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moduli that produce a given period

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The Fibonacci sequence under a modulus: computing all moduli that produce a given period

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The Fibonacci sequence $F = 0, 1, 1, 2, 3, 5, 8, 13, \ldots$, when reduced modulo m is periodic. For example, $F \bmod 4 = 0, 1, 1, 2, 3, 1, 0, 1, 1, 2, \ldots$. The period of $F \bmod m$ is denoted by $\pi(m)$, so $\pi(4) = 6$. In this paper we present an algorithm that, given a period k, produces all m such that $\pi(m) = k$. For efficiency, the algorithm employs key ideas from a 1963 paper by John Vinson on the period of the Fibonacci sequence. We present output from the algorithm and discuss the results.

1. The problem

Consider the usual Fibonacci sequence $F = 0, 1, 1, 2, 3, 5, 8, \ldots$, with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. When reduced modulo m, the Fibonacci sequence is periodic. For example, $F \bmod 4 = 0, 1, 1, 2, 3, 1, 0, 1, 1, \ldots$. The period of $F \bmod m$ is denoted by $\pi(m)$, so we see that $\pi(4) = 6$. The properties of $\pi(m)$ have been studied extensively; see, e.g., [Gupta et al. 2012; Robinson 1963; Vinson 1963; Wall 1960]. One might ask, of course, if there are any other values of m such that $\pi(m) = 6$. The answer is no (you can verify this by hand), but it turns out that there are 10 different moduli m such that $\pi(m) = 24$ (namely, 6, 9, 12, 16, 18, 24, 36, 48, 72, 144). Our goal is to construct an efficient algorithm that, given a period k, produces all m such that $\pi(m) = k$.

It is instructive to first consider how one might solve the problem by brute force. If $\pi(m) = k$, then $F_k \equiv 0 \pmod{m}$ and $F_{k+1} \equiv 1 \pmod{m}$. That is, m divides both F_k and $F_{k+1} - 1$. For brute force, we fix k, find all common divisors of F_k and $F_{k+1} - 1$, and then apply the π function to these divisors to see which ones produce the desired value of k. Computing $\pi(m)$ is not difficult but it requires factoring m as a product of primes, then factoring $p \pm 1$ for each prime p that divides m. See [Wall 1960] for theorems on $\pi(m)$ and [Flanagan et al. 2015] for an algorithm for $\pi(m)$ (as well as many other facts about the Fibonacci sequence under a modulus).

By employing key ideas from a 1963 paper by John Vinson on the period of the Fibonacci sequence, we were able to produce an algorithm that does not require computing $\pi(m)$. Instead, the moduli we seek can be produced with simple divisibility tests.

2. The algorithm

In this section we present Theorem 2.1 on which our algorithm is based, pseudo-code for the algorithm, and some output. In the next section we provide a proof of Theorem 2.1.

First, we note that $\pi(2) = 3$ but it is known that for $m > 2$, $\pi(m)$ must be even. By inspecting a few small cases, it is easy to see that no moduli produce a period of 4, and the smallest even period is 6. Let $L = 2, 1, 3, 4, 7, \ldots$ denote the Lucas sequence: $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$. It is well-known that $L_n = F_{2n}/F_n = F_{n-1} + F_{n+1}$.

Theorem 2.1. *Given any even $k \geq 6$:*

(1) *If $k \equiv 2 \pmod 4$, then $\pi(m) = k$ if and only if $m \mid L_{k/2}$, and $m \nmid F_q$ for all q such that $q \mid k$ and $q \neq k$.*

(2) *If $k \equiv 4 \pmod 8$, then $\pi(m) = k$ if and only if $m \mid F_{k/2}$, and $m \nmid L_{k/4}$, and $m \nmid F_q$ for all q such that $q \mid \frac{k}{2}$ and $q \neq \frac{k}{2}$ or $\frac{k}{4}$.*

(3) *If $k \equiv 0 \pmod 8$, then $\pi(m) = k$ if and only if $m \mid F_{k/2}$, and $m \nmid F_q$ for all q such that $q \mid \frac{k}{2}$ and $q \neq \frac{k}{2}$.*

The algorithm follows immediately from the theorem.

Algorithm 2.2. Given an integer $k \geq 2$, to produce the set of all m such that $\pi(m) = k$:

```
Input:  an integer k ≥ 2
If k = 3, then return {2}.
If k ∈ {2, 4} or if k is odd, then return {}.
If k mod 4 = 2:
   Let M = {m : m | L_{k/2}}.
   Let F = {F_q : q | k and q ≠ k}.
If k mod 8 = 4:
   Let M = {m : m | F_{k/2} and m ∤ L_{k/4}}.
   Let F = {F_q : q | k/2 and q ≠ k/2 and q ≠ k/4}.
If k mod 8 = 0:
   Let M = {m : m | F_{k/2}}.
   Let F = {F_q : q | k/2 and q ≠ k/2}.
Return {m ∈ M : m ∤ f for all f ∈ F}
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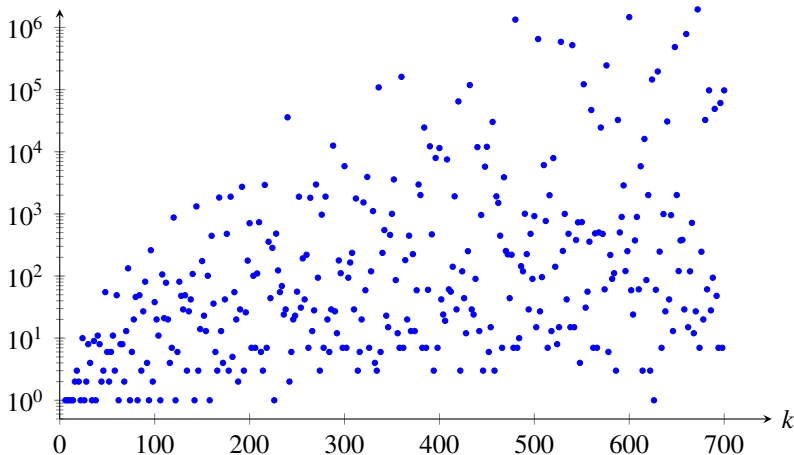
Figure 1. The number of m such that $\pi(m) = k$ for a given k.

Figure 1 shows the results when the algorithm is run on all even k from 6 to 700 and the size of the output set is calculated. The value of k appears on the horizontal axis, and the number of moduli m such that $\pi(m) = k$ is expressed on the vertical axis.

What surprised us most in this study was the incredible number of moduli that can produce a given period. For example, $\pi(m) = 600$ for 1,466,812 different values of m.

Moreover, the algorithm above has much greater speed than simple brute force. When we computed the moduli for all even periods k from 6 to 300, the brute force algorithm took 180.28 seconds, whereas Algorithm 2.2 completed the task in 0.62 seconds. We used the online Sage computer algebra system for our computations [Stein et al. 2016].

3. Proof of Theorem 2.1

The zeros in F mod m are evenly spaced. For example, consider F mod 5:

$$F \bmod 5 = 0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, \ldots.$$

(To see why the zeros are evenly spaced, we can use the identities

$$F_{s+t} = F_{s-1}F_t + F_s F_{t+1},$$
$$F_{s-t} = (-1)^t (F_s F_{t+1} - F_{s+1}F_t).$$

If $F_s \equiv F_t \equiv 0$, then $F_{s+t} \equiv 0$ and $F_{s-t} \equiv 0$.)

The *rank* of F mod m, denoted by $\alpha(m)$, is the least index $i > 0$ such that $F_i \equiv 0 \pmod{m}$. We can deduce, for example, that if $m \mid F_i$, then $\alpha(m) \mid i$. The

order of $F \bmod m$, denoted by $\omega(m)$, is $\pi(m)/\alpha(m)$ (which is an integer since the zeros are evenly spaced). We see above that $\pi(5) = 20$, $\alpha(5) = 5$, and $\omega(5) = 4$.

It turns out that $\pi(2) = 3$, but for all $m > 2$, $\pi(m)$ must be even. As we see in the mod 5 example, $\alpha(m)$ need not be even. It is a remarkable fact that for any m, $\omega(m) = 1$, 2, or 4; this is proven in [Vinson 1963]. In that paper, Vinson studies the relationship between the period, rank, and order. Based on the Vinson paper, Renault was able find several other consequences, and the following theorem is a direct result of Theorem 3.35 and Corollary 3.38 in [Renault 1996].

Theorem 3.1. *For any modulus $m > 2$:*

(1) $\pi(m) \equiv 2 \pmod 4$ *if and only if $\omega(m) = 1$. In this case, $\alpha(m) \equiv 2 \pmod 4$.*

(2) *If $\pi(m) \equiv 4 \pmod 8$, then $\omega(m) = 2$ or 4. In this case, $\alpha(m) \equiv 2 \pmod 4$ or $\alpha(m)$ is odd, respectively.*

(3) *If $\pi(m) \equiv 0 \pmod 8$, then $\omega(m) = 2$. In this case, $\alpha(m) \equiv 0 \pmod 4$.*

Since $\pi(m)$ is even for $m > 2$, the above theorem describes all possible cases for $\pi(m)$. Also, even though the "in this case" portions follow obviously from their preceding statements, we can use them to draw conclusions. For example, we can see from the theorem that $\alpha(m) \equiv 0 \pmod 4$ if *and only if* $\pi(m) \equiv 0 \pmod 8$. We proceed now to the proof of Theorem 2.1.

Proof of Theorem 2.1(1). ($\Rightarrow$) Assume $k \equiv 2 \pmod 4$ and $\pi(m) = k$. Since $k \equiv 2 \pmod 4$, Theorem 3.1 tells us that $\omega(m) = 1$. Thus, $m \nmid F_q$ for all q such that $1 \le q < k$. In particular, $m \nmid F_q$ for any q such that $q \mid k$ and $q \neq k$.

It remains to show that $m \mid L_{k/2}$. By the fact that $\pi(m) = k$ and the identity $F_{-n} = (-1)^{n+1} F_n$, we see that $F_{k-n} \equiv F_{-n} \equiv (-1)^{n+1} F_n \pmod m$. Then, since $\frac{k}{2}$ is odd,

$$F_{k/2-1} = F_{k-(k/2+1)} \equiv -F_{k/2+1} \pmod m.$$

Consequently, $m \mid F_{k/2-1} + F_{k/2+1}$. But by the identity $L_n = F_{n-1} + F_{n+1}$, this is exactly $m \mid L_{k/2}$, as required.

($\Leftarrow$) Assume $k \equiv 2 \pmod 4$ and (a) $m \mid L_{k/2}$ and (b) $m \nmid F_q$ for any q such that $q \mid k$ and $q \neq k$. We must show that $\pi(m) = k$.

By (a), $m \mid F_k$, so $\alpha(m) \mid k$. By (b) we find that in fact, $\alpha(m) = k$. Thus, $\pi(m) = k$, $2k$, or $4k$.

If $\pi(m) = 4k$, then $\omega(m) = 4$ and by Theorem 3.1, $\alpha(m)$ must be odd. However, $\alpha(m) \equiv 2 \pmod 4$, so this can't be the case.

If $\pi(m) = 2k$, then $\pi(m) \equiv 4 \pmod 8$, and so Theorem 2.1(2)($\Rightarrow$) implies $m \nmid L_{\pi(m)/4}$; that is, $m \nmid L_{k/2}$. But this contradicts our hypothesis (a) that $m \mid L_{k/2}$, and so $\pi(m) \neq 2k$.

We must conclude that $\pi(m) = k$ and the proof is complete. $\qquad\square$

Proof of Theorem 2.1(2). ($\Rightarrow$) Assume $k \equiv 4 \pmod 8$ and $\pi(m) = k$. Since $\pi(m) \equiv 4 \pmod 8$, by Theorem 3.1 we know that $\omega(m) = 2$ or 4. In either case, $m \mid F_{k/2}$ and $m \nmid F_q$ where $q \mid \frac{k}{2}$ and $q \neq \frac{k}{2}, \frac{k}{4}$. Thus, it only remains to prove that $m \nmid L_{k/4}$.

For ease of notation, let $s = F_{k/2+1}$, let $a = F_{k/4+1}$, and observe that $s \not\equiv 1 \pmod m$.

Claim 1. $F_{k/4-1} \equiv -sa \pmod m$.

Proof of Claim 1. Modulo m, the Fibonacci sequence starting at $F_{k/2}$ is $0, s, s, 2s, 3s, 5s, \ldots$, and in general, $F_{k/2+n} \equiv sF_n \pmod m$. In particular, $F_{(3k)/4+1} \equiv sa$. The identity $F_{-n} = (-1)^{n+1}F_n$ implies $F_{k-n} \equiv F_{-n} \equiv (-1)^{n+1}F_n \pmod m$. Since $\frac{k}{4}$ is odd, we find,

$$F_{k/4-1} \equiv F_{k-((3k)/4+1)} \equiv -F_{(3k)/4+1} \equiv -sa \pmod m.$$

Claim 2. $(a, m) = 1$.

Proof of Claim 2. We have $(F_{k/4-1}, F_{k/4+1}) = F_{(k/4-1,k/4+1)} = F_2 = 1$. So, there exist integers u and v such that $F_{k/4-1}u + F_{k/4+1}v = 1$. Thus, $-sau + av \equiv 1 \pmod m$, and so $a(-su + v) \equiv 1 \pmod m$ and we find that a is invertible mod m. That is, $(a, m) = 1$.

Consider the identity $L_n = F_{n-1} + F_{n+1}$. For contradiction,

$$m \mid L_{k/4} \implies m \mid F_{k/4-1} + F_{k/4+1} \implies -sa + a \equiv 0 \pmod m$$
$$\implies a(1 - s) \equiv 0 \pmod m \implies s \equiv 1 \pmod m.$$

The last implication is due to the fact that $(a, m) = 1$, and we've arrived at a contradiction since $s \not\equiv 1 \pmod m$. We conclude $m \nmid L_{k/4}$, as needed.

($\Leftarrow$) Assume $k \equiv 4 \pmod 8$, (a) $m \mid F_{k/2}$, (b) $m \nmid L_{k/4}$, and (c) $m \nmid F_q$ for all $q \mid \frac{k}{2}$ where $q \neq \frac{k}{2}$ or $\frac{k}{4}$. We must prove that $\pi(m) = k$. By (a) and (c), $\alpha(m) = \frac{k}{4}$ or $\frac{k}{2}$. We know that the only possible values for $\omega(m)$ are $1, 2$, or 4.

Case 1: $\alpha(m) = \frac{k}{4}$.
If $\omega(m) = 2$, then $\pi(m) = \frac{k}{2} \equiv 2 \pmod 4$. However this contradicts Theorem 3.1 since $\pi(m) \equiv 2 \pmod 4$ if and only if $\omega(m) = 1$.

If $\omega(m) = 1$, then $\pi(m) = \frac{k}{4} \equiv 1 \pmod 2$. Again, this contradicts Theorem 3.1 since $\omega(m) = 1$ if and only if $\pi(m) \equiv 2 \pmod 4$.

Thus, in Case 1 we find that $\omega(m) = 4$ and we conclude $\pi(m) = k$.

Case 2: $\alpha(m) = \frac{k}{2}$.
If $\omega(m) = 4$, then $\pi(m) = 2k \equiv 0 \pmod 8$. But by Theorem 3.1, if $\pi(m) \equiv 0 \pmod 8$, then $\omega(m) = 2$, a contradiction.

If $\omega(m) = 1$, then $\pi(m) = \frac{k}{2} \equiv 2 \pmod 4$. We can now apply Theorem 2.1(1)($\Rightarrow$), and we find $m \mid L_{\pi(m)/2} = L_{k/4}$. However, this contradicts our hypothesis (b).

Thus, in Case 2 we find $\omega(m) = 2$ and we conclude $\pi(m) = k$. $\qquad\square$

Proof of Theorem 2.1(3). ($\Rightarrow$) Assume $k \equiv 0 \pmod 8$ and $\pi(m) = k$. Since $\pi(m) \equiv 0 \pmod 8$, Theorem 3.1 tells us that $\omega(m) = 2$, and so $\alpha(m) = \frac{k}{2}$. Thus, $m \mid F_{k/2}$ and $m \nmid F_q$ for any q such that $1 \leq q < \frac{k}{2}$. In particular, $m \nmid F_q$ for all q such that $q \mid \frac{k}{2}$ and $q \neq \frac{k}{2}$, and this direction of the proof is complete.

($\Leftarrow$) Assume $k \equiv 0 \pmod 8$, and (a) $m \mid F_{k/2}$, and (b) $m \nmid F_q$ for all q such that $q \mid \frac{k}{2}$ and $q \neq \frac{k}{2}$. We must prove that $\pi(m) = k$. By (a), we see $\alpha(m) \mid \frac{k}{2}$, and by (b), we deduce that in fact $\alpha(m) = \frac{k}{2}$. Thus $\alpha(m) \equiv 0 \pmod 4$. By Theorem 3.1, this can only happen when $\omega(m) = 2$. Thus $\pi(m) = k$. □

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