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Analytical solution of a one-dimensional thermistor problem with Robin boundary condition

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A one-dimensional nonlinear heat conduction equation of steady-state Joule heating in the presence of an electric field in a metal with temperature-dependent conductivities is considered. A technique developed by Young (1986) is adapted and used to derive an analytical solution for the problem with a Robin boundary condition.

1. Introduction

A thermal resistor, or thermistor, is a type of resistor with a highly temperature-dependent electrical conductivity. Thermistors are used as temperature-control elements in a range of equipment, such as spacecraft and air conditioning units, and have applications in the medical field, meteorology, and the chemical industry [Ng 1995; Macklen 1979]. The thermistor problem has been a source of significant mathematical interest and research but, due to the nonlinear nature of the problem, this research has been largely concerned with numerical solutions or existence proofs for a solution [Antontsev and Chipot 1994; Fowler et al. 1992; Howison et al. 1993; Shi et al. 1993; Sidi Ammi and Torres 2008; Xu 2004a; 2004b; Zhou and Westbrook 1997], rather than analytical solutions. In this paper, the thermistor problem is modeled as a nonlinear heat conduction equation of steady-state Joule heating in the presence of an electric field in a metal with temperature-dependent electrical and thermal conductivities. This paper extends the solution found in [Young 1986] to a more general case by introducing a Robin boundary condition on the temperature at an endpoint of the thermistor. This establishes the existing solution in [Young 1986] as a special case.

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Keywords: thermistor problem, Robin boundary condition, Joule heating, analytical solution.

2. Formulation of the problem

Assuming that the electrical conductivity $\sigma(T)$ and the thermal conductivity $\kappa(T)$ are smooth functions, the heat conduction in an electrical conductor in the presence of Joule heating due to current can be shown [Young 1986] to satisfy the following two equations, one for the potential Φ and one for the temperature T:

$$\nabla^2\Phi = -\frac{1}{\sigma}\nabla\sigma\cdot\nabla\Phi \quad \text{in } \Omega, \tag{1}$$

$$\frac{d^2T}{d\Phi^2} + \left(\frac{1}{\kappa}\frac{d\kappa}{dT} - \frac{1}{\sigma}\frac{d\sigma}{dT}\right)\left(\frac{dT}{d\Phi}\right)^2 = -\frac{\sigma}{\kappa} \quad \text{in } \Omega, \tag{2}$$

with some appropriate boundary conditions. Throughout this section and Section 3 we assume that the given domain Ω lies in $\mathbb{R}^n$. Equations (1) and (2) respectively describe conservation of charge and the steady diffusion of heat in the presence of Joule heating due to electric current (see [Young 1986] for more details).

3. Derivation of the solution in the general case

For the sake of convenience, we recreate the technique developed in [Young 1986] for obtaining a solution to (1) and (2). Equation (2) can be simplified by introducing a new variable

$$\frac{\sigma(T)}{\kappa(T)} = e^{-\xi(T)}. \tag{3}$$

Differentiating both sides of (3), and after some manipulations, (2) can be written as

$$\frac{d^2T}{d\Phi^2} + \frac{d\xi}{dT}\left(\frac{dT}{d\Phi}\right)^2 = -e^{-\xi}. \tag{4}$$

Next, setting

$$\theta = \frac{dT}{d\Phi}, \tag{5}$$

(4) becomes

$$\theta\frac{d\theta}{dT} + \frac{d\xi}{dT}\theta^2 = -e^{-\xi}. \tag{6}$$

Observing

$$\tfrac{1}{2}e^{-2\xi}\frac{d}{dT}(e^{2\xi}\theta^2) = \theta\frac{d\theta}{dT} + \frac{d\xi}{dT}\theta^2$$

allows us to rewrite (6) as

$$\tfrac{1}{2}e^{-2\xi}\frac{d}{dT}(e^{2\xi}\theta^2) = -e^{-\xi}. \tag{7}$$

Integrating (7), we get

$$e^{2\xi(T)}\theta^2 = C - 2\int e^{\xi(T)}\,dT, \tag{8}$$

where C is a constant of integration. Solving for θ^2 in (8) and taking into account (5), we have

$$\left(\frac{dT}{d\Phi}\right)^2 = \frac{C - 2\int e^{\xi(T)}\,dT}{e^{2\xi(T)}}.$$

Finally, the following equation for Φ is obtained:

$$\Phi = \int \frac{(\kappa(T)/\sigma(T))\,dT}{\sqrt{C - 2\int (\kappa(s)/\sigma(s))\,ds}} + C', \tag{9}$$

where the integration constant of the integral under the square root is absorbed into C. It turns out that for many metals, the ratio of conductivities is proportional to the absolute temperature of the metal. This relationship is known as the Wiedemann–Franz–Lorenz (WFL) law [Berman 1976; Meaden 1965],

$$\frac{\kappa(T)}{\sigma(T)} = \alpha T, \tag{10}$$

where α is the Lorenz number for a given metal and may have slightly different valuesfor different metals. Once the ratio of conductivities is specified using the WFL law, (9) can be integrated to obtain the temperature in terms of the potential, $T(\Phi)$,

$$T(\Phi) = \frac{1}{\sqrt{\alpha}}[C - (\Phi - C')^2]^{1/2}. \tag{11}$$

For (11) to be of any help, we must determine Φ that solves (1). This issue can be dealt with by introducing an auxiliary potential Ψ that satisfies Laplace's equation [Young 1986; Flynn 1969]. Namely, define this auxiliary potential Ψ as

$$\sigma_0 \nabla \Psi \equiv \sigma[T(\Phi)]\nabla\Phi,$$

where σ_0 is the electrical conductivity at some conveniently chosen reference temperature. Then clearly Ψ satisfies Laplace's equation $\nabla^2\Psi = 0$, and it is easily seen, by isolating $\nabla\Psi$ and integrating, that

$$\Psi = \frac{1}{\sigma_0}\int \sigma[T(\Phi)]\,d\Phi. \tag{12}$$

Knowing how σ depends on T and using (11) will enable us to perform integration in (12). This will give us Ψ in terms of Φ, and therefore finding the inverse of this function will result in an expression for Φ in terms of a function that satisfies Laplace's equation. In the next section, we will also solve for the constants C and C' in the expression. This ends the derivation of a solution to (1) and (2) when conductivities obey the WFL law.

4. Solution in one dimension with Robin boundary condition

In this section we adapt the technique described in Section 3 for a one-dimensional problem with a Robin boundary condition at one endpoint. Namely, consider a thin rod of length L, where the potential and temperature satisfy (1) and (2), respectively. In addition, the endpoints at $z = 0$ and $z = L$ are held at potentials V_0 and 0, respectively. The boundary condition for T at the right endpoint of the rod $z = L$ is given by a Robin boundary condition, whereas the left endpoint is held at the constant temperature $T = T_0$. Note that it is reasonable to consider a Robin boundary condition for at least one end of the rod, as it models the cooling effect of that end of the thermistor through Newton's law of cooling [Howison 2005]. The boundary conditions are summarized below:

$$T = T_0, \quad \Phi = V_0 \quad \text{at } z = 0, \tag{13}$$

$$\frac{dT}{dz} + \beta(T - T_0) = 0, \quad \Phi = 0 \quad \text{at } z = L. \tag{14}$$

Observe that when β approaches infinity, the boundary condition for T at $z = L$ reduces to $T = T_0$ at $z = L$, which corresponds to that in [Young 1986]. Recall that

$$T(\Phi) = \frac{1}{\sqrt{\alpha}}[C - (\Phi - C')^2]^{1/2}.$$

Now we use the boundary conditions (13) and (14) to determine the constants C and C'. From (13), it is immediate that

$$C - C'^2 = \alpha T_0^2 + V_0^2 - 2V_0C'. \tag{15}$$

First, we find

$$\begin{aligned}\frac{dT}{dz} &+ \beta(T - T_0)\\ &= \frac{C' - \Phi}{\sqrt{\alpha}} \frac{1}{[C - (\Phi - C')^2]^{1/2}} \frac{d\Phi}{dz} + \beta\left(\frac{1}{\sqrt{\alpha}}[C - (\Phi - C')^2]^{1/2} - T_0\right)\end{aligned} \tag{16}$$

and using (14) to evaluate (16) at $z = L$ gives us

$$\frac{C'}{\sqrt{\alpha}} \frac{\Phi_0}{[C - C'^2]^{1/2}} + \beta\left(\frac{1}{\sqrt{\alpha}}[C - C'^2]^{1/2} - T_0\right) = 0, \tag{17}$$

where we defined $\Phi_0 := \Phi'(L)$. Note that a new parameter Φ_0 has been introduced into the problem. We will address this issue later. Rewriting (17) as

$$\frac{C'}{\beta} \frac{\Phi_0}{[C - C'^2]^{1/2}} + [C - C'^2]^{1/2} = \sqrt{\alpha} T_0 \tag{18}$$

and squaring both sides of (18), we get

$$\left(\frac{C'\Phi_0}{\beta}\right)^2 + \frac{2\Phi_0 C'}{\beta}(C - C'^2) + (C - C'^2)^2 = \alpha T_0^2 (C - C'^2).$$

Now using (15) and grouping the result by C'^2, we obtain the following quadratic equation for C':

$$C'^2\left[\left(\frac{\Phi_0}{\beta}\right)^2 - \frac{4V_0\Phi_0}{\beta} + 4V_0^2\right] + C'\left[\frac{2\Phi_0}{\beta}(\alpha T_0^2 + V_0^2) - 4V_0^3 - 2\alpha V_0 T_0^2\right] + \alpha T_0^2 V_0^2 + V_0^4 = 0.$$

Defining

$$A = \left(\frac{\Phi_0}{\beta}\right)^2 - \frac{4V_0\Phi_0}{\beta} + 4V_0^2,$$
$$B = \frac{2\Phi_0}{\beta}(\alpha T_0^2 + V_0^2) - 4V_0^3 - 2\alpha V_0 T_0^2,$$
$$D = \alpha T_0^2 V_0^2 + V_0^4,$$

the solution for C' is given by

$$C' = \frac{-B \pm \sqrt{B^2 - 4AD}}{2A}, \tag{19}$$

where

$$B^2 = \frac{4\Phi_0(\alpha T_0^2 + V_0^2)^2 - 4\Phi_0\beta(\alpha T_0^2 + V_0^2)(4V_0^3 + 2\alpha V_0 T_0^3) + \beta^2(4V_0^3 + 2\alpha V_0 T_0^2)^2}{\beta^2},$$

$$4AD = \frac{4\Phi_0^2(\alpha T_0^2 + V_0^2)V_0^2 - 16\Phi_0\beta V_0(\alpha T_0^2 + V_0^2)V_0^2 + \beta^2 16 V_0^2(\alpha T_0^2 V_0 + V_0^3)V_0}{\beta^2},$$

so that

$$B^2 - 4AD = \frac{4\Phi_0^2\alpha T_0^2(\alpha T_0^2 + V_0^2) - 8\Phi_0\beta\alpha V_0 T_0^2(\alpha T_0^2 + V_0^2) + 4\beta^2\alpha^2 V_0^2 T_0^4}{\beta^2}$$
$$= \frac{4\Phi_0^2\alpha T_0^2(\alpha T_0^2 + V_0^2)}{\beta^2} - \frac{8\Phi_0\alpha V_0 T_0^2(\alpha T_0^2 + V_0^2)}{\beta} + 4\alpha^2 V_0^2 T_0^4.$$

Now it can be verified that this solution to the quadratic equation will match that in [Young 1986] when $\beta \to \infty$. In this case, the first two terms above disappear and we have

$$\lim_{\beta\to\infty} A = 4V_0^2, \quad \lim_{\beta\to\infty} B = -4V_0^3 - 2\alpha V_0 T_0^2, \quad \lim_{\beta\to\infty} [B^2 - 4AD] = 4\alpha^2 V_0^2 T_0^4.$$

We are thus left with

$$\lim_{\beta\to\infty} C' = \lim_{\beta\to\infty}\left[\frac{-B\pm\sqrt{B^2-4AD}}{2A}\right] = \frac{4V_0^3+2\alpha V_0 T_0^2 \pm\sqrt{4\alpha^2 V_0^2 T_0^4}}{8V_0^2}.$$

Taking the negative square root, we get $C' = \frac{1}{2}V_0$, as in [Young 1986]. The constants C and C' in terms of β are as follows:

$$\begin{aligned} C' &= \frac{-B - 2T_0\sqrt{\alpha}\sqrt{(\Phi_0-\beta V_0)^2(\alpha T_0^2+V_0^2)-\beta^2 V_0^4}}{2A}, \\ C &= \alpha T_0^2 + V_0^2 - 2V_0 C' + C'^2, \end{aligned} \tag{20}$$

where A and B are defined above. We take the negative root in the quadratic formula for C' because this is the root that reduces to Young's solution when $\beta\to\infty$. Now we use the auxiliary potential Ψ, given in (12),

$$\Psi = \frac{1}{\sigma_0}\int \sigma[T(\Phi)]\,d\Phi.$$

First, we note that it is an experimentally verified fact that the thermal conductivity κ varies very little with temperature for many metals; see [Young 1986]. Therefore, it is physically reasonable to assume that $\kappa(T) = \kappa_0$, where κ_0 is a constant. Now, taking $\sigma[T(\Phi)]$ to obey the WFL law (10), we have

$$\sigma[T(\Phi)] = \frac{\kappa[T(\Phi)]}{\alpha T(\Phi)} = \frac{\kappa_0}{\alpha T(\Phi)}. \tag{21}$$

Substituting (21) into (12), we get

$$\Psi = \frac{\kappa_0}{\alpha\sigma_0}\int\frac{d\Phi}{T(\Phi)} = \frac{\kappa_0}{\alpha\sigma_0}\int\frac{d\Phi}{(1/\sqrt{\alpha})[C-(\Phi-C')^2]^{1/2}} = \frac{\kappa_0}{\alpha\sigma_0}\sin^{-1}\left(\frac{\Phi-C'}{\sqrt{C}}\right).$$

Since $\nabla^2\Psi = 0$, it follows that Ψ is a linear function of z. We set $\Psi(z) = a + bz$ and absorb the constant $\kappa_0/(\alpha\sigma_0)$, as well as the integration constant of $\Psi(z)$, into the coefficients of the linear function. Hence,

$$\sin^{-1}\left(\frac{\Phi-C'}{\sqrt{C}}\right) = a + bz.$$

Using the boundary conditions for Φ to determine a and b,

$$\begin{aligned} \Phi(z{=}0) &= \sqrt{C}\sin(a) + C' = V_0, \\ \Phi(z{=}L) &= \sqrt{C}\sin(a+bL) + C' = 0, \end{aligned}$$

we obtain the expression for the general solution $\Phi(z)$,

$$\Phi(z) = \sqrt{C}\sin(a+bz) + C', \tag{22}$$

fluid 1	transmission surface	fluid 2	β	Φ_0
air	cast iron	air	5.7	-0.03864
air	mild steel	air	7.9	-0.03834
steam	cast iron	air	11.3	-0.03809
steam	mild steel	air	14.2	-0.03796
steam	copper	air	17	-0.03788

Table 1. Φ_0 for realistic values of β.

where

$$a = \sin^{-1}\left(\frac{V_0 - C'}{\sqrt{C}}\right), \quad b = \frac{1}{L}\left[\sin^{-1}\left(\frac{-C'}{\sqrt{C}}\right) - a\right],$$

with C and C' given by (20), and where Φ_0 is determined numerically from the equation

$$\Phi_0 = \Phi'(L) = b\sqrt{C}\cos(a + bL). \tag{23}$$

Equation (23) was obtained by differentiating (22) with respect to z and then evaluating the derivative at $z = L$. Note that since the right-hand side of (23) also contains Φ_0, we view (23) as an equation where the unknown is Φ_0. Even though (23) cannot be solved analytically for Φ_0, as it enters the right-hand side of (23) in a complicated way, we can still solve (23) numerically by choosing physically realistic values for the parameters of the problem. Table 1 gives values of Φ_0 for realistic values of β, provided in [Engineering ToolBox 2003] for transmission surfaces between various combinations of fluids. The units of β are $W/(m^2K)$ and the units of Φ_0 are V/m.

To complete the general solution, we substitute (22) back into (11) to obtain $T(z)$:

$$T(z) = \frac{1}{\sqrt{\alpha}}\sqrt{C}\cos(a + bz). \tag{24}$$

Finally, as is expected, $\Phi(z)$ in (22) tends to the one found in [Young 1986] as $\beta \to \infty$. Indeed, we have

$$\lim_{\beta\to\infty} C' = \tfrac{1}{2}V_0,$$

$$\lim_{\beta\to\infty} C = \alpha T_0^2 + \tfrac{1}{4}V_0^2,$$

$$\lim_{\beta\to\infty} a = \sin^{-1}\left(\frac{\frac{1}{2}V_0}{\sqrt{\alpha T_0^2 + \frac{1}{4}V_0^2}}\right) = \Omega,$$

$$\lim_{\beta\to\infty} b = \frac{1}{L}\left[\sin^{-1}\left(\frac{-\frac{1}{2}V_0}{\sqrt{\alpha T_0^2 + \frac{1}{4}V_0^2}}\right) - \Omega\right] = \frac{1}{L}[-\Omega - \Omega] = -\frac{2\Omega}{L},$$

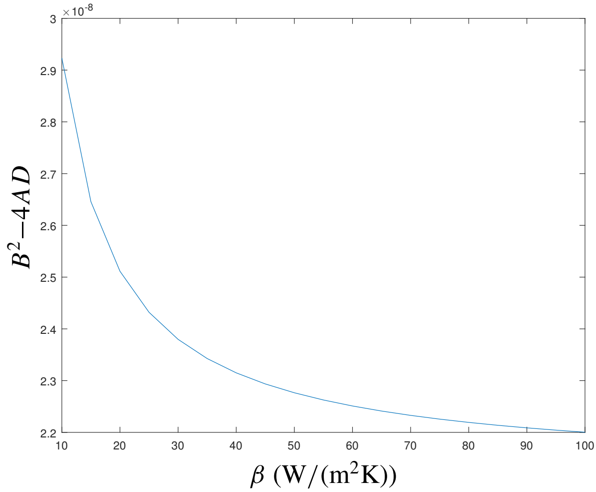

Figure 1. B^2-4AD as a function of β for $T_0 = 273\,\mathrm{K}$, $\alpha = 2.445 \cdot 10^{-8}\,(\mathrm{V/K})^2$, $V_0 = 40\,\mathrm{mV}$.

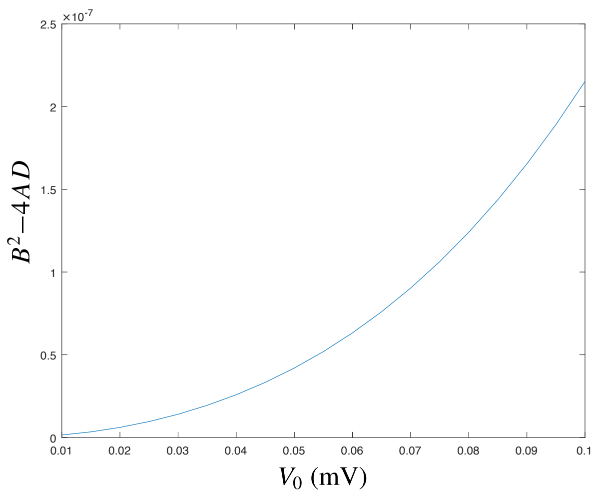

Figure 2. B^2-4AD as a function of V_0 for $T_0 = 273\,\mathrm{K}$, $\alpha = 2.445 \cdot 10^{-8}\,(\mathrm{V/K})^2$, $\beta = 17\,\mathrm{W/(m^2K)}$.

so that

$$\lim_{\beta\to\infty} \Phi(z) = \tfrac{1}{2}V_0 + \sqrt{\alpha T_0^2 + \tfrac{1}{4}V_0^2}\,\sin\Big[\Omega\Big(1 - 2\frac{z}{L}\Big)\Big],$$

which coincides with the expression derived in [Young 1986]. Similarly, it can be shown that (24) tends to the expression for $T(z)$ found in [Young 1986] as $\beta \to \infty$.

Figures 1 and 2 show the graphs of B^2-4AD as a function of β and V_0, respectively.

These figures show that the expression B^2-4AD under the square root in (19) is always greater than zero for a physically meaningful range of parameters $\beta > 0$ and $V_0 > 0$. This, in turn, guarantees that there is no "nonexistence" of solution to the given problem.

We can also derive a solution for the case $\beta = 0$ and verify that the general solution (22) reduces to this solution as $\beta \to 0$. Indeed, when $\beta = 0$, the boundary conditions (13) and (14) are reduced to the boundary conditions

$$\begin{aligned} T &= T_0, \quad \Phi = V_0 \quad \text{at } z = 0, \\ \frac{dT}{dz} &= 0, \quad \Phi = 0 \quad \text{at } z = L. \end{aligned}$$

With the same steps as before, the following expressions for $\Phi(z)$ and $T(z)$ can be derived:

$$\Phi(z) = \sqrt{\alpha T_0^2 + V_0^2}\,\sin\Big[\widetilde{\Omega}\Big(1 - \frac{z}{L}\Big)\Big], \tag{25}$$

$$T(z) = \frac{1}{\sqrt{\alpha}}\sqrt{\alpha T_0^2 + V_0^2}\,\cos\Big[\widetilde{\Omega}\Big(1 - \frac{z}{L}\Big)\Big], \tag{26}$$

where

$$\widetilde{\Omega} := \sin^{-1}\bigg(\frac{V_0}{\sqrt{\alpha T_0^2 + V_0^2}}\bigg).$$

It can be easily verified that (22) and (24) approach (25) and (26), respectively, as $\beta \to 0$.

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