involve
a journal of mathematics

On the minuscule representation of type B_n

William J. Cook and Noah A. Hughes

msp

2018 vol. 11, no. 5

INVOLVE 11:5 (2018)
dx.doi.org/10.2140/involve.2018.11.721

On the minuscule representation of type B_n

William J. Cook and Noah A. Hughes

(Communicated by Ravi Vakil)

We study the action of the Weyl group of type B_n acting as permutations on the set of weights of the minuscule representation of type B_n (also known as the spin representation). Motivated by a previous work, we seek to determine when cycle structures alone reveal the irreducibility of these minuscule representations. After deriving formulas for the simple reflections viewed as permutations, we perform a series of computer-aided calculations in GAP. We are then able to establish that, for certain ranks, the irreducibility of the minuscule representation cannot be detected by cycle structures alone.

1. Introduction

The original motivation for this project was to extend results found in [Cook et al. 2005]. In that paper the authors present a constructive method for solving the inverse problem in differential Galois theory. This problem seeks to determine if certain groups can appear as differential Galois groups of systems of linear differential equations and, if so, given that group, determine such a system of equations.

In [Cook et al. 2005] the authors present a construction which relies on the existence of minuscule modules whose irreducibility can be detected by examining the cycle structures of the corresponding Weyl group viewed as permutations of weights. While each simple Lie algebra has infinitely many isomorphism classes of finite-dimensional irreducible representations, not every simple Lie algebra possesses a minuscule representation. Those which do, have only a handful.

Minuscule representations have the interesting property that all of their weights lie in a single Weyl group orbit. This then implies that all of the weight spaces are 1-dimensional. The irreducibility of such a module is guaranteed by the transitive action of the Weyl group. We set out to find when this transitivity (and thus irreducibility) can be seen from the cycle structures of the Weyl group elements (viewed as permutations) alone.

MSC2010: primary 17B10; secondary 20F55.
Keywords: Lie algebra, minuscule representation, Weyl group.

The authors in [Cook et al. 2005] were able to show that each algebra of type A_n ($n \geq 1$), C_n ($n \geq 3$), D_n ($n \geq 4$), E_6, or E_7 possesses a minuscule representation having the desired property. Since E_8, F_4, and G_2 have no minuscule representations at all, these cases must be discarded. This leaves type B_n as the final case to be considered. Using calculations performed in Maple (a computer algebra system), the authors were able to show that B_2, B_3, B_5, and B_7 have a conforming minuscule representation. They also showed that B_4's irreducibility cannot be seen from cycle structures alone. The status of the other type-B_n cases were left open.

In this paper, we focus on simple Lie algebras of type B_n. Such algebras have only one minuscule representation which is also known as the spin representation. After some introductory material, we explicitly determine the action of the Weyl group of type B_n on the weights of its minuscule representation. We then produce results obtained from calculations performed in [GAP 2017]; our code can be found in the online supplement. We are able to show that the irreducibility of the minuscule representation of type B_n can be detected by cycle structures alone when $n = 1, 2, 3, 5$, and 7 and that irreducibility cannot be detected when $n = 4, 6, 8, 9, \dots, 14$. We conjecture that this continues to be true for all higher ranks as well.

2. Simple Lie algebras

We give a brief account of the background needed to discuss minuscule representations. We recommend [Erdmann and Wildon 2006] for a gentle introduction to this material or the texts [Humphreys 1972] and [Carter 2005] for more complete discussions.

A *Lie algebra* is a vector space $\mathfrak{g}$ (over $\mathbb{C}$) equipped with a bilinear multiplication $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, which is alternating, i.e., $[x, x] = 0$ for all $x \in \mathfrak{g}$, and satisfies the Jacobi identity $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathfrak{g}$. For each $g \in \mathfrak{g}$ we define $\mathrm{ad}(g) : \mathfrak{g} \to \mathfrak{g}$ to be left multiplication by g: $\mathrm{ad}(g)(x) = [g, x]$. A *subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ which is closed under the Lie bracket; i.e., $\mathfrak{h} \subseteq \mathfrak{g}$ such that for all $x, y \in \mathfrak{h}$ we have $[x, y] \in \mathfrak{h}$. An *ideal* $\mathfrak{i}$ of $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ which absorbs multiplication by elements of $\mathfrak{g}$; i.e., $\mathfrak{i} \subseteq \mathfrak{g}$ such that for all $x \in \mathfrak{i}$ and $g \in \mathfrak{g}$ we have $[g, x] \in \mathfrak{i}$. We call $\mathfrak{g}$ *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. A nonabelian Lie algebra with no proper nontrivial ideals is called *simple*. This means that $\mathfrak{g}$ is simple if $[\mathfrak{g}, \mathfrak{g}] \neq \mathbf{0}$ and if $\mathfrak{i}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{i} = \mathbf{0}$ or $\mathfrak{g}$.

As an example, $\mathbb{R}^3$ equipped with the familiar cross product is a 3-dimensional simple Lie algebra (over the field of real numbers $\mathbb{R}$). If we let $\mathfrak{gl}_n$ denote the $n \times n$ complex matrices, then $\mathfrak{gl}_n$ becomes the *general linear* Lie algebra when given the *commutator bracket* $[A, B] = AB - BA$. The set of all trace-zero $n \times n$ complex matrices is called the *special linear* Lie algebra $\mathfrak{sl}_n$. It is a subalgebra of $\mathfrak{gl}_n$ and turns out to be simple when $n \geq 2$.

Let $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a linear map between two Lie algebras. We call φ a *homomorphism* if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}_1$. Of course, a bijective homomorphism is an *isomorphism*.

One of the early triumphs of Lie theory was Killing and Cartan's classification of all finite-dimensional simple Lie algebras (over $\mathbb{C}$). Killing and Cartan were able to show that each finite-dimensional simple Lie algebra was isomorphic to one of the algebras on their list:

$$A_n \ (n \geq 1), \quad B_n \ (n \geq 2), \quad C_n \ (n \geq 3), \quad D_n \ (n \geq 4),$$
$$E_6, \ E_7, \ E_8, \quad F_4, \quad \text{and} \quad G_2.$$

Algebras of types A through D are called *classical algebras*. Those of types E, F, and G are called *exceptional algebras*. We refer the reader to [Erdmann and Wildon 2006] for an accessible introduction to this classification.

A *Cartan subalgebra* $\mathfrak{h}$ of a simple Lie algebra $\mathfrak{g}$ is a subalgebra which is nilpotent, i.e.,

$$\underbrace{[[\cdots[[\mathfrak{h}, \mathfrak{h}], \mathfrak{h}], \ldots], \mathfrak{h}]}_{k-\text{times}} = \mathbf{0}$$

for some integer $k > 0$, and self-normalizing, i.e., if $x \in \mathfrak{g}$, $y \in \mathfrak{h}$, and $[x, y] \in \mathfrak{h}$ then $x \in \mathfrak{h}$. Equivalently, a Cartan subalgebra is a maximal toral subalgebra (a *toral* subalgebra is a subalgebra $\mathfrak{h}$ such that for all $h \in \mathfrak{h}$, the linear endomorphism $\text{ad}(h) : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable). Every Cartan subalgebra of a finite-dimensional simple Lie algebra $\mathfrak{g}$ has the same dimension. This dimension is called the *rank* of the simple Lie algebra.

Since all toral subalgebras $\mathfrak{h}$ are abelian, we have that for all $x, y \in \mathfrak{h}$, the maps $\text{ad}(x)$ and $\text{ad}(y)$ commute and so the space of endomorphisms $\text{ad}(\mathfrak{h})$ can be simultaneously diagonalized. Thus $\mathfrak{g}$ decomposes into a collection of simultaneous eigenspaces for $\text{ad}(\mathfrak{h})$ for any toral subalgebra $\mathfrak{h}$. By choosing $\mathfrak{h}$ to be maximal toral, our eigenspaces are in some sense maximally refined.

For what follows, let $\mathfrak{g}$ be a simple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $n = \dim(\mathfrak{h})$ be the rank of $\mathfrak{g}$. Since $\text{ad}(\mathfrak{h})$ is simultaneously diagonalizable, $\mathfrak{g} = \prod_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{h}^* = \{f : \mathfrak{g} \to \mathbb{C} \mid f \text{ is linear}\}$ is the dual space of $\mathfrak{h}$ and $\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h, g] = \alpha(h)g \text{ for all } h \in \mathfrak{h}\}$ when $\alpha \in \mathfrak{h}^*$. When nontrivial, $\mathfrak{g}_\alpha$ is a simultaneous eigenspace corresponding to the eigenvalue $\alpha(h)$ for each $h \in \mathfrak{h}$. Since $\mathfrak{h}$ is abelian and self-normalizing, $\mathfrak{g}_{\mathbf{0}} = \mathfrak{h}$. If $\mathbf{0} \neq \alpha \in \mathfrak{h}^*$ and $\mathfrak{g}_\alpha \neq \mathbf{0}$, we call α a *root* and $\mathfrak{g}_\alpha$ a *root space* of $\mathfrak{g}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$.

Given a set of roots Δ, there exists a subset $\Pi \subseteq \Delta$ such that each root can be expressed as a nonpositive or nonnegative integral linear combination of elements of Π. In this case we call the elements of Π *simple roots*. A root system may have many equivalent collections of simple roots. The cardinality of a set of simple roots is exactly the rank of $\mathfrak{g}$ (i.e., the dimension of $\mathfrak{h}$). Let us fix such a set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\} \subseteq \Delta$. So for each $\alpha \in \Delta$ there exists $c_1, \dots, c_n \in \mathbb{Z}$ such that $\alpha = c_1\alpha_1 + \cdots + c_\ell\alpha_n$ with either all $c_i \geq 0$ (for a *positive root*) or all $c_i \leq 0$ (for a *negative root*).

3. The Weyl group and irreducible modules

The simple roots, $\Pi = \{\alpha_1, \dots, \alpha_n\}$, form a basis for $\mathfrak{h}^*$. The *fundamental weights* $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ form another important basis for $\mathfrak{h}^*$. The root and weight bases are related by the *Cartan matrix* of $\mathfrak{g}$. In particular, if $A = (a_{ij})_{1\leq i,j\leq n}$ is the Cartan matrix, then $\alpha_i = a_{i1}\lambda_1 + a_{i2}\lambda_2 + \cdots + a_{in}\lambda_n$ for $1 \leq i \leq n$.

For each $1 \leq i \leq n$, we define $\sigma_i : \mathfrak{h}^* \to \mathfrak{h}^*$ by $\sigma_i(\lambda_j) = \lambda_j - \delta_{ij}\alpha_i$ and extend linearly (where δ_{ij} is the Kronecker delta). The map σ_i is called the *simple reflection* associated with the simple root α_i. Let $\mathfrak{W}(\mathfrak{g}) = \langle \sigma_1, \sigma_2, \dots, \sigma_n \rangle$ be the group generated by the simple reflections (generated as a subgroup of, for example, $\mathrm{GL}(\mathfrak{h}^*)$). This is called the *Weyl group* of $\mathfrak{g}$.

A (finite-dimensional) vector space M (over $\mathbb{C}$) equipped with a bilinear $\mathfrak{g}$-action $(g, \boldsymbol{v}) \mapsto g \cdot \boldsymbol{v}$ is a $\mathfrak{g}$-*module* if $[x, y] \cdot \boldsymbol{v} = x \cdot (y \cdot \boldsymbol{v}) - y \cdot (x \cdot \boldsymbol{v})$ for all $x, y \in \mathfrak{g}$ and $\boldsymbol{v} \in M$. A homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(M)$ (where $\mathfrak{gl}(M)$ is equipped with the commutator bracket) is called a *representation*. It is not hard to show that every module gives rise to a representation and vice versa. Specifically, given a module action or representation, one can define the other structure as $x \cdot \boldsymbol{v} = (\varphi(x))(\boldsymbol{v})$. For what follows, we will treat the words "module" and "representation" as synonyms.

Let $\varphi : M_1 \to M_2$ be a linear map between two $\mathfrak{g}$-modules. If $\varphi(g \cdot \boldsymbol{v}) = g \cdot \varphi(\boldsymbol{v})$ for all $g \in \mathfrak{g}$ and $\boldsymbol{v} \in M_1$, then φ is a $\mathfrak{g}$-module map. A bijective module map is called a ($\mathfrak{g}$-module) isomorphism.

A subspace closed under the action of $\mathfrak{g}$ is called a *submodule*. A nontrivial module ($M \neq \mathbf{0}$) which has no nontrivial proper submodules (if N is a submodule, then $N = \mathbf{0}$ or $N = M$) is called an *irreducible* module. If M is a $\mathfrak{g}$-module and $\lambda \in \mathfrak{h}^*$, we define $M_\lambda = \{\boldsymbol{v} \in M \mid h \cdot \boldsymbol{v} = \lambda(h)\boldsymbol{v} \text{ for all } h \in \mathfrak{h}\}$. If $M_\lambda \neq \mathbf{0}$, we say that M_λ is a *weight space* (whose elements are *weight vectors*) with *weight* λ. Just as $\mathfrak{g}$ is a direct sum of root spaces, $\mathfrak{g}$-modules are direct sums of weight spaces: $M = \coprod_{\lambda\in\mathfrak{h}^*} M_\lambda$.

Let M be an irreducible $\mathfrak{g}$-module. There exists a (unique) weight $\lambda \in \mathfrak{h}^*$ of M such that given any other weight $\mu \in \mathfrak{h}^*$ we have $\mu = \lambda - \sum_{i=1}^n b_i\alpha_i$, where $b_i \in \mathbb{Z}$ and $b_i \geq 0$. So every other weight is obtained by subtracting certain collections of positive roots from this weight. Such a weight, λ, is unique and is called the *highest weight* of M. If $\lambda \in \mathfrak{h}^*$ and there exists $c_i \in \mathbb{Z},\ c_i \geq 0$ such that $\lambda = \sum_{i=1}^n c_i\lambda_i$ (the λ_i's are the fundamental weights), then λ is a *dominant integral weight*.

Highest weights of finite-dimensional irreducible modules are dominant integral. Conversely, each dominant integral weight is the highest weight of some finite-dimensional irreducible module. Two irreducible modules with the same highest weight are isomorphic, so we have a bijection between the set of dominant integral weights and the isomorphism classes of finite-dimensional irreducible modules.

Let λ be a dominant integral weight for some simple Lie algebra of type X_n. We denote the irreducible highest-weight X_n-module with highest weight λ by $L(X_n, \lambda)$ or just $L(\lambda)$ when the algebra is understood.

4. Minuscule representations

There are many equivalent ways of defining minuscule weights. In fact, six equivalent conditions are given in [Bourbaki 2005, Chapter VIII, Section 7.3]. The following definition best fits our purposes:

Definition 4.1. Suppose $L(\lambda)$ is an irreducible finite-dimensional $\mathfrak{g}$-module with nonzero highest weight $\lambda \in \mathfrak{h}^*$. Then λ is a *minuscule weight* and $L(\lambda)$ is a *minuscule module* if the Weyl group $\mathfrak{W}(\mathfrak{g})$ acts transitively on the set of weights of $L(\lambda)$, i.e., $\mathfrak{W}(\mathfrak{g}) \cdot \lambda$ is the set of all weights of $L(\lambda)$.

Given a $\mathfrak{g}$-module M, we know M decomposes into weight spaces M_λ for $\lambda \in \mathfrak{h}^*$. The dimension of a weight space M_λ is called the *multiplicity* of the weight λ.

If $\mu = w \cdot \lambda$ for $\mu, \lambda \in \mathfrak{h}^*$ and $w \in \mathfrak{W}(\mathfrak{g})$, then M_μ and M_λ have the same dimension. Therefore, weights lying in an orbit of the Weyl group all have the same multiplicity. Thus since the weights of a minuscule module all lie in a single Weyl group orbit, the weight spaces in a minuscule module must all have the same multiplicity as the highest weight. But the highest-weight space for an irreducible module is always 1-dimensional. Therefore, all the weight spaces in a minuscule module are 1-dimensional and the dimension of a minuscule module is the same as the number of its weights.

Both [Humphreys 1972, Section 13, p. 72, Exercise 13] and [Bourbaki 2005, Chapter VIII, Section 7.3, p. 132] give the following table of minuscule weights for finite-dimensional simple Lie algebras:

type	A_n	B_n	C_n	D_n	E_6	E_7
minuscule weights	$\lambda_1, \ldots, \lambda_n$	λ_n	λ_1	$\lambda_1,\ \lambda_{n-1},\ \lambda_n$	$\lambda_1,\ \lambda_6$	λ_7

Note that algebras of types F_4, E_8, and G_2 have no minuscule representations.

For further information about minuscule representations we direct the reader to either [Bourbaki 2005, Chapter VII, Section 7.3] or the book [Green 2013], which is entirely devoted to the study of minuscule representations and contains a wealth of information about them.

5. Strictly transitive sets

Recall that the original motivation for this project was to extend results found in [Cook et al. 2005]. Following that paper, let us denote the conjugacy class of a permutation σ by $\bar{\sigma}$. We say a collection of conjugacy classes, $\{C_1, \ldots, C_\ell\}$ of the symmetric group S_m is *strictly transitive* if for any choice of $\tau_i \in C_i$ $(i = 1, \ldots, \ell)$ the subgroup generated by $\tau_1, \ldots, \tau_\ell$ acts transitively. Lemma 3.7 in [Cook et al. 2005] states that $\{C_1, \ldots, C_\ell\}$ is strictly transitive if and only if for some (and therefore any) set of representatives $\{\tau_1, \ldots, \tau_\ell\}$ (with $\tau_i \in C_i$) and for any $1 \le j \le m-1$, there is an element τ_k leaving no set of cardinality j invariant.

As an example, working in S_4, $\{\overline{(1234)}\}$ is strictly transitive by itself (leaving only the empty set and $\{1, 2, 3, 4\}$ invariant). Also, $\{\overline{(123)}, \overline{(12)(34)}\}$ is strictly transitive since an element from $\overline{(123)}$ only allows invariant sets of cardinalities 0, 1, 3, and 4 whereas elements in $\overline{(12)(34)}$ only allow invariant sets of sizes 0, 2, and 4. So putting these two criteria together, cardinalities 1, 2, and 3 are ruled out. On the other hand, $\{\overline{(1)}, \overline{(12)}, \overline{(12)(34)}\}$ is not strictly transitive since selecting the permutations (1), (12), and (12)(34) allows the set $\{1, 2\}$ (of cardinality 2) to remain invariant.

Recall that the Weyl group permutes the weights of a representation. Thus if $\mathfrak{g}$ is a simple Lie algebra and M is a $\mathfrak{g}$-module with $\dim(M) = m$, then $\mathfrak{W}(\mathfrak{g})$ can be viewed as a subgroup of the symmetric group S_m, say

$$\mathfrak{W}(\mathfrak{g}) \cong W \subseteq S_m.$$

For the construction in [Cook et al. 2005] to work for a Lie group with corresponding Lie algebra $\mathfrak{g}$, the authors needed an irreducible representation where the conjugacy classes of the corresponding permutation representation of the Weyl group form a strictly transitive set.

To have any hope of W having a strictly transitive set of conjugacy classes we must have that the weights of M lie in a single orbit of $\mathfrak{W}(\mathfrak{g}) \cong W$. This means that the construction cannot go through unless M is a minuscule representation. This in turn implies that the construction cannot work for algebras of type E_8, F_4, or G_2 (where there are not minuscule representations).

Now let M (with $\dim(M) = m$) be a minuscule $\mathfrak{g}$-module with corresponding Weyl group W (viewed as permutations of the weights of M). The conjugacy classes of W form a strictly transitive set if and only if the cycle structures in W do not allow invariant sets of cardinality j for $1 \le j \le m-1$. Essentially this means that the conjugacy classes of W form a strictly transitive set only if the irreducibility of M is visible directly from the cycle structures of W. So for the construction in [Cook et al. 2005] to go through we need a representation whose irreducibility can be established by examining the cycle structures of the Weyl group elements acting as permutations on the weights of this representation.

6. Seeing irreducibility from cycle structures

The problem of identifying a minuscule representation with corresponding Weyl group action possessing a strictly transitive set of conjugacy classes was solved in [Cook et al. 2005] for a simple Lie algebra of type A_n, C_n, D_n, E_6, or E_7. Again, algebras of types F_4, E_8, and G_2 have no minuscule representations so there are no strictly transitive sets associated with representations there. We will briefly review the results found in [Cook et al. 2005]. For more detail we refer the reader to Section 4 of that paper.

Recall that $L(A_n, \lambda_i)$ (where $n = 1, 2, \dots$) is minuscule for all $i = 1, \dots, n$. Focusing on $i = 1$, the minuscule module $L(A_n, \lambda_1)$ (where $n = 1, 2, \dots$) is $(n+1)$-dimensional. It turns out that the Coxeter element (i.e., the product of all of the simple reflections) of the Weyl group is represented by an $(n+1)$-cycle, since such a cycle leaves only sets of cardinalities 0 and $n + 1$ invariant. Thus we have a strictly transitive set, and so the irreducibility of $L(A_n, \lambda_1)$ is visible from cycle structures alone.

For type C_n (where $n = 3, 4, \dots$), the only minuscule module is the $(2n)$-dimensional representation $L(C_n, \lambda_1)$. As with type A_n, it turns out that the Coxeter element is represented by a $(2n)$-cycle. This means that the irreducibility of $L(C_n, \lambda_1)$ is visible from cycle structures alone.

Each algebra of type D_n (where $n = 4, 5, \dots$) possesses three minuscule modules: $L(D_n, \lambda_1)$, $L(D_n, \lambda_{n-1})$, and $L(D_n, \lambda_n)$. The first of these, $L(D_n, \lambda_1)$, is $(2n)$-dimensional. If the weights are suitably labeled by $1, 2, \dots, 2n$, it turns out that the product of the first $n - 1$ simple reflections yields the permutation $\tau_1 = (1, 2, \dots, n)(n + 1, \dots, 2n)$ and the Coxeter element is $\tau_2 = (1, \dots, n - 1, n+1, \dots, 2n-1)(n, 2n)$. Representatives from the class $\bar{\tau}_1$ leave sets of cardinalities $0, n$, and $2n$ invariant whereas representatives from $\bar{\tau}_2$ leave sets of cardinalities $0, 2, 2n - 2$, and $2n$ invariant. Since $n \geq 4$, intersecting these two criteria leaves just 0 and $2n$. Therefore, $\{\bar{\tau}_1, \bar{\tau}_2\}$ is a strictly transitive set and so the irreducibility of $L(D_n, \lambda_1)$ is visible from cycle structures alone.

The algebra of type E_6 possess two minuscule modules: $L(E_6, \lambda_1)$ and $L(E_6, \lambda_6)$. These are both 27-dimensional. The corresponding permutation representations of the Weyl group possess elements τ_1 and τ_2 with respective cycle structures $12 + 12 + 3$ (two 12-cycles and a 3-cycle) and $9 + 9 + 9$ (three 9-cycles). This means that elements from $\bar{\tau}_2$ only allow invariant sets of cardinality 0, 9, 18, and 27. Notice that cardinalities 9 and 18 are not allowed by elements of $\bar{\tau}_1$. Therefore, $\{\bar{\tau}_1, \bar{\tau}_2\}$ is a strictly transitive set.

The only minuscule module of E_7 is the 56-dimensional representation $L(E_7, \lambda_7)$. The corresponding permutation representation of the Weyl group possesses elements τ_1 and τ_2 with respective cycle structures $18 + 18 + 18 + 2$ (three 18-cycles and a transposition) and $14+14+14+14$ (four 14-cycles). This means that elements from $\bar{\tau}_2$ only allow invariant sets of cardinality 0, 14, 28, 42 and 56. Notice that cardinalities 14, 28 and 42 are not allowed by elements of $\bar{\tau}_1$. Therefore, $\{\bar{\tau}_1, \bar{\tau}_2\}$ is a strictly transitive set.

Finally, algebras of type B_n (where $n = 2, 3, \dots$) only have one minuscule representation: $L(B_n, \lambda_n)$. This is a 2^n-dimensional representation and the focus of this project. In [Cook et al. 2005], it is stated that when $n = 2, 3, 5$, and 7 the Weyl group corresponding to the minuscule module $L(B_n, \lambda_n)$ possesses a strictly transitive set. However, the Weyl group in the case $n = 4$ does not. For other ranks the problem is left open.

7. The action of $\mathfrak{W}(B_n)$ on the minuscule representation

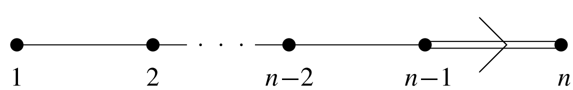
We now focus on simple Lie algebras of type B_n (where $n = 2, 3, \dots$). Algebras of type B_n can be realized as the *special orthogonal* Lie algebras $\mathfrak{so}_{2n+1}$. Specifically, letting I_n denote the $n \times n$ identity matrix, we have that the special orthogonal Lie algebra is the following set of $(2n+1) \times (2n+1)$ complex matrices:

$$\mathfrak{so}_{2n+1} = \left\{ X \in \mathfrak{gl}_{2n+1} \;\middle|\; X^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix} X \right\}.$$

This is a $(2n^2+n)$-dimensional simple Lie algebra of rank n. Let us fix a collection of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and corresponding fundamental weights $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ for this algebra. We have that the Cartan matrix (the change of basis matrix from Λ to Π) is

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -2 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

with corresponding Dynkin diagram



Explicitly we have the following relationships between our fundamental weights and simple roots:

$$\alpha_1 = 2\lambda_1 - \lambda_2, \quad \alpha_2 = -\lambda_1 + 2\lambda_2 - \lambda_3, \quad \dots,$$

$$\alpha_{n-2} = -\lambda_{n-3} + 2\lambda_{n-2} - \lambda_{n-1}, \quad \alpha_{n-1} = -\lambda_{n-2} + 2\lambda_{n-1} - 2\lambda_n, \quad \alpha_n = -\lambda_{n-1} + 2\lambda_n.$$

Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis for $\mathbb{R}^n$. In addition, consider $\alpha_i = 4(\epsilon_i - \epsilon_{i+1})$ for $i = 1, \ldots, n-1$ and $\alpha_n = 4\epsilon_n$. By Lemma 5.1 in [Green 2008], $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a set of simple roots for a root system of type B_n.

Recall, see [Humphreys 1972, Section 13.2, Table 1, p. 69], that for type B_n,

$$\lambda_i = \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \cdots + \alpha_{n-1} + \alpha_n) \quad \text{for } i = 1, \ldots, n-1,$$
$$\lambda_n = \tfrac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n).$$

In terms of the standard basis we have that $\lambda_i = 4(\epsilon_1 + \cdots + \epsilon_i)$ for $i = 1, \ldots, n-1$ and $\lambda_n = 2(\epsilon_1 + \cdots + \epsilon_n)$. This in turn implies $\epsilon_1 = \frac{1}{4}\lambda_1$, $\epsilon_j = \frac{1}{4}\lambda_j - \frac{1}{4}\lambda_{j-1}$ (where $j = 2, \ldots, n-1$), and $\epsilon_n = \frac{1}{2}\lambda_n - \frac{1}{4}\lambda_{n-1}$.

Recall that the Weyl group is generated by the simple reflections: $\sigma_i(\lambda_j) = \lambda_j - \delta_{ij}\alpha_i$ $(i = 1, \ldots, n)$. Notice that ϵ_j only involves λ_{j-1} and λ_j for $j = 2, \ldots, n$ and ϵ_1 only involves λ_1. Therefore, since $\sigma_i(\lambda_k) = \lambda_k$ for $k \neq i$, we have $\sigma_i(\epsilon_j) = \epsilon_j$ if $j \neq i$ or $i + 1$.

For $1 < i < n$,

$$\begin{aligned}\sigma_i(\epsilon_i) &= \sigma_i\left(\tfrac{1}{4}\lambda_i - \tfrac{1}{4}\lambda_{i-1}\right) = \tfrac{1}{4}\sigma_i(\lambda_i) - \tfrac{1}{4}\sigma_i(\lambda_{i-1}) \\ &= \tfrac{1}{4}\lambda_i - \tfrac{1}{4}\alpha_i - \tfrac{1}{4}\lambda_{i-1} = \epsilon_i - \tfrac{1}{4}\alpha_i = \epsilon_i - (\epsilon_i - \epsilon_{i+1}) = \epsilon_{i+1}.\end{aligned}$$

Likewise, $\sigma_i(\epsilon_{i+1}) = \epsilon_i$. Therefore, for $i = 2, \ldots, n-1$, we see σ_i switches ϵ_i and ϵ_{i+1} and leaves the other ϵ_j fixed. A similar calculation shows that σ_1 switches ϵ_1 and ϵ_2 leaving the other basis vectors fixed.

Notice $\sigma_n(\epsilon_j) = \epsilon_j$ for $j = 1, \ldots, n-1$. Finally, consider

$$\begin{aligned}\sigma_n(\epsilon_n) &= \sigma_n\left(\tfrac{1}{2}\lambda_n - \tfrac{1}{4}\lambda_{n-1}\right) = \tfrac{1}{2}\sigma_n(\lambda_n) - \tfrac{1}{4}\sigma_n(\lambda_{n-1}) \\ &= \tfrac{1}{2}\lambda_n - \tfrac{1}{2}\alpha_n - \tfrac{1}{4}\lambda_{n-1} = \epsilon_n - \tfrac{1}{2}\alpha_n = \epsilon_n - 2\epsilon_n = -\epsilon_n.\end{aligned}$$

Thus σ_n leaves all but the last basis vector fixed and switches the sign of the final basis vector.

If we label $\epsilon_1, \ldots, \epsilon_n$ by $1, \ldots, n$, then we have that the Weyl group is acting as signed permutations on $\{\pm 1, \ldots, \pm n\}$. In fact, the permutation representation of the Weyl group $\mathfrak{W}(C_n)$ acting on the weights of the minuscule $L(C_n, \lambda_1)$ can be realized in this way. This is part of the reason it was relatively easy for the authors of [Cook et al. 2005] to resolve the type C_n case.

Even though types B_n and C_n have isomorphic Weyl groups (both groups are isomorphic to the group of signed permutations on $\{1, \ldots, n\}$), the permutation representation of $\mathfrak{W}(B_n)$ acting on the weights of the minuscule representation $L(B_n, \lambda_n)$ is much more complicated than $\mathfrak{W}(C_n)$ acting on the weights of $L(C_n, \lambda_1)$.

Let Ψ be the set of 2^n vectors of the form $(\pm 2, \ldots, \pm 2)$. By Proposition 5.2 in [Green 2008], Ψ is a set of roots for $L(B_n, \lambda_n)$. Notice that

$$\lambda_n = 2(\epsilon_1 + \cdots + \epsilon_n) = (2, \ldots, 2)$$

is the highest weight. We know that $\mathfrak{W}(B_n)$ permutes the elements of Ψ. Consider the signs of the coordinates of an element of Ψ. We can treat these like reversed binary digits (interpret $+$ as 0 and $-$ as 1) then add 1 to this number. For example: $(-2, +2, +2)$ is interpreted as $001_2 + 1 = 2$ and $(+2, -2, -2)$ is interpreted as $110_2 + 1 = 7$.

Then σ_i for $i = 1, \ldots, n-1$ has the effect (after adjusting for the addition of 1) of switching the j and $(j+1)$-th digits of the reversed binary number and σ_n has the effect of flipping the final digit of the reversed binary number. This gives us the following:

Theorem 7.1. *The simple reflections of the Weyl group $\mathfrak{W}(B_n)$ acting on the weights of the minuscule representation $L(B_n, \lambda_n)$ can be represented by the permutations*

$$\sigma_j = \prod_{p=0}^{2^{(n-j-1)}-1} \prod_{k=1}^{2^{j-1}} (p2^{j+1} + 2^{j-1} + k,\ p2^{j+1} + 2^j + k), \quad 1 \le j \le n-1,$$

$$\sigma_n = \prod_{k=1}^{2^{n-1}} (k,\ 2^{n-1} + k).$$

8. Experimental results for type B_n

Using Theorem 7.1 and [GAP 2017], for $n \le 14$, we were able to find complete lists of cycle structures for the elements in $\mathfrak{W}(B_n)$ viewed as permutations of weights of the minuscule module. (Our GAP code can be found in the online supplement.) These lists allowed us to conclude that the cycle structures for types B_n when $n = 1, 2, 3, 5$, and 7 yield strictly transitive sets. Thus the irreducibility of $L(B_n, \lambda_n)$ can be seen from cycle structure alone when $n = 1, 2, 3, 5$, and 7.

The same cannot be concluded for other values of n. Below we elaborate on our method for determining irreducibility from cycle structures by examining the cycle structures of B_n for the ranks $n = 1, 2, 3, 4$, and 5.

Note that, viewed as permutations, $\mathfrak{W}(B_1) = \{(1), (12)\}$. For our purposes we describe the cycle structures in this group by $1 + 1$ for the identity (two 1-cycles) and 2 for the transposition (12) (a single 2-cycle). This identification allows us to read off the possible dimensions of invariant subspaces allowed by each cycle structure. If we can find a cycle structure (or a collection of cycle structures) that only allows for dimensions of 0 and 2^n we know we can conclude irreducibility from the cycle structures alone. In this case, the 2-cycle structure guarantees the irreducibility of our minuscule representation. We will understand why after the following examples.

When $n = 2$, we have $\mathfrak{W}(B_2) = \langle (23), (13)(24) \rangle$ with cycle structures

$$1+1+1+1 \;=\; 1+1+2 \;=\; 2+2 \;=\; 4.$$

So every element in $\mathfrak{W}(B_2)$ viewed as a permutation is of the form four 1-cycles, two 1-cycles and a 2-cycle, two 2-cycles or a 4-cycle. Any partial sum of a type of cycle structure is a possible dimension for an invariant subspace of our minuscule representation allowed by that cycle structure. So the cycle structure $1+1+2$ allows for possible dimensions of 0, 1, 2, $3 = 1+2$ and $4 = 1+1+2$. However, the pair of cycles $2+2$ only allows dimensions 0, 2, and $4 = 2+2$. Critically, we also have that the cycle structure 4 (a 4-cycle) allows for dimensions of only 0 and 4. Hence, we conclude that any invariant subspace of our minuscule representation must be of dimension 0 or 4. So irreducibility of our minuscule representation is visible from examining cycle structures alone.

Next $\mathfrak{W}(B_3) = \langle (23)(67),\ (35)(46),\ (15)(26)(37)(48)\rangle$ and has cycle structures

$$\begin{aligned} 1+1+\cdots+1 &= 1+1+1+1+2+2 = 1+1+3+3 \\ = 2+2+2+2 &= 2+6 = 4+4. \end{aligned}$$

In this case there is no structure of the form $2^3 = 8$ to guarantee irreducibility. Instead we may consider the structures $2+6$ and $4+4$ simultaneously: $2+6$ allows for the possible dimensions 0, 2, 6, and 8, while $4+4$ allows for 0, 4, and 8. These lists of possible dimensions of invariant subspaces intersect at just 0 and 8. Hence, irreducibility follows from cycle structures.

The first case in which this method fails is that of $n = 4$:

$$\begin{aligned} \mathfrak{W}(B_4) = \big\langle &(2,3)(6,7)(10,11)(14,15),\ (3,5)(4,6)(11,13)(12,14), \\ &(5,9)(6,10)(7,11)(8,12),\ (1,9)(2,10)\cdots(8,16)\big\rangle. \end{aligned}$$

In this realization of $\mathfrak{W}(B_4)$ we find the cycle structures

$$\begin{aligned} 1+1+\cdots+1 &= 1+1+\cdots+1+2+2+2+2 \\ = 1+1+2+4+4+4 &= 1+1+1+1+3+3+3+3 \\ = 2+2+\cdots+2 &= 1+1+1+1+2+2+\cdots+2 \\ = 2+2+6+6 &= 4+4+4+4 = 8+8. \end{aligned}$$

Each of these cycle structures allows for an invariant subspace of dimension 8. So even though B_4's minuscule module is irreducible, cycle structures alone will not reveal this to us.

For B_5, we have that $\mathfrak{W}(B_5)$ has cycles structures of the forms $8+8+8+8$ and $2+10+10+10$. The form $8+8+8+8$ only allows for submodules of dimensions 0, 8, 16, 24, and 32, whereas $2+10+10+10$ only allows for submodules of dimensions 0, 2, 10, 12, 20, 22, 30, and 32. Thus, only 0 and 32 are allowed, so irreducibility follows.

Table 1 sums up the results for ranks $6 \le n \le 12$. We see that the cycle structures for B_7 imply the irreducibility of its minuscule representation.

rank	invariant subspace dimensions allowed by cycle structures
6	0, 24, 40, 64
7	0, 128
8	0, 16, 32, 112, 128, 144, 224, 240, 256
9	0, 144, 224, 288, 368, 512
10	0, 64, 144, 224, 240, 320, 400, 464, 480, 544, 560, 624, 704, 784, 800, 880, 960, 1024
11	0, 288, 464, 528, 640, 704, 1344, 1408, 1520, 1584, 1760, 2048
12	0, 48, 112, 176, 224, 288, 352, 400, 464, 528, 576, 640, 704, 752, 816, 880, 928, 992, 1056, 1104, 1168, 1232, 1280, 1344, 1408, 1456, 1520, 1584, 1632, 1696, 1760, 1808, 1872, 1936, 1984, 2048, 2112, 2160, 2224, 2288, 2336, 2400, 2464, 2512, 2576, 2640, 2688, 2752, 2816, 2864, 2928, 2992, 3040, 3104, 3168, 3216, 3280, 3344, 3392, 3456, 3520, 3568, 3632, 3696, 3744, 3808, 3872, 3920, 3984, 4048, 4096
13	0, 624, 704, 1328, 1456, 2160, 2288, 2912, 2992, 3616, 3744, 4448, 4576, 5280, 5904, 6032, 6736, 6864, 7488, 7568, 8192
14	0, 368, 704, 1456, 2160, 2912, 3616, 3696, 4368, 5072, 5152, 5824, 6528, 5200, 6608, 6864, 8064, 8320, 9520, 9776, 9856, 10560, 11232, 11312, 12016, 12688, 12768, 13472, 14224, 14928, 15680, 16016, 16384

Table 1. Summary of results for B_n, where $6 \leq n \leq 12$.

We were not able to get GAP to complete calculations for any higher-rank cases. The problem is that Weyl groups grow very fast as rank is increased. In fact $\mathfrak{W}(B_n)$ is isomorphic to a semidirect product of S_n and $(\mathbb{Z}_2)^n$, so $|\mathfrak{W}(B_n)| = 2^n \cdot n!$. Even at rank 14 we have a group of order $2^{14} \cdot 14!$ acting on a set of $2^{14} = 16384$ weights! However, by randomly sampling $\mathfrak{W}(B_n)$ for ranks of up to $n = 23$, we obtained strong evidence that the number of allowed invariant subspace dimensions blows up as rank is increased. We conjecture that the irreducibility of the minuscule representation cannot be seen from cycle structures alone after rank 7. We found this quite surprising given the nature of the minuscule representations for the other types of algebras.

References

[Bourbaki 2005] N. Bourbaki, *Lie groups and Lie algebras, Chapters 7–9*, Springer, 2005. MR Zbl

[Carter 2005] R. W. Carter, *Lie algebras of finite and affine type*, Cambridge Studies in Advanced Mathematics **96**, Cambridge University Press, 2005. MR Zbl

[Cook et al. 2005] W. J. Cook, C. Mitschi, and M. F. Singer, "On the constructive inverse problem in differential Galois theory", *Comm. Algebra* **33**:10 (2005), 3639–3665. MR Zbl

[Erdmann and Wildon 2006] K. Erdmann and M. J. Wildon, *Introduction to Lie algebras*, Springer, 2006. MR Zbl

[GAP 2017] "GAP – Groups, Algorithms, and Programming", version 4.8.7, 2017, available at http://www.gap-system.org.

[Green 2008] R. M. Green, "Representations of Lie algebras arising from polytopes", *Int. Electron. J. Algebra* **4** (2008), 27–52. MR Zbl

[Green 2013] R. M. Green, *Combinatorics of minuscule representations*, Cambridge Tracts in Mathematics **199**, Cambridge University Press, 2013. Zbl

[Humphreys 1972] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics **9**, Springer, 1972. MR Zbl

Received: 2014-04-23 Revised: 2017-11-06 Accepted: 2017-11-20

cookwj@appstate.edu *Department of Mathematical Sciences, Appalachian State University, Boone, NC, United States*

noah.hughes@uconn.edu *Department of Mathematics, University of Connecticut, Storrs, CT, United States*

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2018 is US $190/year for the electronic version, and $250/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2018 vol. 11 no. 5

On the minuscule representation of type B_n 721
WILLIAM J. COOK AND NOAH A. HUGHES
Pythagorean orthogonality of compact sets 735
PALLAVI AGGARWAL, STEVEN SCHLICKER AND RYAN SWARTZENTRUBER
Different definitions of conic sections in hyperbolic geometry 753
PATRICK CHAO AND JONATHAN ROSENBERG
The Fibonacci sequence under a modulus: computing all moduli that produce a given period 769
ALEX DISHONG AND MARC S. RENAULT
On the faithfulness of the representation of $\mathrm{GL}(n)$ on the space of curvature tensors 775
COREY DUNN, DARIEN ELDERFIELD AND RORY MARTIN-HAGEMEYER
Quasipositive curvature on a biquotient of $\mathrm{Sp}(3)$ 787
JASON DEVITO AND WESLEY MARTIN
Symmetric numerical ranges of four-by-four matrices 803
SHELBY L. BURNETT, ASHLEY CHANDLER AND LINDA J. PATTON
Counting eta-quotients of prime level 827
ALLISON ARNOLD-ROKSANDICH, KEVIN JAMES AND RODNEY KEATON
The k-diameter component edge connectivity parameter 845
NATHAN SHANK AND ADAM BUZZARD
Time stopping for Tsirelson's norm 857
KEVIN BEANLAND, NOAH DUNCAN AND MICHAEL HOLT
Enumeration of stacks of spheres 867
LAUREN ENDICOTT, RUSSELL MAY AND SIENNA SHACKLETTE
Rings isomorphic to their nontrivial subrings 877
JACOB LOJEWSKI AND GREG OMAN
On generalized MacDonald codes 885
PADMAPANI SENEVIRATNE AND LAUREN MELCHER
A simple proof characterizing interval orders with interval lengths between 1 and k 893
SIMONA BOYADZHIYSKA, GARTH ISAAK AND ANN N. TRENK

1944-4176(2018)11:5;1-4