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On the faithfulness of the representation of GL(n) on the space of curvature tensors

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We prove that the standard representation of GL(n) on the space of algebraic curvature tensors is almost faithful by showing that the kernel of this representation contains only the identity map and its negative. We additionally show that the standard representation of GL(n) on the space of algebraic covariant derivative curvature tensors is faithful.

1. Introduction

Let V be a finite-dimensional real vector space. An *algebraic curvature tensor* on V (or ACT for short) is a multilinear function

$$R : V \times V \times V \times V \to \mathbb{R}$$

that satisfies the following for all $x, y, z, w \in V$:

$$\begin{gathered} R(x, y, z, w) = -R(y, x, z, w), \quad R(x, y, z, w) = R(z, w, x, y), \\ 0 = R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w). \end{gathered} \tag{1-a}$$

The last of these is called the *first Bianchi identity*. Let $\mathcal{A}(V)$ be the set of all algebraic curvature tensors on V. As a set of real-valued functions, it is easy to check that $\mathcal{A}(V)$ is a vector space under the usual operations of summing the functions and scaling by real numbers; see [Gilkey 2001, p. 23].

There is another multilinear function on V that we study here. An *algebraic covariant derivative curvature tensor* on V (or ACDCT for short) is a multilinear function

$$R_1 : V \times V \times V \times V \times V \to \mathbb{R}$$

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that satisfies the following for all $x, y, z, w, v \in V$:

$$
\begin{aligned}
R_1(x, y, z, w; v) &= -R_1(y, x, z, w; v),\\
R_1(x, y, z, w; v) &= R_1(z, w, x, y; v),\\
0 &= R_1(x, y, z, w; v) + R_1(z, x, y, w; v) + R_1(y, z, x, w; v),\\
0 &= R_1(x, y, z, w; v) + R_1(x, y, v, z; w) + R_1(x, y, w, v; z).
\end{aligned} \tag{1-b}
$$

The first three properties of R_1 are similar to those of R, while the last property is referred to as the *second Bianchi identity*. Let $\mathcal{A}_1(V)$ be the set of ACDCT on V. $\mathcal{A}_1(V)$ is similar to $\mathcal{A}(V)$ in that it is a vector space as well; see [Gilkey 2001, p. 26].

These multilinear objects play a central role in the area of differential geometry. If g is a pseudo-Riemannian metric on a manifold M, then the curvature tensor R^g and its covariant derivative ∇R^g have the same symmetries of R and R_1, respectively, upon restriction to a point of the manifold (when one uses the Levi-Civita connection to construct them).

Let the *general linear group*, denoted $\mathrm{GL}(n)$, be the set of all invertible linear transformations $A : V \to V$. There is a natural action of $\mathrm{GL}(n)$ on both $\mathcal{A}(V)$ and $\mathcal{A}_1(V)$ that defines representations ρ and ρ_1 of $\mathrm{GL}(n)$ on $\mathcal{A}(V)$ and $\mathcal{A}_1(V)$, respectively. Define

$$
\begin{aligned}
\rho(A)(R)(x, y, z, w) &= R(A^{-1}x, A^{-1}y, A^{-1}z, A^{-1}w),\\
\rho_1(A)(R_1)(x, y, z, w; v) &= R_1(A^{-1}x, A^{-1}y, A^{-1}z, A^{-1}w; A^{-1}v).
\end{aligned} \tag{1-c}
$$

For convenience, we simply express these actions of precomposition by the inverse of A by $\rho(A)(R) = A^*R$, and $\rho_1(A)(R_1) = A^*R_1$.

These representations have been studied by previous authors. The representation of the orthogonal group on $\mathcal{A}(V)$ decomposes into eight irreducible subspaces, see [Gilkey 2007; Blažić et al. 2006], with geometric significance. For example, one of these irreducible subspaces is the space of Weyl conformal curvature tensors. The action of $\mathrm{GL}(n)$ on the space $\mathcal{A}_1(V)$ was studied in [Strichartz 1988].

By definition, if G is a group, W is a vector space, and τ is a representation of G on W (that is, τ is a homomorphism from G to the endomorphisms of W), then τ is a *faithful* representation if $\ker(\tau)$ is trivial. In addition, τ is *almost faithful* if $\ker(\tau)$ is a discrete subgroup of G (in the event G is a Lie group, this is equivalent to $\ker(\tau)$ being a zero-dimensional subgroup of G).

It is our goal to investigate the faithfulness of the representations ρ and ρ_1 described above in (1-c). After establishing some supporting lemmas in Section 2, we establish the following theorem in Section 3:

Theorem 1.1. *The representation ρ in* (1-c) *is almost faithful. In fact*, $\ker(\rho) = \{\pm I\}$.

We go on to prove the following result concerning ρ_1.

Theorem 1.2. *The representation* ρ_1 *in* (1-c) *is faithful.*

We describe an immediate corollary and application to these main results concerning groups of symmetries of curvature tensors. Following [Dunn et al. 2015], we define the *structure group* G_T of an ACT or ACDCT T to be the following subgroup of $\mathrm{GL}(n)$:

$$G_T = \{A \in \mathrm{GL}(n) \mid A^*T = T\}.$$

One is interested in any data concerning structure groups for a variety of reasons, although one main purpose would be in constructing invariants — these invariants are then used to study the manifolds that these objects are derived from. See [Dunn 2009; Gilkey 2007] for more on the development of invariants from structure groups.

Corollary 1.3. *Let* G_R *be the structure group of the ACT* R, *and* G_{R_1} *be the structure group of the ACDCT* R_1. *If* $I : V \to V$ *is the identity map*, *then*

$$\bigcap_{R\in\mathcal{A}(V)} G_R = \{\pm I\} \quad \textit{and} \quad \bigcap_{R_1\in\mathcal{A}_1(V)} G_{R_1} = \{I\}.$$

Put differently, Theorems 1.1 and 1.2 demonstrate in this corollary that with exception to $\pm I$ (and only in the ACT case), there is no subgroup of $\mathrm{GL}(n)$ that preserves every ACT or every ACDCT.

2. Preliminary results

There are three preliminary results we shall need to establish our main results. The first two (Lemmas 2.1 and 2.3) concern a construction of ACTs and ACDCTs. The final preliminary result (Theorem 2.7) and a needed corollary (Corollary 2.8) concern the Jordan decomposition of a matrix.

Tensor constructions. Let $S^k(V)$ be the (vector) space of k-multilinear functions

$$\varphi : \times^k V \to \mathbb{R}$$

that are symmetric in every slot. For example, $S^2(V)$ is the set of symmetric bilinear forms, and $S^3(V)$ is the set of totally symmetric trilinear forms. If $\varphi \in S^2(V)$ and $\psi \in S^3(V)$, define

$$\begin{aligned} R_\varphi(x, y, z, w) &= \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w), \\ (R_1)_{\varphi,\psi}(x, y, z, w; v) &= \varphi(x, w)\psi(y, z, v) + \varphi(y, z)\psi(x, w, v) \\ &\qquad - \varphi(x, z)\psi(y, w, v) - \varphi(y, w)\psi(x, z, v). \end{aligned} \tag{2-a}$$

The R_φ and $(R_1)_{\varphi,\psi}$ described in (2-a) are referred to as *canonical* ACTs or ACDCTs. It can be shown that $R_\varphi \in \mathcal{A}(V)$ and $(R_1)_{\varphi,\psi} \in \mathcal{A}_1(V)$; see [Gilkey

2007]. In fact, it is known [Gilkey 2007, p. 47] that

$$\mathcal{A}(V) = \operatorname{span}\{R_\varphi \mid \varphi \in S^2(V)\}, \quad \mathcal{A}_1(V) = \operatorname{span}\{(R_1)_{\varphi,\psi} \mid \varphi \in S^2(V),\, \psi \in S^3(V)\}.$$

Moreover, these canonical ACTs and ACDCTs have geometric significance since they arise as the curvature tensor and its covariant derivative of a hypersurface embedding [Gilkey 2007].

We use the construction found in (2-a) to produce certain ACTs and CDACTs that will be of use to us.

Lemma 2.1. *Let $\{e_1, \dots, e_n\}$ be a basis for V. Let i, j and k be given distinct indices*:

(1) *There exists $R \in \mathcal{A}(V)$ such that*, *up to the symmetries listed in* (1-a), *the only nonzero term is $R(e_i, e_j, e_j, e_i) = 1$.*

(2) *There exists $R \in \mathcal{A}(V)$ such that*, *up to the symmetries listed in* (1-a), *the only nonzero term is $R(e_i, e_j, e_k, e_i) = 1$.*

(3) *Given constants $c_{i,j}$ and $c_{i,j,k}$, there exists $R \in \mathcal{A}(V)$ such that*, *up to the symmetries listed in* (1-a), *the only nonzero terms of R are*

$$R(e_i, e_j, e_j, e_i) = c_{ij} \quad \text{and} \quad R(e_i, e_j, e_k, e_i) = c_{ijk}.$$

Proof. We prove these results by using the construction in (2-a). To prove the first assertion, define $\varphi \in S^2(V)$ by setting $\varphi(e_i, e_i) = \varphi(e_j, e_j) = 1$ and all other entries equal to zero. It is a now a routine check that $R_\varphi(e_i, e_j, e_j, e_i) = 1$ and all other curvature entries up to the symmetries listed in (1-a) are zero.

To prove the second assertion, define φ_1 to have the nonzero entries

$$\varphi_1(e_i, e_i) = \varphi_1(e_j, e_k) = \varphi_1(e_k, e_j) = 1.$$

We now have the following nonzero entries of R_{φ_1} up to the symmetries listed in (1-a):

$$R_{\varphi_1}(e_i, e_j, e_k, e_i) = 1 \quad \text{and} \quad R_{\varphi_1}(e_j, e_k, e_k, e_j) = -1.$$

By the first assertion, there exists an ACT $\tilde{R}$ such that the only nonzero entry up to the symmetries listed in (1-a) is $\tilde{R}(e_j, e_k, e_k, e_j) = 1$. We now complete the second assertion by defining $R = R_{\varphi_1} + \tilde{R}$.

To prove the final assertion, let the constants c_{ij} and c_{ijk} be given, and for every i, j, and k, using the previous assertions define the ACTs $R_{ij}, R_{ijk} \in \mathcal{A}(V)$ such that up to the symmetries listed in (1-a), the only nonzero entries of these ACTs are

$$R_{ij}(e_i, e_j, e_j, e_i) = 1 \quad \text{and} \quad R_{ijk}(e_i, e_j, e_k, e_i) = 1.$$

We can now define $R = \sum_{i,j} c_{ij} R_{ij} + \sum_{i,j,k} c_{ijk} R_{ijk}$. □

Remark 2.2. The notation R_{ij} and R_{ijk} will be used in the proof of Theorem 1.1.

We can prove a similar result concerning the construction of an ACDCT that has certain prescribed entries.

Lemma 2.3. *Let $\{e_1, \dots, e_n\}$ be a basis for V. Let i, j be given distinct indices*:

(1) *There exists $R_1 \in \mathcal{A}_1(V)$ such that*, *up to the symmetries listed in* (1-a), *the only nonzero term is $R_1(e_i, e_j, e_j, e_i; e_j) = 1$.*

(2) *Given constants c_1 and c_2, there exists an $R_1 \in \mathcal{A}_1(V)$ such that*

$$R_1(e_1, e_2, e_2, e_1; e_1) = c_1 \quad \textit{and} \quad R_1(e_1, e_2, e_2, e_1; e_2) = c_2.$$

Proof. We use the construction in (2-a). To prove the first assertion, define $\varphi \in S^2(V)$ and $\psi \in S^3(V)$ by having the nonzero values

$$\varphi(e_i, e_i) = 1, \quad \psi(e_j, e_j, e_j) = 1.$$

It is now a routine check that $(R_1)_{\varphi,\psi}(e_i, e_j, e_j, e_i; e_j) = 1$ is the only nonzero entry up to the symmetries listed in (1-a):

To prove the second assertion, let ${}^1R_1$, ${}^2R_1 \in \mathcal{A}(V)$ be given such that the only nonzero entries up to the symmetries listed in (1-a) are

$${}^1R_1(e_1, e_2, e_2, e_1; e_1) = 1 \quad \text{and} \quad {}^2R_1(e_1, e_2, e_2, e_1; e_2) = 1.$$

We then define $R_1 = c_1({}^1R_1) + c_2({}^2R_1)$, which satisfies the given conditions. □

Remark 2.4. According to the symmetries in (1-b), we have

$$R_1(e_i, e_j, e_j, e_i; e_j) = R_1(e_j, e_i, e_i, e_j; e_j).$$

So the ACDCT guaranteed to exist from Lemma 2.3 can be chosen to have the final index match the fourth index, or chosen to match the third, provided the first four indices are of the form (i, j, j, i) — or any of the other dependent forms derivable from the symmetries in (1-b).

Remark 2.5. The notation ${}^1R_1$ and ${}^2R_1$ will be used in the proof of Theorem 1.2.

Jordan normal form. We recall a familiar result from linear algebra concerning the Jordan normal form of a matrix; see [Adkins and Weintraub 1992] for details. To properly state this result, we make the following definitions.

Definition 2.6. Let $\lambda \in \mathbb{R}$, and let $a + b\sqrt{-1} \in \mathbb{C}$ with $a, b \in \mathbb{R}$ and $b > 0$. The *real Jordan block of size k corresponding to* λ is the $k \times k$ matrix $J(\lambda, k)$ of real numbers

$$J(\lambda, k) = \begin{bmatrix} \lambda & 1 & 0 & 0 & \\ 0 & \lambda & 1 & 0 & \cdots \\ 0 & 0 & \lambda & 1 & \\ & \vdots & & & \ddots \end{bmatrix}.$$

The *complex Jordan block of size k corresponding to $a+b\sqrt{-1}$* is the $2k \times 2k$ matrix $J(a,b,k)$ of real numbers

$$J(a,b,k) = \begin{bmatrix} a & b & 1 & 0 & 0 & 0 & \\ -b & a & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & a & b & 1 & 0 & \\ 0 & 0 & -b & a & 0 & 1 & \\ & \vdots & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

We briefly recall the direct sum operation on matrices before we state Theorem 2.7. If A and B are matrices of any size, then one may create the new matrix $A \oplus B$ by defining

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where the 0 entries above denote the 0-matrix of appropriate size.

We can now state the famous result concerning the Jordan decomposition of a linear transformation.

Theorem 2.7 (Jordan normal form of a linear transformation). *Let $A : V \to V$ be any linear transformation. There exists a basis $\mathcal{B}$ for V such that the matrix representation $[A]_{\mathcal{B}}$ of A on $\mathcal{B}$ is the direct sum of Jordan blocks corresponding to the eigenvalues of A. Furthermore, the unordered collection of Jordan blocks is uniquely determined by A.*

Note that the expression of A into a direct sum of Jordan blocks is referred to as "expressing A in its Jordan normal form". We shall need the following corollary in Section 4.

Corollary 2.8. *Let $A : V \to V$ be any linear transformation. There exists a basis $\{e_1, \dots, e_n\}$ for V such that* $\operatorname{span}\{e_1, e_2\}$ *is A-invariant.*

Proof. Find a basis for V that expresses A in its Jordan normal form, which is possible by Theorem 2.7. If A has any complex eigenvalues, rearrange this basis so that the corresponding complex Jordan block appears first. The result is now true in this case, since $Ae_1 = ae_1 - be_2$ and $Ae_2 = be_1 + ae_2$.

If A has no complex eigenvalues, then the Jordan normal form of A is a direct sum of real Jordan blocks of various sizes, and these real Jordan blocks are all upper triangular. Hence, their direct sum is upper triangular. Now we have

$$Ae_i \in \operatorname{span}\{e_1, \dots, e_i\}$$

for every $i \geq 1$. In particular, this holds for $i = 2$. □

3. The representation on $\mathcal{A}(V)$ is almost faithful

Here, we establish Theorem 1.1. The general method of proof of Theorem 1.1 (and with minor adjustments, Theorem 1.2 in the next section) is as follows. If $A \in \mathrm{GL}(n)$ is given and $A \neq \pm I$, then we produce an ACT R such that $A^*R \neq R$. By expressing A^{-1} in its Jordan normal form, this comes down to a number of cases, and in each case we use Lemma 2.1 (or Lemma 2.3 in the next section) to produce the needed ACT (or ACDCT).

Proof of Theorem 1.1. Since any ACT R inputs four entries and is multilinear, we have $(\pm I)^*R = R$ for every R. In the language of representations, $\rho(\pm I)$ is the identity map on $\mathcal{A}(V)$; hence $\pm I \in \ker\rho$. We prove that if $A \neq \pm I$, then $A \notin \ker(\rho)$ by finding an ACT R for which $\rho(A)(R) \neq R$, demonstrating that $\rho(A)$ is not the identity on $\mathcal{A}(V)$.

Note that since $A \neq \pm I$ and each of these is self-inverse, $A^{-1} \neq \pm I$. We decompose A^{-1} into its Jordan normal form and proceed by cases depending on the first Jordan block in this form. Recall that since $A \in \mathrm{GL}(n)$, none of its eigenvalues are equal to 0.

(1) **The first Jordan block of A^{-1} is $J(\lambda, 1)$.** We break this case into several subcases.[1] Note that in what follows, $\lambda \in \mathbb{R}$.

(a) *The second Jordan block of A^{-1} is $J(\eta, 1)$.* Note that $\eta \in \mathbb{R}$. There are now three possibilities for the third Jordan block of A^{-1}:

(i) *All other Jordan blocks of A^{-1} are real and of size* 1. Since $A^{-1} \neq \pm I$, not all of the eigenvalues are 1 and not all of the eigenvalues are -1. Thus there is at least one real eigenvalue γ of A^{-1} that differs from either λ or η. Without loss of generality, suppose $\gamma \neq \lambda$, and $J(\gamma, 1)$ is the third Jordan block of A^{-1}. Then for an arbitrary ACT R, we have

$$
\begin{aligned}
A^*R(e_1, e_2, e_2, e_1) &= \lambda^2\eta^2 R(e_1, e_2, e_2, e_1),\\
A^*R(e_1, e_3, e_3, e_1) &= \lambda^2\gamma^2 R(e_1, e_3, e_3, e_1),\\
A^*R(e_2, e_3, e_3, e_2) &= \eta^2\gamma^2 R(e_2, e_3, e_3, e_2),\\
A^*R(e_2, e_1, e_3, e_2) &= \eta^2\lambda\gamma R(e_2, e_1, e_3, e_2).
\end{aligned}
$$

Using the notation of Lemma 2.1, choose $R = R_{12} + R_{13} + R_{23} + R_{213}$. Then if $A^*R = R$, the above equations results in the system of equations

$$\lambda^2\eta^2 = \lambda^2\gamma^2 = \eta^2\gamma^2 = \eta^2\lambda\gamma = 1.$$

[1] It is not surprising that this is the most complicated case: $J(\lambda, 1)$ is a Jordan block of $\pm I$ when $\lambda = \pm 1$ and so further distinguishing features of A^{-1} are needed.

Since these eigenvalues are nonzero, we see from the first three of these that $\lambda^2 = \eta^2 = \gamma^2$, and so $\lambda^4 = \eta^4 = \gamma^4 = 1$. Thus, each of these is either ± 1. But $\lambda \neq \gamma$, so they must be of opposite signs. In this case, $\lambda\gamma = -1$, but then in the final expression above $\eta^2\lambda\gamma = -1$, a contradiction.

(ii) *In the remaining Jordan blocks, there exists a real Jordan block of size greater than or equal to 2.* Suppose the next Jordan block is $J(\gamma, k)$ for $k \geq 1$. Notice that for any ACT R we would then have

$$A^*R(e_1, e_3, e_4, e_1) = \lambda^2\gamma R(e_1, e_3, e_3, e_1) + \lambda^2\gamma^2 R(e_1, e_3, e_4, e_1).$$

So, if $R = R_{13}$, then $R_{13}(e_1, e_3, e_4, e_1) = 0$, while

$$A^*R_{13}(e_1, e_3, e_4, e_1) = \lambda^2\gamma \neq 0,$$

a contradiction if $A^*R = R$.

(iii) *The remaining Jordan blocks of A^{-1} are complex.* Suppose the next Jordan block of A^{-1} is $J(a + b\sqrt{-1}, k)$ for $k \geq 1$. If R is an arbitrary ACT, then

$$\begin{aligned} &A^*R(e_1, e_3, e_3, e_1) \\ &\quad = \lambda^2 a^2 R(e_1, e_3, e_3, e_1) - 2\lambda^2 ab R(e_1, e_3, e_4, e_1) + \lambda^2 b^2 R(e_1, e_4, e_4, e_1). \end{aligned}$$

Then we recall that $b \neq 0$ and notice that if $R = R_{13} + a/(2b)R_{134}$, then $A^*R = R$ implies the left side of this equation is 1, while the right side is 0, a contradiction.

(b) *The second Jordan block is real and of size at least* 2. Suppose the second Jordan block is $J(\eta, k)$ for $k \geq 2$. It will be helpful in comparison to Case (2a) later to note here that our assumptions have

$$A^{-1}(e_1) = \lambda e_1, \quad A^{-1}(e_2) = \eta e_2, \quad A^{-1}(e_3) = \eta e_3 + e_2. \tag{3-a}$$

Now if R is an arbitrary ACT, we have

$$\begin{aligned} &A^*R(e_1, e_3, e_3, e_1) \\ &\quad = \lambda^2 R(e_1, e_2, e_2, e_1) + \lambda^2\eta^2 R(e_1, e_3, e_3, e_1) + 2\lambda^2\eta R(e_1, e_2, e_3, e_1). \end{aligned}$$

So if $R = -\eta^2 R_{12} + R_{13}$ and $A^*R = R$, then the left side of the equation is 1, while the right side is 0, another contradiction.

(c) *The second Jordan block is complex.* Suppose the second Jordan block is $J(a + b\sqrt{-1}, k)$ for $k \geq 1$. Then for an arbitrary ACT R, we have

$$\begin{aligned} &A^*R(e_1, e_2, e_2, e_1) \\ &\quad = \lambda^2 a^2 R(e_1, e_2, e_2, e_1) + \lambda^2 b^2 R(e_1, e_3, e_3, e_1) + 2\lambda^2 ab R(e_1, e_2, e_3, e_1). \end{aligned}$$

Recalling that $b \neq 0$, if $R = R_{12} + a/(2b)R_{123}$ and $A^*R = R$, then the left side of the equation is 1, while the right side is 0, another contradiction.

(2) **The first Jordan block of A^{-1} is $J(\lambda, 2)$.** There are two cases to consider concerning the second Jordan block:

(a) *The remaining Jordan blocks of A^{-1} are all real.* If the second Jordan block is $J(\eta, k)$ for $k \geq 1$, then we have

$$A^{-1}(e_1) = \lambda e_1, \quad A^{-1}(e_2) = \lambda e_2 + e_1, \quad A^{-1}(e_3) = \eta e_3. \tag{3-b}$$

Comparing (3-b) to (3-a), one sees that under a permutation of the basis vectors, one reproduces the Case (1b) above.

(b) *There exists a complex Jordan block in A^{-1}.* Suppose the second Jordan block is $J(a + b\sqrt{-1}, k)$ for some k. Then for an arbitrary ACT R, we have

$$\begin{aligned} &A^*R(e_1, e_3, e_3, e_1) \\ &\qquad = \lambda^2 b^2 R(e_1, e_4, e_4, e_1) + \lambda^2 a^2 R(e_1, e_3, e_3, e_1) + 2\lambda^2 ab R(e_1, e_3, e_4, e_1). \end{aligned}$$

Recalling that $b \neq 0$, if $R = R_{13} + a/(2b)R_{134}$ and $A^*R = R$, then the left side of the equation is 1, while the right side is 0, another contradiction.

(3) **The first Jordan block of A^{-1} is $J(\lambda, m)$ for $m \geq 3$.** For an arbitrary ACT R, we have

$$A^*R(e_1, e_3, e_3, e_1) = \lambda^2 R(e_1, e_2, e_2, e_1) + 2\lambda^3 R(e_1, e_2, e_3, e_1) + \lambda^4 R(e_1, e_3, e_3, e_1).$$

If $R = -\lambda^2 R_{12} + R_{13}$ and $A^*R = R$, then the left side of the equation is 1, while the right side is 0, another contradiction.

(4) **The first Jordan block of A^{-1} is $J(a, b, 1)$.** There are two cases to consider:

(a) *All remaining Jordan blocks of A^{-1} are real.* Permuting basis vectors only reorders the Jordan blocks in A^{-1}. Thus, if there are other real Jordan blocks after one complex Jordan block, one may reorder the basis vectors to have the real Jordan blocks appear first. Thus, we reproduce one of the previous cases.

(b) *There exists another complex Jordan block in A^{-1}.* For an arbitrary ACT R, we have

$$\begin{aligned} A^*R(e_1, e_2, e_3, e_1) = {}& ac(a^2+b^2)R(e_1, e_2, e_3, e_1) + bc(a^2+b^2)R(e_2, e_1, e_3, e_2) \\ & - ad(a^2+b^2)R(e_1, e_2, e_4, e_1) - bd(a^2+b^2)R(e_2, e_1, e_4, e_2). \end{aligned}$$

Recall that b and d are nonzero. If $R = ac/(bd)R_{214} + R_{123}$ and $A^*R = R$, then the left side of the equation is 1, while the right side is 0, another contradiction.

(5) **The first Jordan block of A^{-1} is $J(a, b, m)$ for $m \geq 2$.** For an arbitrary ACT R, we have

$$\begin{aligned}A^*R(e_1, e_2, e_3, e_1) = b(a^2+b^2)R(e_1, e_2, e_2, e_1)+a^2(a^2+b^2)R(e_1, e_2, e_3, e_1)\\ +ab(a^2+b^2)R(e_2, e_1, e_3, e_2)-ab(a^2+b^2)R(e_1, e_2, e_4, e_1)\\ -b^2(a^2+b^2)R(e_2, e_1, e_4, e_2).\end{aligned}$$

If $R = (a^2/b^2)R_{214} + R_{123}$ and $A^*R = R$, then the left side of the equation is 1, while the right side is 0, another contradiction.

To summarize, when given any Jordan decomposition of A^{-1} and $A \neq \pm I$, there exists an ACT R for which $A^*R \neq R$. Since $(\pm I)^*R = R$, the only ACT for which $A^*R = R$ for all R is when $A = \pm I$. As a result, $\rho(A)$ is the identity endomorphism on the space of algebraic curvature tensors precisely when $A = \pm I$. □

4. The representation on $\mathcal{A}_1(V)$ is faithful

We conclude the paper by establishing Theorem 1.2.

Proof of Theorem 1.2. Unlike the proof of Theorem 1.1, by Corollary 2.8 we only need to consider three possible Jordan forms that occupy the upper 2×2 part of the matrix A^{-1}.

(1) **There is a complex Jordan block in A^{-1}.** Suppose the first Jordan block of A^{-1} is $J(a, b, k)$. For an arbitrary ACDCT R_1, we have

$$A^*R_1(e_1,e_2,e_2,e_1;e_1) = (a^2+b^2)^2\big(aR_1(e_1,e_2,e_2,e_1;e_1) - bR_1(e_1,e_2,e_2,e_1;e_2)\big).$$

Recall that $b \neq 0$. Using the notation of Lemma 2.3, if $R_1 = ({}^1R_1) + a/b({}^2R_1)$ and $A^*R_1 = R_1$, then the left side of the equation is 1, while the right side is 0, a contradiction.

(2) **There are only real Jordan blocks, and there is at least one of size 2 or more.** Suppose the first Jordan block of A^{-1} is $J(\lambda, k)$ for $k \geq 2$. For an arbitrary ACDCT R_1, we have

$$A^*R_1(e_1, e_2, e_2, e_1; e_2) = \lambda^4(R_1(e_1, e_2, e_2, e_1; e_1)) + \lambda^5(R_1(e_1, e_2, e_2, e_1; e_2)).$$

If $R_1 = -\lambda({}^1R_1) + ({}^2R_1)$ and $A^*R_1 = R_1$, then the left side of the equation is 1, while the right side is 0, another contradiction.

(3) **There are only real Jordan blocks, all of which have size 1.** Suppose without loss of generality that the first Jordan block of A^{-1} is $J(\lambda, 1)$. The next Jordan block $J(\eta, 1)$, by assumption, is a real one of size 1, and hence we have the relations

$A^{-1}e_1 = \lambda e_1$, and $A^{-1}e_2 = \eta e_2$. For an arbitrary CDACT R_1, we have

$$A^*R_1(e_1, e_2, e_2, e_1; e_1) = \lambda^3\eta_2 R_1(e_1, e_2, e_2, e_1; e_1),$$
$$A^*R_1(e_1, e_2, e_2, e_1; e_2) = \lambda^2\eta_3(\lambda R_1(e_1, e_2, e_2, e_1; e_2)).$$

If $R_1 = ({}^1R_1)$ and $A^*R_1 = R_1$, we conclude $\lambda^3\eta^2 = 1$. If $R_1 = {}^2R_1$, then we conclude $\lambda^2\eta^3 = 1$. Since both must happen simultaneously, we have $\lambda = \eta$, and $\lambda^5 = 1$, so $\lambda = \eta = 1$. We have shown that for *any* other real Jordan block $J(\eta, 1)$, for any k, $\eta = \lambda = 1$. Thus since there are only real Jordan blocks of size 1, the only way $A^*R_1 = R_1$ for all $R_1 \in \mathcal{A}(V)$ is if all Jordan blocks of A^{-1} are $J(1, 1)$ and as a result $A^{-1} = A = I$. □

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