place of n and k in (16) to get the maximum number of interior roots for $\tilde{p}(z)$:

$$2\left\lceil\frac{n/g+k/g}{6}\right\rceil-1=2\left\lceil\frac{n+k}{6g}\right\rceil-1.$$

The maximum number of interior roots for $p(z)$, then, is $2g\lceil(n+k)/(6g)\rceil-g$, which is exactly (2).

Using the same procedure, it can be shown that, when $\gcd(n,k)=g>1$, (19) morphs to give $n-2g\lfloor(n+k)/(6g)\rfloor-g$ as the maximum number of exterior roots for $p(z)$, which is exactly (3).

To complete our analysis we note that, when there are no unimodular roots, (2) and (3), when added together, give the maximum number of roots for $p(z)$:

$$\left(2g\left\lceil\frac{n+k}{6g}\right\rceil-g\right)+\left(n-2g\left\lfloor\frac{n+k}{6g}\right\rfloor-g\right). \tag{20}$$

When $p(z)$ has unimodular roots, Theorem 1 guarantees that the maximum number of roots it has is

$$2g+\left(2g\left\lceil\frac{n+k}{6g}\right\rceil-g\right)+\left(n-2g\left\lfloor\frac{n+k}{6g}\right\rfloor-g\right). \tag{21}$$

But according to Theorem 1, unimodular roots occur precisely when $6g$ divides $n+k$. Thus, $\lceil(n+k)/(6g)\rceil=\lfloor(n+k)/(6g)\rfloor+1$ in (20), and $\lceil(n+k)/(6g)\rceil=\lfloor(n+k)/(6g)\rfloor$ in (21). In both cases, then, the expressions sum to n, which equals the total number of roots for $p(z)$. Because interior and exterior regions are the only possible locations for interior and exterior roots, the maximum numbers of interior and exterior roots as expressed in (2) and (3) must be attained.

The enumeration of interior, exterior, and unimodular roots of trinomials $p(z)$ is now complete. For convenience, we summarize the results.

Theorem 3. *For $n\geq 2$, $1\leq k\leq n-1$, and $g=\gcd(n,k)$, the trinomial $p(z)=z^n+z^k-1$ has $2g\lceil(n+k)/(6g)\rceil-g$ interior roots, $n-2g\lfloor(n+k)/(6g)\rfloor-g$ exterior roots, and, when $6g$ divides $n+k$, it has $2g$ unimodular roots.*

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Received: 2017-07-21 Accepted: 2017-08-29

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2018 vol. 11 no. 4