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On the covering number of S_{14}

Ryan Oppenheim and Eric Swartz

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If all elements of a group G are contained in the set-theoretic union of proper subgroups $H_1, \dots, H_n$, then we define this collection to be a cover of G. When such a cover exists, the cardinality of the smallest possible cover is called the covering number of G, denoted by $\sigma(G)$. Maróti determined $\sigma(S_n)$ for odd $n \neq 9$ and provided an estimate for even n. The second author later determined $\sigma(S_n)$ for $n \equiv 0 \pmod 6$ when $n \geqslant 18$, while joint work of the second author with Kappe and Nikolova-Popova also verified that Maróti's rule holds for $n = 9$ and established the covering numbers of S_n for various other small n. Currently, $n = 14$ is the smallest value for which $\sigma(S_n)$ is unknown. In this paper, we prove the covering number of S_{14} is 3096.

1. Introduction

For a group G, a set $\mathcal{H}$ of proper subgroups of G is a *cover* of G if and only if $\bigcup_{A \in \mathcal{H}} A = G$. Further, supposing a cover for G exists, define the *covering number* of G, denoted by $\sigma(G)$, to be the cardinality of the smallest possible cover of G; that is, $\sigma(G)$ is the size of a minimal cover of G.

Based on the work of Neumann [1954], who showed that a group has a finite cover if and only if it has a finite noncyclic homomorphic image, it suffices to consider covers of finite groups. Covers have enjoyed some degree of attention in recent years, particularly given the property that $\sigma(G)$ serves as an upper bound for $\omega(G)$, defined as the largest integer m such that some subset S of G exists where $|S| = m$ and any two distinct elements of S generate G. This and other related problems have garnered much of the current interest in covering numbers; see [Blackburn 2006; Holmes and Maróti 2010], and, for a general survey of such problems, [Serena 2003].

Tomkinson [1997] determined the covering number for a given solvable group and suggested that it would be of interest to investigate minimal covers of nonsolvable groups. The symmetric and alternating groups have naturally attracted special attention, and there has been significant work to derive formulae for the covering numbers of A_n and S_n. Regarding alternating groups, Maróti [2005] established

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that $\sigma(A_n) \geqslant 2^{n-2}$, where $n \neq 7, 9$ (and $\sigma(A_n) = 2^{n-2}$ if and only if $n \equiv 2 \pmod 4$). Turning our attention to the symmetric groups, Maróti also showed in the same paper that $\sigma(S_n) = 2^{n-1}$ for odd $n \neq 9$. Later, Kappe, Nikolova-Popova, and the second author [Kappe et al. 2016] showed that this rule holds when $n = 9$ as well, and ascertained the covering numbers of S_8, S_{10}, and S_{12}. The second author also demonstrated in [Swartz 2016] that $\sigma(S_{18}) = 36773$, and that

$$\sigma(S_n) = \frac{1}{2}\binom{n}{n/2} + \sum_{i=0}^{n/3-1} \binom{n}{i}$$

when $n \equiv 0 \pmod 6$ and $n \geqslant 24$; given that $\sigma(S_6)$ and $\sigma(S_{12})$ were already known, this accounts for all multiples of 6. In pursuit of formulae for all yet-unknown $\sigma(S_n)$, this paper is intended to begin the process of finding the general covering number when $n \equiv 2 \pmod 6$. In determining $\sigma(S_{14})$, or indeed any group whose covering number is unknown, we must establish both the existence of a certain cover of S_{14} and show that no smaller set of proper subgroups could contain among them every element of S_{14}. When considering those groups for which a cover exists (i.e., noncyclic groups), it trivially suffices to consider only maximal subgroups.

The following notation will be used throughout this paper in the discussion of the elements of symmetric groups. We say that $g \in S_n$ has cycle structure $(n_1, \dots, n_k)$ if g can be written as the product of disjoint cycles $g_1, \dots, g_k$, where the length of each cycle g_i is n_i and $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$. For example, the element $g = (1\,2\,3\,4\,5\,6\,7)(8\,9\,10\,11\,12\,13) \in S_{14}$ has cycle structure $(1, 6, 7)$.

In Section 2, we demonstrate a cover of S_{14} containing 3096 subgroups and prove that $\sigma(S_{14}) = 3096$ by showing that this cover is in fact minimal. The GAP code used in the proof can be found in the online supplement.

2. Covering S_{14}

Let $\mathcal{C}_{14}$ be the set of those maximal subgroups of S_{14} isomorphic to one of A_{14}, $S_7 \operatorname{wr} S_2$ (here wr denotes the wreath product), S_{13}, $S_3 \times S_{11}$, or $S_4 \times S_{10}$.

Lemma 2.1. *The set $\mathcal{C}_{14}$ is a cover of S_{14}.*

Proof. Any 14-cycle is contained in some subgroup isomorphic to $S_7 \operatorname{wr} S_2$, and any element of S_{14} that fixes some element of $\{1, \dots, 14\}$ is contained in a subgroup isomorphic to S_{13}. Furthermore, any element without a fixed point that is the product of two cycles is covered by A_{14}, meaning that some element $g \in S_{14}$ could only fail to be covered if it consists of three or more cycles and fixes no points. If the length of one of these cycles is 3 or 4, then g is covered by $S_3 \times S_{11}$ or $S_4 \times S_{10}$, respectively; similarly, if there are two cycles of length 2, then g is covered by $S_4 \times S_{10}$. Furthermore, any element of cycle structure $(2, 6, 6)$ or

isomorphism type	class size
A_{14}	1
S_7 wr S_2	1716
S_2 wr S_7	135135
$S_1 \times S_{13}$	14
$S_2 \times S_{12}$	91
$S_3 \times S_{11}$	364
$S_4 \times S_{10}$	1001
$S_5 \times S_9$	2002
$S_6 \times S_8$	3003
$\mathrm{PGL}_2(13)$	39916800

Table 1. Conjugacy classes of maximal subgroups of S_{14}.

$(2, 5, 7)$ stabilizes a decomposition of $\{1, \ldots, 14\}$ into two subsets of cardinality 7 and thus is contained in a subgroup isomorphic to S_7 wr S_2. Since any element of S_{14} which is the product of three or more disjoint cycles must contain a cycle of length 4 or smaller, and we have covered all such elements, we have shown that $\mathcal{C}_{14}$ is indeed a cover. □

We note that $\mathcal{C}_{14}$ contains 3096 subgroups (see Table 1). We will show that $\mathcal{C}_{14}$ is in fact a minimal cover.

Lemma 2.2. *Any minimal cover of S_{14} contains all subgroups isomorphic to one of A_{14} or S_{13}.*

Proof. We note that $\sigma(A_{14}) = \sigma(S_{13}) = 2^{12} > 3096$, where 3096 is our established upper bound for $\sigma(S_{14})$. Lemma 1 of [Garonzi 2013] states that a maximal subgroup H of a group G with $\sigma(H) > \sigma(G)$ is included in any minimal cover of G containing only maximal subgroups. Thus every minimal cover of the elements of S_{14} must contain every subgroup isomorphic to either A_{14} or S_{13}. □

Lemma 2.2 shows that we can restrict ourselves to finding a minimal cover of the elements not contained in a subgroup isomorphic to either A_{14} or S_{13}. Let Π denote the set of all $g \in S_{14}$ with cycle structure (14), $(3, 5, 6)$, or $(4, 5, 5)$. We will divide the elements of Π as follows: Π_0 will be the set of 14-cycles, Π_3 the set of cycles with structure $(3, 5, 6)$, and Π_4 the set of cycles with structure $(4, 5, 5)$. The distribution of these elements among maximal subgroups of S_{14} is shown in Table 2. In Table 2, if the entry in the row indexed by maximal subgroup M_i and column indexed by cycle structure (j) is "n_m", then a subgroup isomorphic to M_i contains n elements with cycle structure (j), and each element with cycle structure (j) is contained in m maximal subgroups isomorphic to M_i. If the entry in the row indexed by maximal subgroup M_i and the column indexed by cycle

isomorphism type	(14)	(3, 5, 6)	(4, 5, 5)
A_{14}	0	0	0
$S_7 \operatorname{wr} S_2$	3628800, P	0	0
$S_2 \operatorname{wr} S_7$	46080, P	0	0
$S_1 \times S_{13}$	0	0	0
$S_2 \times S_{12}$	0	0	0
$S_3 \times S_{11}$	0	2661120, P	0
$S_4 \times S_{10}$	0	0	435456, P
$S_5 \times S_9$	0	483840, P	435456_2
$S_6 \times S_8$	0	322560, P	0
$\mathrm{PGL}_2(13)$	468_3	0	0

Table 2. Elements of a given cycle structure in S_{14} in each maximal subgroup of a given isomorphism type.

structure (j) is "n, P", then a subgroup isomorphic to M_i contains n elements with cycle structure (j), and the elements with cycle structure (j) are partitioned among the maximal subgroups isomorphic to M_i.

Let $\mathcal{C}'_{14}$ be the set of all subgroups isomorphic to one of $S_7 \operatorname{wr} S_2$, $S_3 \times S_{11}$, or $S_4 \times S_{10}$. By showing that the set $\mathcal{C}'_{14}$ is a minimal cover of the elements of Π, we will show that $\mathcal{C}_{14}$ is also a minimal cover of S_{14}.

Lemma 2.3. *Any minimal cover of* Π *contains all subgroups isomorphic to* $S_7 \operatorname{wr} S_2$.

Proof. Let $\mathcal{B}$ be a minimal cover of S_{14}. Any cover of S_{14} must contain some mix of subgroups conjugate to $S_7 \operatorname{wr} S_2$, $S_2 \operatorname{wr} S_7$, or $\mathrm{PGL}_2(13)$ to cover the elements of Π_0. Examining Table 2, if M is a maximal subgroup of S_{14} and $M \cap \Pi_0 \neq \varnothing$, then $M \cap \Pi = M \cap \Pi_0$. Hence any minimal cover of the elements of Π must contain a minimal cover of the elements of Π_0, which is precisely all subgroups isomorphic to $S_7 \operatorname{wr} S_2$. □

Lemmas 2.2 and 2.3 show that it suffices to restrict our attention to subgroups isomorphic to one of $S_3 \times S_{11}$, $S_4 \times S_{10}$, $S_5 \times S_9$, or $S_6 \times S_8$ covering elements of $\Pi_3 \cup \Pi_4$ when determining a minimal cover of the permutations in Π. We define $H_1 := \mathrm{Sym}(\{1, 2, 3\}) \times \mathrm{Sym}(\{4, \dots, 14\})$ and will use this notation henceforth.

Lemma 2.4. *If a minimal cover* $\mathcal{B}$ *of the elements of* Π *does not contain a subgroup isomorphic to* $S_3 \times S_{11}$, *then there are at least* 11 *subgroups isomorphic to* $S_3 \times S_{11}$ *not contained in* $\mathcal{B}$.

Proof. Let $\mathcal{B}$ be a minimal cover of the elements of Π. Since we know that $\mathcal{C}_{14}$ is a cover of Π, we can compare $\mathcal{B}$ to $\mathcal{C}_{14}$. Define $\mathcal{B}' := \mathcal{B} \backslash \mathcal{C}_{14}$ and $\mathcal{C}' := \mathcal{C}_{14} \backslash \mathcal{B}$. This implies

$$\mathcal{B} = (\mathcal{B} \cap \mathcal{C}_{14}) \cup \mathcal{B}',$$
$$\mathcal{C}_{14} = (\mathcal{B} \cap \mathcal{C}_{14}) \cup \mathcal{C}'.$$

Since $\mathcal{B}$ is a minimal cover of the elements of Π, we have $|\mathcal{B}'| \leqslant |\mathcal{C}'|$. By Lemmas 2.2 and 2.3, $\mathcal{B}'$ consists only of subgroups isomorphic to either $S_5 \times S_9$ or $S_6 \times S_8$, and $\mathcal{C}'$ consists only of subgroups isomorphic to either $S_3 \times S_{11}$ or $S_4 \times S_{10}$. Moreover, we will assume that $\mathcal{C}'$ consists of c_3 subgroups isomorphic to $S_3 \times S_{11}$ and c_4 subgroups isomorphic to $S_4 \times S_{10}$. This means that

$$|\mathcal{B}'| \leqslant |\mathcal{C}'| = c_3 + c_4,$$

and we want to show that if $c_3 \geqslant 1$, then $c_3 \geqslant 11$.

Since we are assuming that $\mathcal{B}$ does not contain a subgroup isomorphic to $S_3 \times S_{11}$, without loss of generality we may assume that $H_1 := \mathrm{Sym}(\{1, 2, 3\}) \times \mathrm{Sym}(\{4, \ldots, 14\}) \notin \mathcal{B}$. This means that the subgroups in $\mathcal{B}'$ must cover every element with cycle structure $(3, 5, 6)$ in H_1. Let $\{4, \ldots, 14\} = A \cup A^c$, where $|A| = 5$. If $\mathcal{B}$ is a cover of Π, then, for each such set A, either $\mathrm{Sym}(A) \times \mathrm{Sym}(A^c \cup \{1, 2, 3\})$ or $\mathrm{Sym}(A^c) \times \mathrm{Sym}(A \cup \{1, 2, 3\})$ is contained in $\mathcal{B}'$. Hence at least $\binom{11}{5} = 462$ subgroups are contained in $\mathcal{B}'$. Let $\mathcal{B}' = \mathcal{D}_1 \cup \mathcal{D}_2$, where $\mathcal{D}_1$ consists of the 462 subgroups needed to cover $\Pi_3 \cap H_1$.

We will now bound from above c_4, the number of groups isomorphic to $S_4 \times S_{10}$ that are in $\mathcal{C}_{14}$ but not in $\mathcal{B}$. From Table 2, we see that, if M_i is a maximal subgroup isomorphic to $S_i \times S_{14-i}$, then $\Pi_4 \cap M_6 = \varnothing$ and

$$|\Pi_4 \cap M_4| = |\Pi_4 \cap M_5| = 435456.$$

Furthermore, the elements of Π_4 are partitioned among the maximal subgroups isomorphic to $S_4 \times S_{10}$. This means that, if there are n_4 total elements with cycle structure $(4, 5, 5)$ contained in the subgroups of $\mathcal{B}'$, then $\mathcal{B}'$ can cover the elements from at most $n_4/435456$ subgroups isomorphic to $S_4 \times S_{10}$; in other words,

$$c_4 \leqslant \frac{n_4}{435456}.$$

To bound n_4 from above, we first observe that $\mathcal{D}_2$ contains at most $435456 \cdot |\mathcal{D}_2|$ distinct elements with cycle structure $(4, 5, 5)$ (in the case when every subgroup of $\mathcal{D}_2$ is isomorphic to $S_5 \times S_9$). Consider now $\mathcal{D}_1$. The subgroups from $\mathcal{D}_1$ cover the most elements with cycle structure $(4, 5, 5)$ when each subgroup is isomorphic to $S_5 \times S_9$, so we will assume that each subgroup of $\mathcal{D}_1$ is isomorphic to $S_5 \times S_9$ to attain an upper bound. Each element with cycle structure $(4, 5, 5)$ is contained in exactly two subgroups isomorphic to $S_5 \times S_9$, and two subgroups $\mathrm{Sym}(A) \times \mathrm{Sym}(\{1, \ldots, 14\} \backslash A)$ and $\mathrm{Sym}(B) \times \mathrm{Sym}(\{1, \ldots, 14\} \backslash B)$ isomorphic to $S_5 \times S_9$ in $\mathcal{D}_1$ overlap in these elements precisely when $A \cap B = \varnothing$. Since both A and B are subsets of $\{4, \dots, 14\}$, and we are assuming that $\mathcal{D}_1$ contains $\mathrm{Sym}(A) \times \mathrm{Sym}(\{1, \dots, 14\} \backslash A)$ for every subset A of $\{4, \dots, 14\}$ of size 5, each subgroup in $\mathcal{D}_1$ intersects exactly $\binom{11-5}{5} = 6$ other subgroups of $\mathcal{D}_1$ in elements of Π_4. Since each element of Π_4 is contained in exactly two subgroups isomorphic to $S_5 \times S_9$, there are exactly

$$\frac{1}{2}\binom{11}{5}\binom{6}{5} \cdot 3! \cdot 4! \cdot 4! = 4790016$$

elements of Π_4 that are contained in two subgroups of $\mathcal{D}_1$. Hence $\mathcal{D}_1$ contains at most $435456 \cdot |\mathcal{D}_1| - 4790016$ elements with cycle structure $(4, 5, 5)$, which implies

$$c_4 \leqslant \frac{n_4}{435456} \leqslant \frac{435456 \cdot |\mathcal{D}_2| + 435456 \cdot |\mathcal{D}_1| - 4790016}{435456} = |\mathcal{D}_2| + |\mathcal{D}_1| - 11 = |\mathcal{B}'| - 11.$$

Therefore,

$$c_3 + c_4 = |\mathcal{C}'| \geqslant |\mathcal{B}'| \geqslant 11 + c_4,$$

and so $c_3 \geqslant 11$, as desired. □

We now further characterize a hypothetical minimal cover $\mathcal{B}$ of the elements of Π.

Lemma 2.5. *Assume that $H_1 \notin \mathcal{B}$, and let the subgroup $H_2 \cong S_3 \times S_{11}$ of S_{14} stabilize the decomposition $B_2 \cup (\{1, \dots, 14\} \backslash B_2)$, where $|B_2| = 3$. If $H_2 \notin \mathcal{B}$, then $\{1, 2, 3\} \cap B_2 \neq \varnothing$.*

Proof. Let B_2 indeed be such a set without overlap with $\{1, 2, 3\}$ — without loss of generality, say it is $\{4, 5, 6\}$. The output of PossibleExtensions([[1, 2, 3], [4, 5, 6]]) in GAP (see Function 7 in the online supplement) shows that, up to an automorphism, $\{1, 2, 4\}$ is the only possibility for B_3, where $H_3 \cong S_3 \times S_{11}$ stabilizes the decomposition of $\{1, \dots, 14\}$ into B_3 and $\{1, \dots, 14\} \backslash B_3$ and $H_3 \notin \mathcal{B}$. The output of PossibleExtensions([[1, 2, 3], [4, 5, 6], [1, 2, 4]]) reveals that no set of four subgroups not in $\mathcal{B}$ can contain two subgroups whose corresponding 3-sets are disjoint. By Lemma 2.4, there are at least 11 subgroups isomorphic to $S_3 \times S_{11}$ not in $\mathcal{B}$, and so, without loss of generality, $\{1, 2, 3\} \cap B_2 \neq \varnothing$. □

We may now use the program PossibleExtensions_2 (see Function 8 in the online supplement), on the presumption that corresponding fixed 3-sets representing groups isomorphic to $S_3 \times S_{11}$ removed from $\mathcal{B}$ must intersect.

Lemma 2.6. *If a collection $H_1, \dots, H_k$ is not in $\mathcal{B}$, where H_i stabilizes a decomposition of the set $\{1, \dots, 14\}$ into $B_i \cup \{1, \dots, 14\} \backslash B_i$, $|B_i| = 3$, and $B_1 = \{1, 2, 3\}$, then we may assume $1 \in \bigcap_{i=1}^{k} B_i$.*

Proof. We observe at the outset that, by Lemma 2.4, $H_1 \notin \mathcal{B}$ implies that $k \geqslant 11$. Again without loss of generality, we let B_2 be one of $\{1, 2, 4\}$ or $\{1, 4, 5\}$, since $|B_1 \cap B_2| \in \{1, 2\}$. We will first examine the case where $B_2 = \{1, 4, 5\}$. The output of PossibleExtensions_2([[1, 2, 3], [1, 4, 5]]) shows that, without loss of generality, the only possibilities for B_3, when $1 \notin B_3$, are $\{2, 3, 4\}$ and $\{2, 4, 6\}$. The output of PossibleExtensions_2([[1, 2, 3], [1, 4, 5], [2, 3, 4]]) then shows that if $B_3 = \{2, 3, 4\}$, the only possibility for B_4 is $\{1, 2, 4\}$, and the output of PossibleExtensions_2([[1, 2, 3], [1, 4, 5], [2, 3, 4], [1, 2, 4]]) shows there is no possibility for B_5. Meanwhile, if $B_3 = \{2, 4, 6\}$, the output of PossibleExtensions_2([[1, 2, 3], [1, 4, 5], [2, 4, 6]]) shows that there is no possible B_4 in this case. Therefore, if $|B_1 \cap B_2| = 1$, then we may assume that $1 \in B_i$ for any i, $1 \leqslant i \leqslant k$.

Now let $B_2 = \{1, 2, 4\}$; i.e., let $B_1 \cap B_2 = \{1, 2\}$. Then up to symmetry, $1 \in B_3$ is equivalent to $2 \in B_3$; thus, assuming $B_3 \cap \{1, 2\} = \varnothing$, without a loss of generality $\{3, 4\} \subseteq B_3$ and $B_3 = \{3, 4, 5\}$. The output of PossibleExtensions_2([[1, 2, 3], [1, 2, 4], [3, 4, 5]]) then shows that $B_4 = \{1, 3, 4\}$. Finally, we see that the output of PossibleExtensions_2([[1, 2, 3], [1, 2, 4], [3, 4, 5], [1, 3, 4]]) shows that there is no possible B_5. Thus, if $B_1 \cap B_2 = \{1, 2\}$, then $B_i \cap \{1, 2\} \neq \varnothing$ for any i, $1 \leqslant i \leqslant k$. Note that this shows that $B_i \cap B_j \cap B_\ell \neq \varnothing$ for any $i, j, \ell \in \{1, \dots, k\}$.

Moreover, if $B_1 \cap B_2 = \{1, 2\}$ and $B_1 \cap B_2 \cap B_3 \cap B_4 = \varnothing$, then without loss of generality we may let $B_3 \cap \{1, 2\} = \{1\}$ and $B_4 \cap \{1, 2\} = \{2\}$. Note that if $B_3 \cap B_1 = \{1\}$, we are done, as in the first case above, as well as if $B_3 \cap B_2 = \{2\}$. Therefore, to continue, we must assume that $B_3 = \{1, 3, 4\}$, and similarly that $B_4 = \{2, 3, 4\}$. However, under these assumptions, PossibleExtensions_2([[1, 2, 3], [1, 2, 4], [1, 3, 4], [2, 3, 4]]) shows that it is impossible to extend the list to a B_5. Therefore, all the B_i have nonempty intersection, and without loss of generality, $1 \in \bigcap_{i=1}^{k} B_i$. □

Lemma 2.7. *$\mathcal{B}$ contains all subgroups isomorphic to $S_3 \times S_{11}$.*

Proof. We again observe at the outset that, by Lemma 2.4, $H_1 \notin \mathcal{B}$ implies $k \geqslant 11$. Lemma 2.6 implies that we may assume each B_i is of the form $\{1, x, y\}$, where $x, y \in \{2, \dots, 14\}$. Hence there are at most $\binom{13}{2} = 78$ subgroups isomorphic to $S_3 \times S_{11}$ omitted from $\mathcal{B}$, meaning that for any potential list, we have that the output of the GAP function 455Shortage([list]) is at most 78 (see Function 5 in the online supplement). However, we also have 455Shortage([[1,2,3],[1,4,5]])= $\frac{286}{3} > 78$, implying that any two subgroups H_i and H_j not in $\mathcal{B}$ must have $|B_i \cap B_j| = 2$. Without loss of generality we may let $B_1 = \{1, 2, 3\}$ and $B_2 = \{1, 2, 4\}$, and assume that $2 \notin B_3$. Then since $|B_1 \cap B_3| = |B_2 \cap B_3| = 2$, necessarily $B_3 = \{1, 3, 4\}$. However, 455Shortage([[1,2,3],[1,2,4],[1,3,4]]) $= 106 > 78$, so without loss of generality all B_i contain $\{1, 2\}$, meaning that for all i, there exists some x such that $B_i = \{1, 2, x\}$. Since there are only 12 such x possible and 455Shortage([[1,2,3],[1,2,4]]) $= 46 > 12$, we have a contradiction. Thus, all 364 subgroups isomorphic to $S_3 \times S_{11}$ are in any minimal cover $\mathcal{B}$ of S_{14}. □

Theorem 2.8. *$\mathcal{C}_{14}$ is a minimal cover of Π (and therefore of S_{14}), and $\sigma(S_{14}) =$ 3096.*

Proof. Since subgroups isomorphic to either $S_4 \times S_{10}$ or $S_5 \times S_9$ contain the same number of Π_4 elements (those with (4, 5, 5) cycle structure) — 435456 — the best-case scenario for covering those elements is the number of such elements divided by 435456, namely $\binom{14}{4}\frac{1}{2}\binom{10}{5} \cdot 3! \cdot 4! \cdot 4!/435456 = 1001$. By Lemmas 2.3 and 2.7, we have already established that every other class of subgroups contained in $\mathcal{C}'_{14}$ is shared by $\mathcal{B}$. Therefore, any minimal cover of $\Pi_3 \cup \Pi_4$ must contain at least $364 + 1001 = 1365$ subgroups, and so any minimal cover of Π (and hence any minimal cover of S_{14}) contains at least $1 + 14 + 1716 + 1365 = 3096$ subgroups. Combined with Lemma 2.1, we have $\sigma(S_{14}) = 3096$. □

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