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# Enumeration of stacks of spheres

Lauren Endicott, Russell May and Sienna Shacklette

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As a three-dimensional generalization of fountains of coins, we analyze stacks of spheres and enumerate two particular classes, so-called “pyramidal” stacks and “Dominican” stacks. Using the machinery of generating functions, we obtain exact formulas for these types of stacks in terms of the sizes of their bases.

## 1. Introduction

Odlyzko and Wilf [1988] analyzed fountains of coins. An $(n, k)$ fountain is an arrangement of $n$ coins into rows so that the bottom row consists of $k$ contiguous coins and each coin in higher rows sits on two coins in the row beneath it. Figure 1 shows a $(25, 12)$ fountain. Two fountains are different if in any row and any position in the row, one fountain has a coin, but the other does not. Their goal was to enumerate the numbers $f_{n,k}$ of $(n, k)$ fountains, and their main result was that the bivariate generating function $F(x, y) = \sum_{n,k} f_{n,k}\, x^n y^k$ was the continued fraction

$$F(x, y) = \cfrac{1}{1 - \cfrac{xy}{1 - \cfrac{x^2 y}{1 - \cfrac{x^3 y}{\ddots}}}}. \tag{1}$$

If the fountains are enumerated only by the number of coins in the bottom row, $g_k = \sum_n f_{n,k}$, then the generating function $G(y) = \sum_k g_k y^k = F(1, y)$ is much simpler. It is straightforward from (1) that the generating function satisfies $G(y) - yG^2(y) = 1$, and so the $g_k$ are the Catalan numbers. Wilf [2006, Example 2.12] also considered a restricted class of “block” fountains having the property that each row must be a contiguous block of coins. If $b_k$ is the number of block fountains with $k$ coins in the bottom row, then $B(y) = \sum_k b_k y^k$ turns out to be $(1 - 2x)/(1 - 3x + x^2)$, which is the generating function for the Fibonacci numbers with odd indices.

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*Keywords:* enumerative combinatorics, generating functions.

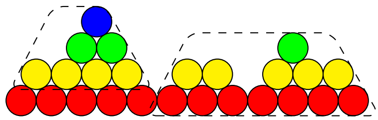

**Figure 1.** A (25, 12) fountain of coins with subfountains around the first missing coin in the second row.

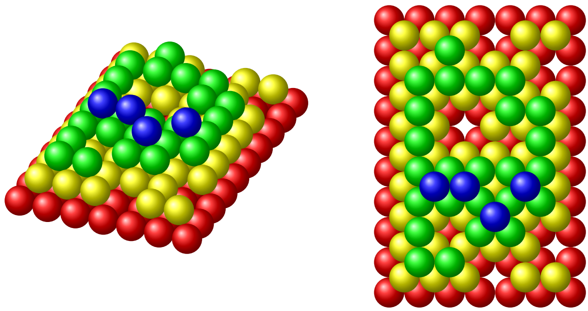

**Figure 2.** A (148, 7, 10) stack of spheres from oblique and top views.

**Figure 3.** Models for stacks: Pyramid of the Sun at Teotihuaca, Mexico (credit: [Lneuw 2006]), and the flag of the Dominican Republic.

As a three-dimensional variant of fountains of coins, we consider stacks of spheres. An $(\ell, m, n)$ stack of spheres is an arrangement of $\ell$ spheres into levels so that the bottom level consists of spheres in an $m \times n$ rectangular grid and each sphere in higher levels sits on four spheres in the level beneath it. Figure 2 shows a (148, 7, 10) stack of spheres. In grocery stores, fruits like oranges and cantaloupes are often arranged into such stacks.

Our goal is to analyze two classes of stacks, *pyramidal* and *Dominican*, and obtain generating functions and exact formulas for the number of stacks in terms of the sizes of their bases. Pyramidal stacks have the property that every level consists of a single rectangular grid of spheres, much like the Pyramid of the Sun at Teotihuaca, shown in Figure 3. Dominican stacks are closer in spirit to general stacks. Their inductive definition closely resembles the color scheme of solid regions and stripes in the flag of the Dominican Republic, also shown in Figure 3.

## 2. Basics of generating functions

Generating functions are a bridge between the discrete world of combinatorics and the continuous world of calculus and complex analysis. Wilf [2006] embraces a five-step method for describing sequences with generating functions:

(1) Find a recurrence relation for the sequence.

(2) Define the generating function.

(3) Convert the recurrence relation to a relation about the generating function.

(4) Solve for the generating function.

(5) Extract or approximate the coefficients of the generating function.

We use a handful of well-known generating functions, based on the geometric series and its derivatives:

$$\sum_{n\geq 0} 1x^n = \frac{1}{1-x}, \tag{2a}$$

$$\sum_{n\geq 0} nx^n = \frac{x}{(1-x)^2}, \tag{2b}$$

$$\sum_{n\geq 0} \binom{n+k}{n} x^n = \frac{1}{(1-x)^{1+k}}. \tag{2c}$$

We also use the product rule: if $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$, then $f(x)\cdot g(x) = \sum_{n\geq 0} c_n x^n$, where $c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}$. An important special case of the product rule is the partial sum rule: $\sum_{n\geq 0}(a_0 + a_1 + \cdots + a_n)x^n = 1/(1-x) f(x)$. We also need the bivariate version of the product rule; namely if $f(x, y) = \sum_{m,n\geq 0} a_{m,n} x^m y^n$ and $g(x, y) = \sum_{m,n\geq 0} b_{m,n} x^m y^n$, then $f(x, y) \cdot g(x, y) = \sum_{m,n\geq 0} c_{m,n} x^m y^n$, where $c_{m,n} = \sum_{m',n'} a_{m',n'} \cdot b_{m-m',n-n'}$. Following standard notation, we define the bivariate coefficient extraction operator as

$$[x^m y^n] \sum_{m,n\geq 0} a_{m,n} x^m y^n = a_{m,n}.$$

## 3. Pyramidal stacks of spheres

One of the simplest types of fountains of coins is a block fountain, defined by the property that each row consists of a single contiguous block of coins. We define a corresponding three-dimensional variant, a *pyramidal stack of spheres*, to be a stack where each level consists of a single rectangular grid of spheres. An example of a pyramidal stack is depicted in Figure 4. Unlike arbitrary stacks, pyramidal stacks are constrained to have only a single spire. We would like to enumerate the pyramidal stacks by the size of their bases. For $m, n \geq 1$, let $p_{m,n}$

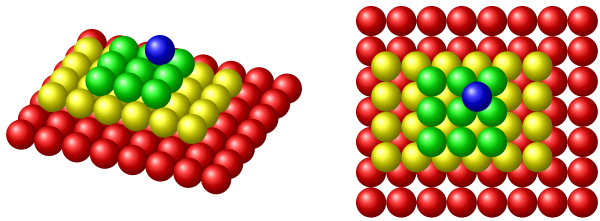

**Figure 4.** An $8 \times 7$ pyramidal stack of spheres from oblique and top views.

| $p_{m,n}$ | $n$=1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $m$=1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 4 | 7 | 11 | 16 | 22 |
| 3 | 1 | 4 | 11 | 24 | 46 | 81 | 134 |
| 4 | 1 | 7 | 24 | 63 | 143 | 294 | 561 |
| 5 | 1 | 11 | 46 | 143 | 376 | 881 | 1894 |
| 6 | 1 | 16 | 81 | 294 | 881 | 2317 | 5534 |
| 7 | 1 | 22 | 134 | 561 | 1894 | 5534 | 14545 |

**Table 1.** Values of $p_{m,n}$ for $m, n \leq 7$.

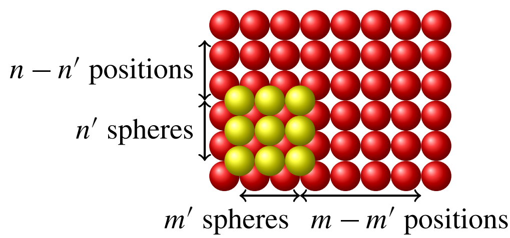


**Figure 5.** Possible positions of an $m' \times n'$ pyramid on top of an $m \times n$ base.

be the number of pyramidal stacks of spheres whose base consists of an $m \times n$ grid of spheres. For convenience, let $p_{m,0} = p_{0,n} = 0$ for all $m, n \geq 0$, and note that by symmetry $p_{m,n} = p_{n,m}$. Then define the bivariate generating function $P(x, y) = \sum_{m,n \geq 0} p_{m,n} x^m y^n$. By hand calculation and assistance from Maple, we computed $p_{m,n}$ for $m, n \leq 7$, shown in Table 1. A pyramidal stack with an $m \times n$ base can either contain nothing on the second level or support another pyramidal stack with an $m' \times n'$ base, where $1 \leq m' < m$ and $1 \leq n' < n$. If the second level is nonempty, it can be shifted horizontally to $m - m'$ positions and vertically to $n - n'$ positions to form different stacks, as shown in Figure 5. Therefore, we have the recurrence relation for pyramidal stacks for $m, n \geq 1$, given as

$$p_{m,n} = 1 + \sum_{\substack{1 \leq m' \leq m-1 \\ 1 \leq n' \leq n-1}} (m-m')(n-n')\, p_{m',n'} = 1 + \sum_{\substack{0 \leq m' \leq m \\ 0 \leq n' \leq n}} (m-m')(n-n')\, p_{m',n'}.$$

The bounds on the sum can be extended from 0 to $m$ and $n$ since $p_{0,n} = p_{m,0} = 0$ and $(m-m') = (n-n') = 0$ when $m' = m$ and $n' = n$. Then, by use of the generating functions in (2a) and (2b) and the product rule we get

$$P(x, y) = \frac{x}{1-x}\frac{y}{1-y} + P(x, y)\frac{xy}{(1-x)^2(1-y)^2}.$$

Solving this equation results in the rational generating function

$$P(x, y) = \frac{xy(1-x)(1-y)}{(1-x)^2(1-y)^2 - xy}. \tag{3}$$

To obtain an exact expression for $p_{m,n}$, we first view $P(x, y)$ as a geometric series:

$$P(x, y) = \frac{\frac{xy}{(1-x)(1-y)}}{1 - \frac{xy}{(1-x)^2(1-y)^2}} = \sum_{\ell \geq 0} \frac{x^{\ell+1} y^{\ell+1}}{(1-x)^{1+2\ell}(1-y)^{1+2\ell}}$$

Then, using (2c),

$$\begin{aligned} p_{m,n} &= [x^m y^n] P(x, y) \\ &= \sum_{\ell \geq 0} [x^{m-\ell-1} y^{n-\ell-1}] \frac{1}{(1-x)^{1+2\ell}(1-y)^{1+2\ell}} \\ &= \sum_{\ell \geq 0} \binom{m+\ell-1}{m-\ell-1}\binom{n+\ell-1}{n-\ell-1}. \end{aligned}$$

This exact expression for $p_{m,n}$ is a sum with $\min(m-1, n-1)$ terms, a significant improvement over the recursion that requires $O(mn)$ computations. Also, note that $g_m$, the number of block fountains with $m$ coins in the bottom row, is equivalent to the $(2m+1)$-th Fibonacci number, which is also expressible as the sum $\sum_\ell \binom{m+\ell-1}{m-\ell-1}$. Therefore, pyramidal stacks of spheres can be viewed as a direct generalization of block fountains of coins.

## 4. Dominican stacks

Pyramidal stacks, having only a single spire, form an extremely restricted class, just as block fountains are to general fountains of coins. We would like to analyze a more robust class that is closer in spirit to general stacks. Dominican stacks are a three-dimensional generalization of arbitrary two-dimensional fountains of coins.

In order to motivate the definition of Dominican stacks, let's review general fountains of coins. A fountain with $m$ coins in the bottom row can be uniquely

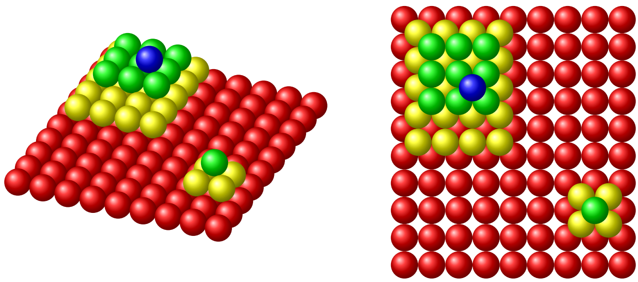

**Figure 6.** A $9 \times 10$ Dominican stack of spheres from oblique and top views.

decomposed into two subfountains by locating the first position in the second row, say at $m'$, where a coin is missing. For instance, the second row of the fountain in Figure 1 has its first missing coin in the fifth position. Thus, a general fountain consists of the subfountain on the left with a base of $m'$ coins, whose second row is full and so consists of an even smaller subfountain with a diminished base of $m' - 1$ coins, and a subfountain on the right with a base of $m - m'$ coins. So, the recurrence relation $g_m = \sum_{m'} g_{m'-1} \cdot g_{m-m'}$ holds for fountains of coins.

We make an analogous definition by induction for stacks of spheres. A *Dominican* stack of spheres is defined by the following cases:

*Base case*: A single level of spheres in a rectangular grid.

*Inductive case*: A multilevel stack of spheres with an $m \times n$ base built from smaller stacks, as follows. It is required that, when viewed from the top, there exist a (necessarily unique) column at position $m'$ and row at position $n'$ devoid of spheres so that the following conditions hold:

*Bottom left*: Every position in the second level above positions $[1, \dots, m'] \times [1, \dots, n']$ has a sphere, and the stack with diminished base of size $(m' - 1) \times (n' - 1)$ from the second level and above is Dominican.

*Top right*: The stack above positions $[m' + 1, \dots, m] \times [n' + 1, \dots, n]$ from the first level and above is Dominican.

*Bottom right and top left*: The stacks above positions $[m'+1, \dots, m] \times [1, \dots, n']$ and $[1, \dots, m'] \times [n' + 1, \dots, n]$ consist solely of rectangular grids of spheres on the first level with nothing above.

An example of a Dominican stack is shown in Figure 6. Informally, a stack of spheres is called Dominican because of its resemblance to the flag of the Dominican Republic. As a visualization depicted in Figure 7, imagine placing a version of the Dominican Republic's flag under a stack of spheres and looking at the stack from above. On the second level, the white stripes of the flag appear in the row and

| $d_{m,n}$ | $n$=1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $m$=1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 1 | 3 | 7 | 12 | 18 | 25 | 33 |
| 4 | 1 | 4 | 12 | 28 | 52 | 85 | 128 |
| 5 | 1 | 5 | 18 | 52 | 122 | 239 | 416 |
| 6 | 1 | 6 | 25 | 85 | 239 | 564 | 1147 |
| 7 | 1 | 7 | 33 | 128 | 416 | 1147 | 2723 |

**Table 2.** Values of $d_{m,n}$ for $m, n \leq 7$.

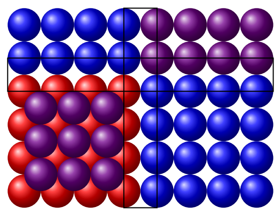


**Figure 7.** Indexing of a Dominican stack with regions of the bottom layer in red and blue and smaller Dominican stacks in purple.

column which are empty but the red region to the bottom-left is full; the substack above this red region is a smaller Dominican stack. The portion of the stack on the top-right above the other red region of the flag also forms a smaller Dominican stack. The portions of the stack above the blue regions are single layers of spheres. Note that if the first empty column ($m' = 1$ from the definition) in the second level forms the boundary, then the corresponding row must also be the first ($n' = 1$), and likewise for the converse. Therefore, the emblem of the flag where the white stripes cross may either be in position $[1, 1]$ of the second level or in a position $[m', n']$ with $2 \leq m' \leq m$ and $2 \leq n' \leq n$.

Let $d_{m,n}$ be the number of Dominican stacks with an $m \times n$ base. For convenience, let $d_{0,n} = d_{m,0} = 1$ for all $m, n \geq 0$, and note that by symmetry $d_{m,n} = d_{n,m}$. We define the bivariate generating function $D(x, y) = \sum_{m,n\geq 0} d_{m,n} x^n y^n$. By hand calculation and assistance from Maple, we computed $d_{m,n}$ for $m, n \leq 7$, shown in Table 2. In order to derive an expression for this generating function, we begin with the recurrence relation for the $d_{m,n}$ that follows immediately from the inductive definition of Dominican stacks. Equation (4b) is just a reindexing of (4a), and (4c) accounts for the inclusion of the terms $m' = 0$ or $n' = 0$ in the sum; see the following:

$$d_{m+1,n+1} = d_{m,n} + \sum_{\substack{2\le m'\le m+1\\ 2\le n'\le n+1}} d_{m'-1,n'-1}\, d_{m+1-m',n+1-n'} \quad (m, n \ge 0) \tag{4a}$$

$$= d_{m,n} + \sum_{\substack{1\le m'\le m\\ 1\le n'\le n}} d_{m',n'} d_{m-m',n-n'} \tag{4b}$$

$$= 2d_{m,n} + \sum_{\substack{0\le m'\le m\\ 0\le n'\le n}} d_{m',n'} d_{m-m',n-n'} - \sum_{0\le m'\le m} d_{m',n} - \sum_{0,\le n'\le n} d_{m,n'}. \tag{4c}$$

Using the generating functions in (2a) and (2b), along with the product rule and the partial sum rule, we obtain the corresponding relation for $D(x, y)$, namely

$$D(x, y) - \frac{x}{1-x} - \frac{y}{1-y} - 1 = 2xyD(x, y) + xyD^2(x, y) - \frac{xy}{1-x}D(x, y) - \frac{xy}{1-y}D(x, y).$$

This relation is quadratic in $D(x, y)$ with coefficients that are polynomials in $x$, $y$. After some algebra, we get a ratio of polynomials with a radical for the generating function:

$$D(x, y) = \frac{(1-xy)(1-x-y+2xy) - \sqrt{(1-xy)^2(1-x-y+2xy)^2 - 4xy(1-x)(1-y)(1-xy)}}{2xy(1-x)(1-y)}. \tag{5}$$

This generating function agrees with the values of $d_{m,n}$ from the recurrence relation. $D(x, y)$ can be viewed as a generalization of the generating function for the Catalan numbers. Unfortunately, asymptotic analysis even of rational bivariate generating functions is difficult, so analysis of this generating function with a radical will require further investigation.

## 5. Further problems

Stacking of spheres lends itself to many one-parameter combinatorial classes. For example, the class of pyramidal stacks can be restricted by the condition that each level forms a *square* grid of spheres. If $s_n$ is the number of such pyramids with an $n \times n$ base, it follows quickly that the generating function $\sum s_n x^n$ is $x(1-x)^3/(1-4x+2x^2-x^3)$. Many geometrical variants with triangles, hexagons, etc., can be formed in this manner and result in single-variable generating functions.

Another restriction leading to single-variable generating functions is to fix the width of the base of a *general* stack. For each base width $m \ge 1$ and length $n \ge 1$, let $a_{m,n}$ be the number of general stacks with an $m \times n$ base and define

$A_m(x) = \sum_n a_{m,n} x^n$. It turns out that

$$A_1(x) = \frac{1}{1-x}, \quad A_2(x) = \frac{x}{1-2x}, \quad A_3(x) = \frac{x(1-x)}{1-5x+3x^2}.$$

The $A_m$ for $m \geq 4$ are more difficult to compute, but would surely shed light on the general case.

However, the main problem of enumerating general stacks of spheres remains unsolved. While it is hoped that recursive methods similar to the pyramidal and Dominican cases will ultimately work out, it is entirely possible that altogether different machinery may be required to enumerate general stacks. The most direct reason that recursive methods may fail is that partitions of rectangles into collections of subrectangles are unwieldy. Additionally, since the generating function of a general fountain of coins (based on the total number of coins) in two dimensions is already significant in complexity as a continued fraction, one must expect the generating function of a general stack of spheres in three dimensions to be an order of difficulty harder.

A more tractable problem may be an asymptotic approximation of the coefficients of the generating functions in (3) and (5). Recent results in [Pemantle and Wilson 2013] may provide insight.

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