

involve

a journal of mathematics

The Fibonacci sequence under a modulus: computing all
moduli that produce a given period

Alex Dishong and Marc S. Renault



The Fibonacci sequence under a modulus: computing all moduli that produce a given period

Alex Dishong and Marc S. Renault

(Communicated by Kenneth S. Berenhaut)

The Fibonacci sequence $F = 0, 1, 1, 2, 3, 5, 8, 13, \dots$, when reduced modulo m is periodic. For example, $F \bmod 4 = 0, 1, 1, 2, 3, 1, 0, 1, 1, 2, \dots$. The period of $F \bmod m$ is denoted by $\pi(m)$, so $\pi(4) = 6$. In this paper we present an algorithm that, given a period k , produces all m such that $\pi(m) = k$. For efficiency, the algorithm employs key ideas from a 1963 paper by John Vinson on the period of the Fibonacci sequence. We present output from the algorithm and discuss the results.

1. The problem

Consider the usual Fibonacci sequence $F = 0, 1, 1, 2, 3, 5, 8, \dots$, with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. When reduced modulo m , the Fibonacci sequence is periodic. For example, $F \bmod 4 = 0, 1, 1, 2, 3, 1, 0, 1, 1, \dots$. The period of $F \bmod m$ is denoted by $\pi(m)$, so we see that $\pi(4) = 6$. The properties of $\pi(m)$ have been studied extensively; see, e.g., [Gupta et al. 2012; Robinson 1963; Vinson 1963; Wall 1960]. One might ask, of course, if there are any other values of m such that $\pi(m) = 6$. The answer is no (you can verify this by hand), but it turns out that there are 10 different moduli m such that $\pi(m) = 24$ (namely, 6, 9, 12, 16, 18, 24, 36, 48, 72, 144). Our goal is to construct an efficient algorithm that, given a period k , produces all m such that $\pi(m) = k$.

It is instructive to first consider how one might solve the problem by brute force. If $\pi(m) = k$, then $F_k \equiv 0 \pmod{m}$ and $F_{k+1} \equiv 1 \pmod{m}$. That is, m divides both F_k and $F_{k+1} - 1$. For brute force, we fix k , find all common divisors of F_k and $F_{k+1} - 1$, and then apply the π function to these divisors to see which ones produce the desired value of k . Computing $\pi(m)$ is not difficult but it requires factoring m as a product of primes, then factoring $p \pm 1$ for each prime p that divides m . See [Wall 1960] for theorems on $\pi(m)$ and [Flanagan et al. 2015] for an algorithm for $\pi(m)$ (as well as many other facts about the Fibonacci sequence under a modulus).

MSC2010: primary 11B39, 11B50; secondary 11Y55.

Keywords: Fibonacci sequence, period, algorithm.

By employing key ideas from a 1963 paper by John Vinson on the period of the Fibonacci sequence, we were able to produce an algorithm that does not require computing $\pi(m)$. Instead, the moduli we seek can be produced with simple divisibility tests.

2. The algorithm

In this section we present [Theorem 2.1](#) on which our algorithm is based, pseudocode for the algorithm, and some output. In the next section we provide a proof of [Theorem 2.1](#).

First, we note that $\pi(2) = 3$ but it is known that for $m > 2$, $\pi(m)$ must be even. By inspecting a few small cases, it is easy to see that no moduli produce a period of 4, and the smallest even period is 6. Let $L = 2, 1, 3, 4, 7, \dots$ denote the Lucas sequence: $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$. It is well-known that $L_n = F_{2n}/F_n = F_{n-1} + F_{n+1}$.

Theorem 2.1. *Given any even $k \geq 6$:*

- (1) *If $k \equiv 2 \pmod{4}$, then $\pi(m) = k$ if and only if $m \mid L_{k/2}$, and $m \nmid F_q$ for all q such that $q \mid k$ and $q \neq k$.*
- (2) *If $k \equiv 4 \pmod{8}$, then $\pi(m) = k$ if and only if $m \mid F_{k/2}$, and $m \nmid L_{k/4}$, and $m \nmid F_q$ for all q such that $q \mid \frac{k}{2}$ and $q \neq \frac{k}{2}$ or $\frac{k}{4}$.*
- (3) *If $k \equiv 0 \pmod{8}$, then $\pi(m) = k$ if and only if $m \mid F_{k/2}$, and $m \nmid F_q$ for all q such that $q \mid \frac{k}{2}$ and $q \neq \frac{k}{2}$.*

The algorithm follows immediately from the theorem.

Algorithm 2.2. Given an integer $k \geq 2$, to produce the set of all m such that $\pi(m) = k$:

```

Input:  an integer  $k \geq 2$ 
If  $k = 3$ , then return  $\{2\}$ .
If  $k \in \{2, 4\}$  or if  $k$  is odd, then return  $\{\}$ .
If  $k \bmod 4 = 2$ :
    Let  $\mathcal{M} = \{m : m \mid L_{k/2}\}$ .
    Let  $\mathcal{F} = \{F_q : q \mid k \text{ and } q \neq k\}$ .
If  $k \bmod 8 = 4$ :
    Let  $\mathcal{M} = \{m : m \mid F_{k/2} \text{ and } m \nmid L_{k/4}\}$ .
    Let  $\mathcal{F} = \{F_q : q \mid \frac{k}{2} \text{ and } q \neq \frac{k}{2} \text{ and } q \neq \frac{k}{4}\}$ .
If  $k \bmod 8 = 0$ :
    Let  $\mathcal{M} = \{m : m \mid F_{k/2}\}$ .
    Let  $\mathcal{F} = \{F_q : q \mid \frac{k}{2} \text{ and } q \neq \frac{k}{2}\}$ .
Return  $\{m \in \mathcal{M} : m \nmid f \text{ for all } f \in \mathcal{F}\}$ 

```

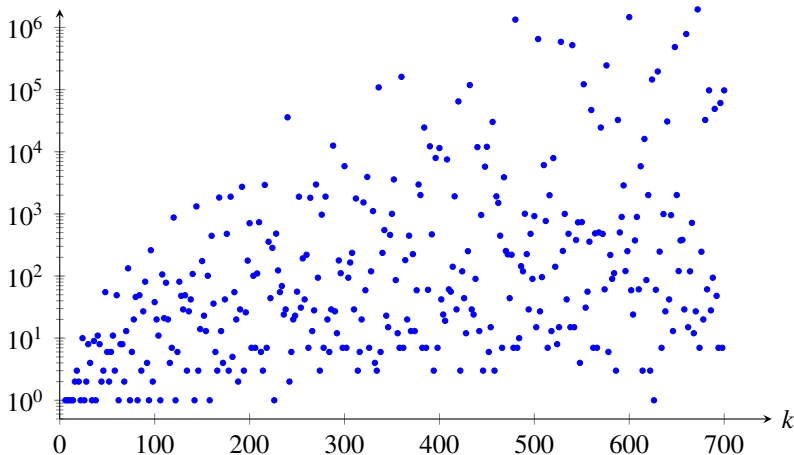


Figure 1. The number of m such that $\pi(m) = k$ for a given k .

Figure 1 shows the results when the algorithm is run on all even k from 6 to 700 and the size of the output set is calculated. The value of k appears on the horizontal axis, and the number of moduli m such that $\pi(m) = k$ is expressed on the vertical axis.

What surprised us most in this study was the incredible number of moduli that can produce a given period. For example, $\pi(m) = 600$ for 1,466,812 different values of m .

Moreover, the algorithm above has much greater speed than simple brute force. When we computed the moduli for all even periods k from 6 to 300, the brute force algorithm took 180.28 seconds, whereas Algorithm 2.2 completed the task in 0.62 seconds. We used the online Sage computer algebra system for our computations [Stein et al. 2016].

3. Proof of Theorem 2.1

The zeros in $F \pmod m$ are evenly spaced. For example, consider $F \pmod 5$:

$$F \pmod 5 = 0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, \dots$$

(To see why the zeros are evenly spaced, we can use the identities

$$F_{s+t} = F_{s-1}F_t + F_sF_{t+1},$$

$$F_{s-t} = (-1)^t(F_sF_{t+1} - F_{s+1}F_t).$$

If $F_s \equiv F_t \equiv 0$, then $F_{s+t} \equiv 0$ and $F_{s-t} \equiv 0$.)

The rank of $F \pmod m$, denoted by $\alpha(m)$, is the least index $i > 0$ such that $F_i \equiv 0 \pmod m$. We can deduce, for example, that if $m \mid F_i$, then $\alpha(m) \mid i$. The

order of $F \pmod m$, denoted by $\omega(m)$, is $\pi(m)/\alpha(m)$ (which is an integer since the zeros are evenly spaced). We see above that $\pi(5) = 20$, $\alpha(5) = 5$, and $\omega(5) = 4$.

It turns out that $\pi(2) = 3$, but for all $m > 2$, $\pi(m)$ must be even. As we see in the mod 5 example, $\alpha(m)$ need not be even. It is a remarkable fact that for any m , $\omega(m) = 1, 2$, or 4 ; this is proven in [Vinson 1963]. In that paper, Vinson studies the relationship between the period, rank, and order. Based on the Vinson paper, Renault was able find several other consequences, and the following theorem is a direct result of Theorem 3.35 and Corollary 3.38 in [Renault 1996].

Theorem 3.1. *For any modulus $m > 2$:*

- (1) $\pi(m) \equiv 2 \pmod 4$ if and only if $\omega(m) = 1$. In this case, $\alpha(m) \equiv 2 \pmod 4$.
- (2) If $\pi(m) \equiv 4 \pmod 8$, then $\omega(m) = 2$ or 4 . In this case, $\alpha(m) \equiv 2 \pmod 4$ or $\alpha(m)$ is odd, respectively.
- (3) If $\pi(m) \equiv 0 \pmod 8$, then $\omega(m) = 2$. In this case, $\alpha(m) \equiv 0 \pmod 4$.

Since $\pi(m)$ is even for $m > 2$, the above theorem describes all possible cases for $\pi(m)$. Also, even though the “in this case” portions follow obviously from their preceding statements, we can use them to draw conclusions. For example, we can see from the theorem that $\alpha(m) \equiv 0 \pmod 4$ if and only if $\pi(m) \equiv 0 \pmod 8$. We proceed now to the proof of Theorem 2.1.

Proof of Theorem 2.1(1). (\Rightarrow) Assume $k \equiv 2 \pmod 4$ and $\pi(m) = k$. Since $k \equiv 2 \pmod 4$, Theorem 3.1 tells us that $\omega(m) = 1$. Thus, $m \nmid F_q$ for all q such that $1 \leq q < k$. In particular, $m \nmid F_q$ for any q such that $q \mid k$ and $q \neq k$.

It remains to show that $m \mid L_{k/2}$. By the fact that $\pi(m) = k$ and the identity $F_{-n} = (-1)^{n+1} F_n$, we see that $F_{k-n} \equiv F_{-n} \equiv (-1)^{n+1} F_n \pmod m$. Then, since $\frac{k}{2}$ is odd,

$$F_{k/2-1} = F_{k-(k/2+1)} \equiv -F_{k/2+1} \pmod m.$$

Consequently, $m \mid F_{k/2-1} + F_{k/2+1}$. But by the identity $L_n = F_{n-1} + F_{n+1}$, this is exactly $m \mid L_{k/2}$, as required.

(\Leftarrow) Assume $k \equiv 2 \pmod 4$ and (a) $m \mid L_{k/2}$ and (b) $m \nmid F_q$ for any q such that $q \mid k$ and $q \neq k$. We must show that $\pi(m) = k$.

By (a), $m \mid F_k$, so $\alpha(m) \mid k$. By (b) we find that in fact, $\alpha(m) = k$. Thus, $\pi(m) = k$, $2k$, or $4k$.

If $\pi(m) = 4k$, then $\omega(m) = 4$ and by Theorem 3.1, $\alpha(m)$ must be odd. However, $\alpha(m) \equiv 2 \pmod 4$, so this can't be the case.

If $\pi(m) = 2k$, then $\pi(m) \equiv 4 \pmod 8$, and so Theorem 2.1(2)(\Rightarrow) implies $m \nmid L_{\pi(m)/4}$; that is, $m \nmid L_{k/2}$. But this contradicts our hypothesis (a) that $m \mid L_{k/2}$, and so $\pi(m) \neq 2k$.

We must conclude that $\pi(m) = k$ and the proof is complete. \square

Proof of Theorem 2.1 (2). (\Rightarrow) Assume $k \equiv 4 \pmod{8}$ and $\pi(m) = k$. Since $\pi(m) \equiv 4 \pmod{8}$, by [Theorem 3.1](#) we know that $\omega(m) = 2$ or 4 . In either case, $m \mid F_{k/2}$ and $m \nmid F_q$ where $q \mid \frac{k}{2}$ and $q \neq \frac{k}{2}, \frac{k}{4}$. Thus, it only remains to prove that $m \nmid L_{k/4}$.

For ease of notation, let $s = F_{k/2+1}$, let $a = F_{k/4+1}$, and observe that $s \not\equiv 1 \pmod{m}$.

Claim 1. $F_{k/4-1} \equiv -sa \pmod{m}$.

Proof of Claim 1. Modulo m , the Fibonacci sequence starting at $F_{k/2}$ is $0, s, s, 2s, 3s, 5s, \dots$, and in general, $F_{k/2+n} \equiv sF_n \pmod{m}$. In particular, $F_{(3k)/4+1} \equiv sa$. The identity $F_{-n} = (-1)^{n+1}F_n$ implies $F_{k-n} \equiv F_{-n} \equiv (-1)^{n+1}F_n \pmod{m}$. Since $\frac{k}{4}$ is odd, we find,

$$F_{k/4-1} \equiv F_{k-(3k)/4+1} \equiv -F_{(3k)/4+1} \equiv -sa \pmod{m}.$$

Claim 2. $(a, m) = 1$.

Proof of Claim 2. We have $(F_{k/4-1}, F_{k/4+1}) = F_{(k/4-1, k/4+1)} = F_2 = 1$. So, there exist integers u and v such that $F_{k/4-1}u + F_{k/4+1}v = 1$. Thus, $-sau + av \equiv 1 \pmod{m}$, and so $a(-su + v) \equiv 1 \pmod{m}$ and we find that a is invertible mod m . That is, $(a, m) = 1$.

Consider the identity $L_n = F_{n-1} + F_{n+1}$. For contradiction,

$$\begin{aligned} m \mid L_{k/4} &\Rightarrow m \mid F_{k/4-1} + F_{k/4+1} \Rightarrow -sa + a \equiv 0 \pmod{m} \\ &\Rightarrow a(1 - s) \equiv 0 \pmod{m} \Rightarrow s \equiv 1 \pmod{m}. \end{aligned}$$

The last implication is due to the fact that $(a, m) = 1$, and we've arrived at a contradiction since $s \not\equiv 1 \pmod{m}$. We conclude $m \nmid L_{k/4}$, as needed.

(\Leftarrow) Assume $k \equiv 4 \pmod{8}$, (a) $m \mid F_{k/2}$, (b) $m \nmid L_{k/4}$, and (c) $m \nmid F_q$ for all $q \mid \frac{k}{2}$ where $q \neq \frac{k}{2}$ or $\frac{k}{4}$. We must prove that $\pi(m) = k$. By (a) and (c), $\alpha(m) = \frac{k}{4}$ or $\frac{k}{2}$. We know that the only possible values for $\omega(m)$ are $1, 2$, or 4 .

Case 1: $\alpha(m) = \frac{k}{4}$.

If $\omega(m) = 2$, then $\pi(m) = \frac{k}{2} \equiv 2 \pmod{4}$. However this contradicts [Theorem 3.1](#) since $\pi(m) \equiv 2 \pmod{4}$ if and only if $\omega(m) = 1$.

If $\omega(m) = 1$, then $\pi(m) = \frac{k}{4} \equiv 1 \pmod{2}$. Again, this contradicts [Theorem 3.1](#) since $\omega(m) = 1$ if and only if $\pi(m) \equiv 2 \pmod{4}$.

Thus, in Case 1 we find that $\omega(m) = 4$ and we conclude $\pi(m) = k$.

Case 2: $\alpha(m) = \frac{k}{2}$.

If $\omega(m) = 4$, then $\pi(m) = 2k \equiv 0 \pmod{8}$. But by [Theorem 3.1](#), if $\pi(m) \equiv 0 \pmod{8}$, then $\omega(m) = 2$, a contradiction.

If $\omega(m) = 1$, then $\pi(m) = \frac{k}{2} \equiv 2 \pmod{4}$. We can now apply [Theorem 2.1](#)(1)(\Rightarrow), and we find $m \mid L_{\pi(m)/2} = L_{k/4}$. However, this contradicts our hypothesis (b).

Thus, in Case 2 we find $\omega(m) = 2$ and we conclude $\pi(m) = k$. □

Proof of Theorem 2.1(3). (\Rightarrow) Assume $k \equiv 0 \pmod{8}$ and $\pi(m) = k$. Since $\pi(m) \equiv 0 \pmod{8}$, [Theorem 3.1](#) tells us that $\omega(m) = 2$, and so $\alpha(m) = \frac{k}{2}$. Thus, $m \mid F_{k/2}$ and $m \nmid F_q$ for any q such that $1 \leq q < \frac{k}{2}$. In particular, $m \nmid F_q$ for all q such that $q \mid \frac{k}{2}$ and $q \neq \frac{k}{2}$, and this direction of the proof is complete.

(\Leftarrow) Assume $k \equiv 0 \pmod{8}$, and (a) $m \mid F_{k/2}$, and (b) $m \nmid F_q$ for all q such that $q \mid \frac{k}{2}$ and $q \neq \frac{k}{2}$. We must prove that $\pi(m) = k$. By (a), we see $\alpha(m) \mid \frac{k}{2}$, and by (b), we deduce that in fact $\alpha(m) = \frac{k}{2}$. Thus $\alpha(m) \equiv 0 \pmod{4}$. By [Theorem 3.1](#), this can only happen when $\omega(m) = 2$. Thus $\pi(m) = k$. \square

References

- [Flanagan et al. 2015] P. Flanagan, M. S. Renault, and J. Updike, “Symmetries of Fibonacci points, mod m ”, *Fibonacci Quart.* **53**:1 (2015), 34–41. [MR](#)
- [Gupta et al. 2012] S. Gupta, P. Rockstroh, and F. E. Su, “Splitting fields and periods of Fibonacci sequences modulo primes”, *Math. Mag.* **85**:2 (2012), 130–135. [MR](#) [Zbl](#)
- [Renault 1996] M. Renault, *The Fibonacci sequence under various moduli*, master’s thesis, Wake Forest University, 1996, available at <http://webspace.ship.edu/msrenault/fibonacci/FibThesis.pdf>.
- [Robinson 1963] D. W. Robinson, “The Fibonacci matrix modulo m ”, *Fibonacci Quart* **1**:2 (1963), 29–36. [MR](#) [Zbl](#)
- [Stein et al. 2016] W. A. Stein et al., *Sage mathematics software*, Version 6.7, Sage Development Team, 2016, available at <http://www.sagemath.org>.
- [Vinson 1963] J. Vinson, “The relation of the period modulo to the rank of apparition of m in the Fibonacci sequence”, *Fibonacci Quart* **1**:2 (1963), 37–45. [MR](#) [Zbl](#)
- [Wall 1960] D. D. Wall, “Fibonacci series modulo m ”, *Amer. Math. Monthly* **67** (1960), 525–532. [MR](#) [Zbl](#)

Received: 2016-06-02

Accepted: 2017-09-09

ajdish@udel.edu

Department of Mathematical Sciences,
University of Delaware, Newark, DE, United States

msrenault@ship.edu

Mathematics Department, Shippensburg University,
Shippensburg, PA, United States

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA	Suzanne Lenhart	University of Tennessee, USA
John V. Baxley	Wake Forest University, NC, USA	Chi-Kwong Li	College of William and Mary, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Robert B. Lund	Clemson University, USA
Martin Bohner	Missouri U of Science and Technology, USA	Gaven J. Martin	Massey University, New Zealand
Nigel Boston	University of Wisconsin, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA	Emil Minchev	Ruse, Bulgaria
Pietro Cerone	La Trobe University, Australia	Frank Morgan	Williams College, USA
Scott Chapman	Sam Houston State University, USA	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Joshua N. Cooper	University of South Carolina, USA	Zuhair Nashed	University of Central Florida, USA
Jem N. Corcoran	University of Colorado, USA	Ken Ono	Emory University, USA
Toka Diagana	Howard University, USA	Timothy E. O'Brien	Loyola University Chicago, USA
Michael Dorff	Brigham Young University, USA	Joseph O'Rourke	Smith College, USA
Sever S. Dragomir	Victoria University, Australia	Yuval Peres	Microsoft Research, USA
Behrouz Emamizadeh	The Petroleum Institute, UAE	Y.-F. S. Pétermann	Université de Genève, Switzerland
Joel Foisy	SUNY Potsdam, USA	Robert J. Plemmons	Wake Forest University, USA
Erin W. Fulp	Wake Forest University, USA	Carl B. Pomerance	Dartmouth College, USA
Joseph Gallian	University of Minnesota Duluth, USA	Vadim Ponomarenko	San Diego State University, USA
Stephan R. Garcia	Pomona College, USA	Bjorn Poonen	UC Berkeley, USA
Anant Godbole	East Tennessee State University, USA	James Propp	U Mass Lowell, USA
Ron Gould	Emory University, USA	József H. Przytycki	George Washington University, USA
Andrew Granville	Université Montréal, Canada	Richard Rebarber	University of Nebraska, USA
Jerrold Griggs	University of South Carolina, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Jim Haglund	University of Pennsylvania, USA	James A. Sellers	Penn State University, USA
Johnny Henderson	Baylor University, USA	Andrew J. Sterge	Honorary Editor
Jim Hoste	Pitzer College, USA	Ann Trenk	Wellesley College, USA
Natalia Hritonenko	Prairie View A&M University, USA	Ravi Vakil	Stanford University, USA
Glenn H. Hurlbert	Arizona State University, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
Charles R. Johnson	College of William and Mary, USA	Ram U. Verma	University of Toledo, USA
K. B. Kulasekera	Clemson University, USA	John C. Wierman	Johns Hopkins University, USA
Gerry Ladas	University of Rhode Island, USA	Michael E. Zieve	University of Michigan, USA

PRODUCTION

Silvio Levy, Scientific Editor


Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2018 is US \$190/year for the electronic version, and \$250/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2018 Mathematical Sciences Publishers

involve

2018

vol. 11

no. 5

On the minuscule representation of type B_n	721
WILLIAM J. COOK AND NOAH A. HUGHES	
Pythagorean orthogonality of compact sets	735
PALLAVI AGGARWAL, STEVEN SCHLICHER AND RYAN SWARTZENTRUBER	
Different definitions of conic sections in hyperbolic geometry	753
PATRICK CHAO AND JONATHAN ROSENBERG	
The Fibonacci sequence under a modulus: computing all moduli that produce a given period	769
ALEX DISHONG AND MARC S. RENAULT	
On the faithfulness of the representation of $GL(n)$ on the space of curvature tensors	775
COREY DUNN, DARIEN ELDERFIELD AND RORY MARTIN-HAGEMEYER	
Quasipositive curvature on a biquotient of $Sp(3)$	787
JASON DEVITO AND WESLEY MARTIN	
Symmetric numerical ranges of four-by-four matrices	803
SHELBY L. BURNETT, ASHLEY CHANDLER AND LINDA J. PATTON	
Counting eta-quotients of prime level	827
ALLISON ARNOLD-ROKSANDICH, KEVIN JAMES AND RODNEY KEATON	
The k -diameter component edge connectivity parameter	845
NATHAN SHANK AND ADAM BUZZARD	
Time stopping for Tsirelson's norm	857
KEVIN BEANLAND, NOAH DUNCAN AND MICHAEL HOLT	
Enumeration of stacks of spheres	867
LAUREN ENDICOTT, RUSSELL MAY AND SIENNA SHACKLETTE	
Rings isomorphic to their nontrivial subrings	877
JACOB LOJEWSKI AND GREG OMAN	
On generalized MacDonal codes	885
PADMAPANI SENEVIRATNE AND LAUREN MELCHER	
A simple proof characterizing interval orders with interval lengths between 1 and k	893
SIMONA BOYADZHIYSKA, GARTH ISAAK AND ANN N. TRENK	