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Time stopping for Tsirelson's norm

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Tsirelson's norm $\|\cdot\|_T$ on c_{00} is defined as the limit of an increasing sequence of norms $(\|\cdot\|_n)_{n=1}^{\infty}$. For each $n \in \mathbb{N}$ let $j(n)$ be the smallest integer satisfying $\|x\|_{j(n)} = \|x\|_T$ for all x with $\max \operatorname{supp} x = n$. We show that $j(n)$ is $O(n^{1/2})$. This is an improvement of the upper bound of $O(n)$ given by P. Casazza and T. Shura in their 1989 monograph on Tsirelson's space.

In 1974 B. Tsirelson [1974] constructed a remarkable reflexive Banach space not containing an isomorphic copy of ℓ_p for any $1 < p < \infty$. T. Figiel and W. B. Johnson [1974] gave an analytic description of the dual Tsirelson's space that was subsequently used to discover many new types of Banach spaces and was very influential in solving many old problems in the isomorphic theory of Banach spaces. A monograph of P. Casazza and T. Shura [1989] contains a detailed analysis of many structural properties of Tsirelson's space and played a critical role in the developments in the mid-1990s. In the last chapter in that book, the authors present FORTRAN code that computes the Tsirelson norm of finite length vectors. In the discussion of this code they state several problems and lines of research that to our knowledge are still open or unexplored. The authors of the current paper became interested in these questions since they relate to the well-known open problem of whether Tsirelson's space is arbitrarily distortable and the "polymath" problem [Gowers 2009], which asks whether every "explicitly defined" Banach space must contain ℓ_p or c_0. Our main result is the first nontrivial step toward finding the computational time for computing the Tsireslon's norm. We should note that although Casazza and Shura's book was written almost 30 years ago, there are still many problems and constructions related to Tsireslon's space that are currently attracting attention. For example, the reader should consult the papers [Argyros et al. 2013; Argyros and Motakis 2014; 2016; Khanaki 2016; Ojeda-Aristizabal 2013; Tan 2012] and the aforementioned blog post of W. T. Gowers.

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Duncan and Holt were undergraduate students at Washington and Lee University when the main result of this paper was proved. The main result in this paper is part of the Washington and Lee Honors thesis of Holt written under the direction of the Kevin Beanland.

The dual of Tsirelson's space T is the completion of c_{00}, the space of all eventually zero scalar sequences, with respect to a norm $\|\cdot\|_T$. This norm is defined as the supremum of an increasing sequence of recursively defined norms $(\|\cdot\|_n)_{n=1}^{\infty}$. We recall all precise definitions in the next section. Casazza and Shura introduced the following *time-stopping* function.

Definition 1. For n a positive integer, let $j(n)$ be the smallest nonnegative integer such that for all $x \in c_{00}$ with $\max \operatorname{supp} x \leq n$ we have $\|x\|_{j(n)} = \|x\|_T$.

In [Casazza and Shura 1989, Problem 2(a)], the authors ask for a "reasonably tight" upper bound for the quantity $j(n)$ and offer the upper bound $\lfloor \frac{1}{2}n \rfloor$ as a starting point. Our main theorem is the following improvement on this upper bound.

Theorem A. *For each $n \in \mathbb{N}$ we have $j(n) \leq \lfloor 2\sqrt{n} + 4 \rfloor$. That is, $j(n)$ is $O(n^{1/2})$.*

In a forthcoming paper we provide a lower bound on the order of $\log_2(n)$. The upper bound on $j(n)$ determines the computation time of the vector of length n. Indeed it is shown by Casazza and Shura that the computational time it takes to go from the n norm to the $n+1$ norm is the same for every n. Therefore if t is that computational time, our theorem shows that the computation time required to calculate the norm of a vector of length n is bounded above by $t\sqrt{n}$.

1. Main result

Let (e_i) and (e_i^*) both denote the standard unit vectors in c_{00}. For $E \subset \mathbb{N}$ and $x = \sum_{i=1}^{\infty} a_i e_i \in c_{00}$ let $Ex = \sum_{i \in E} a_i e_i$. If E, F are subsets of $\mathbb{N}$ we write $E < F$ if $\max E < \min F$. A set $E \subset \mathbb{N}$ is in $\mathcal{S}_1$ if $\min E \geq |E|$ (the cardinality of E). If $\sum_{i=1}^{\infty} a_i e_i \in c_{00}$ then $\operatorname{supp} x = \{i : a_i \neq 0\}$. For $n \in \mathbb{N}$ we say that a sequence $(E_i)_{i=1}^n$ of subsets of $\mathbb{N}$ is called admissible if $E_1 < E_2 < \cdots < E_n$ and $(\min E_i)_{i=1}^n \in \mathcal{S}_1$. We define the norm of Tsirelson's space by defining a certain subset of c_{00} to be the norming functionals for the space. The set W_T is the union of the following subsets of c_{00}. A sequence $(f_i)_{i=1}^d \subset c_{00}$ is called admissible if $(\operatorname{supp} f_i)_{i=1}^d$ is admissible. Let $W_0 = \{\pm e_i^* : i \in \mathbb{N}\}$ and for $k \geq 0$ let

$$W_{k+1} = W_k \cup \left\{ \frac{1}{2} \sum_{i=1}^{d} E f_i : d \in \mathbb{N},\ (f_i)_{i=1}^d \subset W_k \text{ is admissible},\ E \subset \mathbb{N} \right\}.$$

Then $W_T = \bigcup_{k=1}^{\infty} W_k$.

The intermediate norms are defined by $\|x\|_n = \sup\{f(x) : f \in W_n\}$. Here $f(x)$ is the usual inner product of f with x. Tsirelson's norm is defined by

$$\|x\| = \max_n \|x\|_n = \sup\{f(x) : x \in W_T\}.$$

Tsirelson's space is the completion of c_{00} with respect to the above norm, which satisfies the following implicit equation for $x \in c_{00}$:

$$\|x\| = \|x\|_\infty \vee \sup\left\{ \frac{1}{2} \sum_{i=1}^{n} \|E_i x\| : n \in \mathbb{N},\ (E_i)_{i=1}^n \text{ is admissible} \right\}. \tag{1}$$

The following remarks follow from the definition of W_T.

Remark 1.1. Let $f \in W_T$. Then $f \in W_T \setminus W_n$ if and only if there is a $k \in \mathbb{N}$ with $0 < |f(e_k)| \le 1/2^{n+1}$.

Remark 1.2. If $f \in W_T$ then either $f = \pm e_i^*$ for some $i \in \mathbb{N}$ or $f \in W_n \setminus W_0$ and there is an admissible sequence $(f_i)_{i=1}^d \subset W_{n-1}$ with $f = \frac{1}{2} \sum_{i=1}^d f_i$. In particular, if $f \in W_T \setminus W_0$ then $|f(e_k)| \le \frac{1}{2}$ for all $k \in \mathbb{N}$.

Based on the above remark it is easy to see that each functional has a decomposition into a "tree" of functionals. The functionals in the tree are naturally enumerated by tuples in $\mathbb{N}$. Let $\mathbb{N}^{<\mathbb{N}} = \bigcup_{n=1}^\infty \mathbb{N}^n \cup \{\varnothing\}$. For $\sigma \in \mathbb{N}^{<\mathbb{N}}$, if $\sigma = (\sigma(1), \dots, \sigma(k))$, we set $|\sigma| = k$.

Definition 2 (tree index set and decomposition). For each $f \in W_T$ there is a set $\mathcal{T}_f \subseteq \mathbb{N}^{<\mathbb{N}} \cup \{\varnothing\}$ called the *tree index set* and a collection of functionals $(f_\alpha)_{\alpha \in \mathcal{T}_f} \subset W_T$ called a *tree decomposition* of f satisfying:

(1) $\varnothing \in \mathcal{T}_f$ and $f_\varnothing = f$.

(2) $\sigma \in \mathcal{T}_f$ is called a terminal node if $\sigma \frown 1 \notin \mathcal{T}_f$. A node $\sigma \in \mathcal{T}$ is a terminal if and only if $f_\sigma = \pm e_i^*$ for some $i \in \mathbb{N}$.

(3) If $\sigma \in \mathcal{T}_f$ is not a terminal node, then

$$f_\sigma = \frac{1}{2} \sum_{\{k : \sigma \frown k \in \mathcal{T}_f\}} f_{\sigma \frown k},$$

where $\{k : \sigma \frown k \in \mathcal{T}_f\} = \{1, \dots, d_\sigma\}$ for some $d_\sigma \in \mathbb{N}$. Moreover $(f_{\sigma \frown k})_{k=1}^{d_\sigma}$ is admissible.

If $\sigma = (n_1, n_2, \dots, n_k) \in \mathcal{T}_f$ then $\beta = (n_1, n_2, \dots, n_{k-1})$ is the immediate predecessor of σ and σ is an immediate successor of β. To set notation let $E_\sigma = \operatorname{supp} f_\sigma$ for each $\sigma \in \mathcal{T}_f$.

The fact that each $f \in W_T$ has a (not necessarily unique) tree index set $\mathcal{T}_f$ and decomposition follows from the definition of an arbitrary $f \in W_T$.

Lemma 1.3. *Let $f \in W_T$. Then $f \in W_n$ if and only if there is a tree decomposition $(f_\alpha)_{\alpha \in \mathcal{T}_f} \subset W_T$ of f such that $|\sigma| \le n$ for all $\sigma \in \mathcal{T}_f$.*

Proof. Let $f \in W_n$. If $f \in W_0$ then any index set contains only the empty set. Thus we assume $f = \frac{1}{2}\sum_{i=1}^{d} f_i$, where $(f_i)_{i=1}^{d}$ is an admissible block sequence in W_{n-1}. Let $\mathcal{T}_{f_i}$ be the tree index set of f_i for each $i \in \{1, \dots, d\}$ such that $|\sigma| \le n-1$ for each $\sigma \in \bigcup_{i=1}^{d} \mathcal{T}_{f_i}$. The tree index set of f is defined by

$$\mathcal{T}_f = \left\{ i \frown \sigma : i \in \{1, \dots, d\},\ \sigma \in \bigcup_{i=1}^{d} \mathcal{T}_{f_i} \right\} \cup \{\varnothing\}.$$

Alternatively, we proceed by induction. The base case is trivial and so assume the claim for some $n-1 \ge 0$. We will establish the claim for n. Suppose there is a tree decomposition $(f_\alpha)_{\alpha \in \mathcal{T}_f} \subset W_T$ of some $f \in W_T$ so that $\sigma \le n$ for all $\sigma \in \mathcal{T}_f$. Let $d \in \mathbb{N}$ so that $\{k : (k) \in \mathcal{T}_f\} = \{1, \dots, d\}$. For each $1 \le i \le d$, set $\mathcal{T}_{f_{(i)}} = \{\sigma : i \frown \sigma \in \mathcal{T}_f\}$ and let $\{g_\sigma = f_{i \frown \sigma} : \sigma \in \mathcal{T}_{f_{(i)}}\}$ be a tree decomposition for $f_{(i)}$. Then for each $\sigma \in \mathcal{T}_{f_{(i)}}$ with $i \in \{1, \dots, d\}$ we have $|\sigma| \le n-1$. Thus, $f_{(i)} \in W_{n-1}$ and $f \in W_n$ as desired. □

The following is a simple but critical definition for our purposes. For a given $x \in c_{00}$ there may be many functionals in W_T that norm x. The support of some of these functionals may not even be a subset of the support of x, while other norming functionals may have supports disjoint from one another. Our goal is to prove an upper bound on $j(n)$ by minimizing the maximum node length of a tree decomposition for a functional that norms an arbitrary x with $\max \operatorname{supp} x \le n$. In order to minimize this quantity, we discard the parts of a functional that are not required to norm a given vector. To this end, we define for each $x \in c_{00}$ a minimal set for x and a functional that *minimally norms* x. We can then restrict our attention to counting the maximum node length of a tree decomposition for a minimally norming functional for a given x.

Definition 3. Let $x \in c_{00}$. Then a set $E \subset \mathbb{N}$ is minimal for x if $\|Ex\| = \|x\|$ and for each $E' \subsetneq E$, we have $\|E'x\| < \|x\|$.

Let us note that minimal sets need not be unique.

Lemma 1.4. *Suppose $f \in W_T$ norms $x \in c_{00}$ and* $\operatorname{supp} f \subset \operatorname{supp} x$. *Then for all $\alpha \in \mathcal{T}_f$, we have f_σ norms $E_\sigma x$.*

Proof. Assume, via contradiction, we can find a minimal length node $\sigma \in \mathcal{T}_f$ so that $f_\sigma(E_\sigma x) < \|E_\sigma x\|$. By assumption $\sigma \ne \varnothing$ (recall that $E_\varnothing = \operatorname{supp} f$). Find the unique predecessor $\beta \in \mathcal{T}_f$ of σ. Let $i_0 \in \{1, \dots, d_\beta\}$ so that $\sigma = \beta \frown i_0$. Then $f_{\beta \frown i}(E_{\beta \frown i} x) \le \|E_{\beta \frown i} x\|$ for $i \ne i_0$ and $f_{\beta \frown i_0}(E_{\beta \frown i_0} x) < \|E_{\beta \frown i_0} x\|$; however, $f_\beta(E_\beta x) = \|E_\beta x\|$ by the minimality of σ. This leads to the following contradiction:

$$\|E_\beta x\| = f_\beta(E_\beta x) = \frac{1}{2}\sum_{i=1}^{d_\beta} f_{\beta \frown i}(E_{\beta \frown i} x) < \frac{1}{2}\sum_{i=1}^{d_\beta} \|E_{\beta \frown i} x\| \le \|E_\beta x\|.$$

The last inequality follows from the implicit equation (1) for the norm, noting that $(E_{\beta\frown i})_{i=1}^{d_\beta}$ is admissible. $\square$

Definition 4. Let $x \in c_{00}$. We say that $f \in W_T$ *minimally norms* x if $\operatorname{supp} f = E$ is minimal for x and $f(x) = \|x\|$.

Note that if f minimally norms x then $\operatorname{supp} f \subset \operatorname{supp} x$. Also, if E is a minimal set for x there is a $f \in W_T$ that minimally norms x with $\operatorname{supp} f = E$.

Lemma 1.5. *Let $x \in c_{00}$ and suppose that f minimally norms x. Then for each $\sigma \in \mathcal{T}_f$, we have E_σ is a minimal set for $E_\sigma x$ and f_σ minimally norms $E_\sigma x$.*

Proof. Using Lemma 1.4 we know that $f_\sigma(E_\sigma x) = \|E_\sigma x\|$ for each $\sigma \in \mathcal{T}_f$. Find a minimal length node $\sigma \in \mathcal{T}_f$ so that E_σ is not a minimal set for $E_\sigma f$. Again it follows from the hypothesis that $\sigma \neq \varnothing$. Let β be the immediate predecessor of σ and $i_0 \in \{1, \dots, d_\beta\}$ with $\sigma = \beta \frown i_0$. Using our assumption, we can find a $E'_{\beta\frown i_0} \subsetneq E_{\beta\frown i_0}$ with $\|E'_{\beta\frown i_0} x\| = \|E_{\beta\frown i_0} x\|$. Let

$$E'_\beta = \Bigg(\bigcup_{i=1,\, i\neq i_0}^{d_\beta} E_{\beta\frown i} \Bigg) \cup E'_{\beta\frown i_0} \subsetneq E_\beta.$$

We can now show that $\|E_\beta x\| \le \|E'_\beta x\|$ as follows:

$$\begin{aligned} \|E_\beta x\| = f_\beta(E_\beta x) &= \frac{1}{2} \sum_{i=1}^{d_\beta} f_{\beta\frown i}(E_{\beta\frown i} x) = \frac{1}{2} \sum_{i=1}^{d_\beta} \|E_{\beta\frown i} x\| \\ &= \frac{1}{2} \Bigg(\sum_{i=1}^{i_0-1} \|E_{\beta\frown i} x\| + \|E'_{\beta\frown i_0} x\| + \sum_{i=i_0+1}^{d_\beta} \|E_{\beta\frown i} x\| \Bigg) \\ &\le \|E'_\beta x\|. \end{aligned} \tag{2}$$

The last inequality uses that

$$(E_{\beta\frown 1}, E_{\beta\frown 2}, \dots, E_{\beta\frown (i_0-1)}, E'_{\beta\frown i_0}, E_{\beta\frown (i_0+1)}, \dots, E_{\beta\frown d_\beta})$$

is admissible. This contradicts the minimality of σ.

Therefore for each $\sigma \in \mathcal{T}_f$, we have E_σ is a minimal set for $E_\sigma x$. The fact that f_σ minimally norms $E_\sigma x$ follows from Lemma 1.4. $\square$

Lemma 1.6. *Let $x \in c_{00}$ and suppose $f \in W_T$ minimally norms x and* $\operatorname{supp} f \in \mathcal{S}_1$. *Then $f \in W_1$.*

Proof. If $f \in W_T \setminus W_1$ then there is a $k \in \operatorname{supp} f$ with $0 < |f(e_k)| \le \frac{1}{4}$. However since $\operatorname{supp} f \in \mathcal{S}_1$, we know that $g = \frac{1}{2} \sum_{i \in \operatorname{supp} f} \operatorname{sign}(e_i^*(x)) e_i^* \in W_1$. But $g(x) > f(x) = \|x\|$. This is a contradiction. $\square$

Lemma 1.7. *Let $x \in c_{00}$ and suppose $f \in W_T$ minimally norms x. Suppose further that $f \in W_T \setminus W_1$. If $f = \frac{1}{2}\sum_{i=1}^{d} f_i$, where $(f_i)_{i=1}^{d}$ is admissible, then* $\min \operatorname{supp} f = d$.

Proof. By definition, $d \leq \min \operatorname{supp} f =: m$, and so it suffices to show that equality holds. Suppose towards a contradiction that $d < m$. Our goal is to build a functional $f' \in W_T$ so that $f'(x) > f(x)$. This will contradict the assumption that $f(x) = \|x\|$. Since $f \in W_T \setminus W_1$ there is some $i_0 \in \{1, \ldots, d\}$ with $f_{i_0} \notin W_0$. By appealing to Remark 1.1 and Lemma 1.6 we may assume that the supp f_{i_0} has more than one element. Let $k_0 = \min \operatorname{supp} f_{i_0}$. Then $0 < |f(e_{k_0})| \leq \frac{1}{4}$. In particular

$$f(e_{k_0}) = \operatorname{sign} e_{k_0}^*(x)\tfrac{1}{2^n} \quad \text{for some } n > 1.$$

Set $f_{i_0}^1 = \operatorname{sign}(e_{k_0}^*(x))e_{k_0}^*$ and $f_{i_0}^2 = f_{i_0}|_{[k_0+1,\infty)}$. Since $d < m$ and $f_{i_0}^1, f_{i_0}^2 \in W_T$ are successive with $\operatorname{supp} f_{i_0}^1 \cup \operatorname{supp} f_{i_0}^2 = \operatorname{supp} f_{i_0}$, we have that

$$f' = \frac{1}{2}\left(\sum_{i=1}^{i_0-1} f_i + f_{i_0}^1 + f_{i_0}^2 + \sum_{i=i_0+1}^{d} f_i\right) \in W_T.$$

The above holds since $(f_1, \ldots, f_{i_0-1}, f_{i_0}^1, f_{i_0}^2, f_{i_0+1}, \ldots, f_d)$ is admissible. However, f' has the same coordinates as f except at the k_0 position where

$$f'(e_{k_0}) = \operatorname{sign} e_{k_0}^*(x)\tfrac{1}{2} \quad \text{and} \quad f(e_{k_0}) = \operatorname{sign} e_{k_0}^*(x)\tfrac{1}{2^n} \text{ for } n > 1.$$

Since $k_0 \in \operatorname{supp} x$ we have that $f'(x) - f(x) = \left(\frac{1}{2} - \frac{1}{2^n}\right)|e_{k_0}^*(x)| > 0$. Therefore $f'(x) > f(x)$. Since $f' \in W_T$ and $f(x) = \|x\|$, this is the desired contradiction. □

The next lemma is the critical observation that allows us to prove the main theorem. It is essentially an averaging argument that allows us to restrict our attention to a smaller collection of norming functions therefore enabling an upper bound on $j(n)$.

Lemma 1.8. *Let $x \in c_{00}$ and suppose $f \in W_T$ minimally norms x, with* $\operatorname{supp} f = E$. *Suppose that $(f_\sigma)_{\sigma \in \mathcal{T}_f}$ is a tree decomposition for f. If $\sigma \in \mathcal{T}_f$ with $|\sigma| \geq 2$ so that there is a k with $\sigma(k-2) = \sigma(k-1) = 1$ then $|\sigma| \leq k$.*

The above lemma roughly states that for any vector x there is a norming functional f for x so that its tree decomposition has very few consecutive 1s. In particular, if a node σ has two consecutive 1s then they must be contained in the last three coordinates of the node.

Proof. Fix $x \in c_{00}$ and fix $f \in W_T$ that minimally norms x, with $\operatorname{supp} f = E$. Let $(f_\sigma)_{\sigma \in \mathcal{T}_f}$ be a tree decomposition for f and fix $\sigma \in \mathcal{T}_f$ with $|\sigma| \geq 2$ so that there is a k with $\sigma(k-2) = \sigma(k-1) = 1$. For convenience let $g = f_{\sigma|_{k-3}}$ ($\sigma|_{k-3}$ is σ restricted to its first $k-3$ coordinates). In the case that $k = 3$, we have $\sigma|_{k-3} = \varnothing$. Let

$g_i = f_{\sigma|_{k-3}\frown i}$ for $i \in \{1, \ldots, d_{\sigma|_{k-3}}\}$ and $g_{(i,j)} = f_{\sigma|_{k-3}\frown(i,j)}$ for $i \in \{1, \ldots, d_{\sigma|_{k-3}}\}$ and $j \in \{1, \ldots, d_{\sigma|_{k-3}\frown i}\}$.

Set $m = \min \operatorname{supp} g$. Suppose first that $\max \operatorname{supp} g_{(1,1)} < 2m - 1$. This implies that $\operatorname{supp} g_{(1,1)} \in \mathcal{S}_1$. Since f minimally norms x and $g_{(1,1)}$ is a functional in the tree decomposition of f, Lemma 1.4 yields that $g_{(1,1)}$ norms $E_{g_{(1,1)}}x$ (where $\operatorname{supp} g_{(1,1)} = E_{g_{(1,1)}}$). Applying Lemma 1.6 for $g_{(1,1)}$ and $E_{g_{(1,1)}}x$, we conclude that $g_{(1,1)} \in W_1$. Therefore $|\sigma| \le k$. Therefore we may consider the case $\max \operatorname{supp} g_{(1,1)} \ge 2m - 1$. Set $d = d_{\sigma|_{k-3}}$ and $r = d_{\sigma|_{k-3}\frown 1}$. Note that $d, r \le m$. Define

$$h_1 := \tfrac{1}{2}(g_{(1,2)} + \cdots + g_{(1,d)} + g_{(2)} + \cdots + g_{(r)}),$$
$$h_2 := \tfrac{1}{2}(g_{(1,1)} + g_{(2)} + \cdots + g_{(r)}).$$

Observe that $h_2 \in W_T$ and since $d + r - 1 \le 2m \le \min \operatorname{supp} g_{1,2}$ we have $h_1 \in W_T$. Again, by Lemma 1.4, we have that $g(E_g x) = \|E_g x\|$, where $E_g = \operatorname{supp} g$. Since f minimally norms x, Lemma 1.5 yields that g minimally norms $E_g x$. Therefore since $\operatorname{supp} h_1 \subsetneq \operatorname{supp} g$ and $\operatorname{supp} h_2 \subsetneq \operatorname{supp} g$, we know that $h_1(x) < g(x)$ and $h_2(x) < g(x)$. This implies that

$$(g_{(1,2)} + \cdots + g_{(1,d)})(x) < g_{(1)}(x) \quad \text{and} \quad g_{(1,1)}(x) < g_{(1)}(x).$$

However, by definition, $g_{(1)}(x) = \frac{1}{2}(g_{(1,1)}(x)) + \frac{1}{2}(g_{(1,2)} + \cdots + g_{(1,d)})(x)$. This is a contradiction. Therefore the case $\max \operatorname{supp} g_{(1,1)} \ge 2m - 1$ is not possible. □

Corollary 1.9. *For each $x \in c_{00}$ there is an $f \in W_T$ that minimally norms x with a tree decomposition $(f_\sigma)_{\sigma \in \mathcal{T}_f}$ such that for each $\sigma \in \mathcal{T}_f$ either σ has no consecutive* 1*s, the third-to-last and second-to-last coordinates are* 1 *or the final two coordinates are* 1.

Let $\mathcal{T}_a$ be the set of all $\sigma \in \mathbb{N}^{<\mathbb{N}}$ that satisfy the conclusion of Corollary 1.9. For example $\sigma = (1, 1, 2, 3) \notin \mathcal{T}_a$ but $(2, 3, 1, 1, 2) \in \mathcal{T}_a$.

Lemma 1.10. *For each $\sigma \in T_a$ with $|\sigma| = k \ge 3$,*

$$\left(\sum_{i=1}^{k-1} [k - i][\sigma(i) - 1]\right) \ge \left(\frac{k-3}{2}\right)^2. \tag{3}$$

Proof. Suppose $|\sigma| = k = 2d + 1$ for $d \in \mathbb{N}$. By replacing $\sigma(1)$ by 1 and $\sigma(2)$ by 2, the quantity on the left-hand side of (3) does not increase. This new element is still in $\mathcal{T}_a$. Continuing in this manner, we see that the above is minimized by $\sigma = (1, 2, 1, 2, \ldots, 2, 1, 1, 1)$—that is, $d - 2$ many 2s. If $k = 2d$ we may do the same procedure described previously to see that the quantity is minimized by $\sigma = (1, 2, 1, 2, \ldots, 2, 1, 1)$. Plugging these in the above yields $\sum_{i=2}^{d} 2i = d^2 - 1$ in the odd case and $\sum_{i=1}^{d-1} 2i + 1 = d^2 - d$. Both of these quantities are larger than $\frac{1}{4}(k - 3)^2$, as desired. □

The next corollary follows from combining Corollary 1.9 and Lemma 1.10.

Corollary 1.11. *For each $x \in c_{00}$ there is an $f \in W_T$ that minimally norms x having a tree decomposition $(f_\sigma)_{\sigma\in\mathcal{T}_f}$ such that*

$$\min\left\{\sum_{i=1}^{|\sigma|-1}[|\sigma|-i][\sigma(i)-1] : \sigma \in \mathcal{T}_f\right\} \geq \left(\frac{|\sigma|-3}{2}\right)^2. \tag{4}$$

We need one more technical lemma before proceeding to the proof of the main theorem.

Lemma 1.12. *Suppose that $f \in W_T$ and* $\max\operatorname{supp} f \leq n$. *Suppose further that f minimally norms x for some $x \in c_{00}$. Then for $\sigma \in \mathcal{T}_f$ with $f_\sigma \in W_T \setminus W_1$ we have*

$$|\operatorname{supp} f_\sigma| \leq n - \left(\sum_{i=1}^{|\sigma|-1}[|\sigma|-i][\sigma(i)-1]\right). \tag{5}$$

We postpone the proof of Lemma 1.12 to the end of paper. We now recall Theorem A and give its proof.

Theorem 1.13. *For $n \in \mathbb{N}$ and $x \in c_{00}$ with* $\max\operatorname{supp} x = n$ *we have $\|x\|_{\lfloor 2\sqrt{n}+4\rfloor} = \|x\|$. That is, $j(n)$ is $O(n^{1/2})$.*

Proof. Let $x \in c_{00}$ with $\max\operatorname{supp} x = n$. Suppose further that f minimally norms x. Suppose that $\sigma \in \mathcal{T}_f$ with $|\sigma| \geq \lfloor 2\sqrt{n}+3\rfloor$. If $f_\sigma \in W_T \setminus W_1$ then by combining Lemma 1.12 and Corollary 1.11, we know that

$$\begin{aligned}|\operatorname{supp} f_\sigma| &\leq n - \left(\sum_{i=1}^{|\sigma|-1}[|\sigma|-i][\sigma(i)-1]\right) \leq n - \left(\frac{|\sigma|-3}{2}\right)^2 \\ &\leq n - \left(\frac{2\sqrt{n}+3-3}{2}\right)^2 = 0.\end{aligned} \tag{6}$$

Therefore no such σ exists. Thus if $|\sigma| \geq \lfloor 2\sqrt{n}+3\rfloor$ we have $f_\sigma \in W_1$. Therefore $\max\{|\sigma| : \sigma \in \mathcal{T}_f\} \leq \lfloor 2\sqrt{n}+4\rfloor$, which implies that $f \in W_{\lfloor 2\sqrt{n}+4\rfloor}$. Since $f(x) = \|x\|$, this is the desired result. □

We conclude by proving Lemma 1.12.

Proof. Let $x \in c_{00}$ and suppose $f \in W_T$ minimally norms x with $\max\operatorname{supp} f \leq n$. Let $\sigma \in \mathcal{T}_f$ with $f_\sigma \in W_T \setminus W_1$. Set $\ell = \min\operatorname{supp} f$. We will prove the following inequality, which is stronger than the desired estimate:

$$|\operatorname{supp} f_\sigma| \leq n - (|\sigma|+1)(\ell-1) - \left(\sum_{i=1}^{|\sigma|-1}[|\sigma|-i][\sigma(i)-1]\right). \tag{7}$$

First we need the inequality

$$\min \operatorname{supp} f_\sigma \geq \ell + s(\sigma) - |\sigma|. \tag{8}$$

Here $s(\sigma) = \sum_{i=1}^{k} \sigma(i)$ for $|\sigma| = k$. To prove (8), we let $|\sigma| = k$ and use induction on k. Let $\sigma|_{k-1} = (n_1, \dots, n_{k-1})$ if $\sigma = (n_1, \dots, n_{k-1}, n_k)$. In the base case of $|\sigma| = 1$, we know $\min \operatorname{supp} f_\sigma \geq \ell + s(\sigma) - 1$, since there are at least $(s(\sigma)-1)$-many values from ℓ to f_σ's beginning index (worst case being that all prior functionals are in W_0). Now we assume $\min \operatorname{supp} f_\sigma \geq \ell + s(\sigma) - |\sigma|$ for some $|\sigma| = k \in \mathbb{N}$ and show the same inequality holds for $|\sigma| = k+1$:

$$\begin{aligned}
\min \operatorname{supp} f_\sigma &\geq \min \operatorname{supp} f_{\sigma|_k} + \sigma(k+1) - 1 \\
&\geq \ell + s(\sigma|_k) - |\sigma|_k| + \sigma(k+1) - 1 \\
&= \ell + s(\sigma) - (k+1).
\end{aligned}$$

The first inequality relies on the fact that f_σ can have the same minimum support value as $f_{\sigma|_k}$ if $\sigma(k+1) = 1$. The second inequality above follows from the inductive hypothesis. The lone equality above follows from the facts that $s(\sigma|_k) + \sigma(k+1) = s(\sigma)$ and $|\sigma|_k| = k$. Thus, (8) holds.

The proof of the inequality (8) begins with the observation that for all σ with $|\sigma| = k$ we have

$$|\operatorname{supp} f_\sigma| \leq |\operatorname{supp} f_{\sigma|_{k-1}}| - \#\{\text{immediate successor of } \sigma|_{k-1}\} + 1.$$

The fact that f minimally norms x combined with Lemma 1.7 implies that for each $\sigma \in \mathcal{T}$ with $f_\sigma \in W_T \setminus W_1$, the number of immediate successor nodes of $\sigma|_{k-1}$ equals $\min \operatorname{supp} f_{\sigma|_{k-1}}$. Therefore in this case

$$|\operatorname{supp} f_\sigma| \leq |\operatorname{supp} f_{\sigma|_{k-1}}| - (\min \operatorname{supp} f_{\sigma|_{k-1}}) + 1.$$

Now let $|\sigma| = k$ and use induction on k. It follows from the induction hypothesis and rearranging terms that

$$\begin{aligned}
|\operatorname{supp} f_\sigma| &\leq |\operatorname{supp} f_{\sigma|_{k-1}}| - (\min \operatorname{supp} f_{\sigma|_{k-1}}) + 1 \\
&\leq n - k(\ell-1) - \left(\sum_{i=1}^{k-2} [(k-1)-i][\sigma(i)-1]\right) - \ell - s(\sigma|_{k-1}) + (k-1) + 1 \\
&\leq n - (k+1)(\ell-1) - \left(\sum_{i=1}^{k-2} [(k-1)-i][\sigma(i)-1]\right) - \sum_{i=1}^{k-1} [\sigma(i)-1] \\
&= n - (k+1)(\ell-1) - \sum_{i=1}^{k-1} [k-i][\sigma(i)-1].
\end{aligned}$$

This is the desired estimate. □

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