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# On the covering number of $S_{14}$

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If all elements of a group  $G$  are contained in the set-theoretic union of proper subgroups  $H_1, \dots, H_n$ , then we define this collection to be a cover of  $G$ . When such a cover exists, the cardinality of the smallest possible cover is called the covering number of  $G$ , denoted by  $\sigma(G)$ . Maróti determined  $\sigma(S_n)$  for odd  $n \neq 9$  and provided an estimate for even  $n$ . The second author later determined  $\sigma(S_n)$  for  $n \equiv 0 \pmod{6}$  when  $n \geq 18$ , while joint work of the second author with Kappe and Nikolova-Popova also verified that Maróti's rule holds for  $n = 9$  and established the covering numbers of  $S_n$  for various other small  $n$ . Currently,  $n = 14$  is the smallest value for which  $\sigma(S_n)$  is unknown. In this paper, we prove the covering number of  $S_{14}$  is 3096.

## 1. Introduction

For a group  $G$ , a set  $\mathcal{H}$  of proper subgroups of  $G$  is a *cover* of  $G$  if and only if  $\bigcup_{A \in \mathcal{H}} A = G$ . Further, supposing a cover for  $G$  exists, define the *covering number* of  $G$ , denoted by  $\sigma(G)$ , to be the cardinality of the smallest possible cover of  $G$ ; that is,  $\sigma(G)$  is the size of a minimal cover of  $G$ .

Based on the work of Neumann [1954], who showed that a group has a finite cover if and only if it has a finite noncyclic homomorphic image, it suffices to consider covers of finite groups. Covers have enjoyed some degree of attention in recent years, particularly given the property that  $\sigma(G)$  serves as an upper bound for  $\omega(G)$ , defined as the largest integer  $m$  such that some subset  $S$  of  $G$  exists where  $|S| = m$  and any two distinct elements of  $S$  generate  $G$ . This and other related problems have garnered much of the current interest in covering numbers; see [Blackburn 2006; Holmes and Maróti 2010], and, for a general survey of such problems, [Serena 2003].

Tomkinson [1997] determined the covering number for a given solvable group and suggested that it would be of interest to investigate minimal covers of nonsolvable groups. The symmetric and alternating groups have naturally attracted special attention, and there has been significant work to derive formulae for the covering numbers of  $A_n$  and  $S_n$ . Regarding alternating groups, Maróti [2005] established

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that  $\sigma(A_n) \geq 2^{n-2}$ , where  $n \neq 7, 9$  (and  $\sigma(A_n) = 2^{n-2}$  if and only if  $n \equiv 2 \pmod{4}$ ). Turning our attention to the symmetric groups, Maróti also showed in the same paper that  $\sigma(S_n) = 2^{n-1}$  for odd  $n \neq 9$ . Later, Kappe, Nikolova-Popova, and the second author [Kappe et al. 2016] showed that this rule holds when  $n = 9$  as well, and ascertained the covering numbers of  $S_8$ ,  $S_{10}$ , and  $S_{12}$ . The second author also demonstrated in [Swartz 2016] that  $\sigma(S_{18}) = 36773$ , and that

$$\sigma(S_n) = \frac{1}{2} \binom{n}{n/2} + \sum_{i=0}^{n/3-1} \binom{n}{i}$$

when  $n \equiv 0 \pmod{6}$  and  $n \geq 24$ ; given that  $\sigma(S_6)$  and  $\sigma(S_{12})$  were already known, this accounts for all multiples of 6. In pursuit of formulae for all yet-unknown  $\sigma(S_n)$ , this paper is intended to begin the process of finding the general covering number when  $n \equiv 2 \pmod{6}$ . In determining  $\sigma(S_{14})$ , or indeed any group whose covering number is unknown, we must establish both the existence of a certain cover of  $S_{14}$  and show that no smaller set of proper subgroups could contain among them every element of  $S_{14}$ . When considering those groups for which a cover exists (i.e., noncyclic groups), it trivially suffices to consider only maximal subgroups.

The following notation will be used throughout this paper in the discussion of the elements of symmetric groups. We say that  $g \in S_n$  has cycle structure  $(n_1, \dots, n_k)$  if  $g$  can be written as the product of disjoint cycles  $g_1, \dots, g_k$ , where the length of each cycle  $g_i$  is  $n_i$  and  $n_1 \leq n_2 \leq \dots \leq n_k$ . For example, the element  $g = (1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9\ 10\ 11\ 12\ 13) \in S_{14}$  has cycle structure  $(1, 6, 7)$ .

In Section 2, we demonstrate a cover of  $S_{14}$  containing 3096 subgroups and prove that  $\sigma(S_{14}) = 3096$  by showing that this cover is in fact minimal. The GAP code used in the proof can be found in the [online supplement](#).

## 2. Covering $S_{14}$

Let  $\mathcal{C}_{14}$  be the set of those maximal subgroups of  $S_{14}$  isomorphic to one of  $A_{14}$ ,  $S_7 \text{ wr } S_2$  (here  $\text{wr}$  denotes the wreath product),  $S_{13}$ ,  $S_3 \times S_{11}$ , or  $S_4 \times S_{10}$ .

**Lemma 2.1.** *The set  $\mathcal{C}_{14}$  is a cover of  $S_{14}$ .*

*Proof.* Any 14-cycle is contained in some subgroup isomorphic to  $S_7 \text{ wr } S_2$ , and any element of  $S_{14}$  that fixes some element of  $\{1, \dots, 14\}$  is contained in a subgroup isomorphic to  $S_{13}$ . Furthermore, any element without a fixed point that is the product of two cycles is covered by  $A_{14}$ , meaning that some element  $g \in S_{14}$  could only fail to be covered if it consists of three or more cycles and fixes no points. If the length of one of these cycles is 3 or 4, then  $g$  is covered by  $S_3 \times S_{11}$  or  $S_4 \times S_{10}$ , respectively; similarly, if there are two cycles of length 2, then  $g$  is covered by  $S_4 \times S_{10}$ . Furthermore, any element of cycle structure  $(2, 6, 6)$  or

isomorphism type	class size
$A_{14}$	1
$S_7 \text{ wr } S_2$	1716
$S_2 \text{ wr } S_7$	135135
$S_1 \times S_{13}$	14
$S_2 \times S_{12}$	91
$S_3 \times S_{11}$	364
$S_4 \times S_{10}$	1001
$S_5 \times S_9$	2002
$S_6 \times S_8$	3003
$\text{PGL}_2(13)$	39916800

**Table 1.** Conjugacy classes of maximal subgroups of  $S_{14}$ .

$(2, 5, 7)$  stabilizes a decomposition of  $\{1, \dots, 14\}$  into two subsets of cardinality 7 and thus is contained in a subgroup isomorphic to  $S_7 \text{ wr } S_2$ . Since any element of  $S_{14}$  which is the product of three or more disjoint cycles must contain a cycle of length 4 or smaller, and we have covered all such elements, we have shown that  $\mathcal{C}_{14}$  is indeed a cover.  $\square$

We note that  $\mathcal{C}_{14}$  contains 3096 subgroups (see Table 1). We will show that  $\mathcal{C}_{14}$  is in fact a minimal cover.

**Lemma 2.2.** *Any minimal cover of  $S_{14}$  contains all subgroups isomorphic to one of  $A_{14}$  or  $S_{13}$ .*

*Proof.* We note that  $\sigma(A_{14}) = \sigma(S_{13}) = 2^{12} > 3096$ , where 3096 is our established upper bound for  $\sigma(S_{14})$ . Lemma 1 of [Garonzi 2013] states that a maximal subgroup  $H$  of a group  $G$  with  $\sigma(H) > \sigma(G)$  is included in any minimal cover of  $G$  containing only maximal subgroups. Thus every minimal cover of the elements of  $S_{14}$  must contain every subgroup isomorphic to either  $A_{14}$  or  $S_{13}$ .  $\square$

Lemma 2.2 shows that we can restrict ourselves to finding a minimal cover of the elements not contained in a subgroup isomorphic to either  $A_{14}$  or  $S_{13}$ . Let  $\Pi$  denote the set of all  $g \in S_{14}$  with cycle structure  $(14)$ ,  $(3, 5, 6)$ , or  $(4, 5, 5)$ . We will divide the elements of  $\Pi$  as follows:  $\Pi_0$  will be the set of 14-cycles,  $\Pi_3$  the set of cycles with structure  $(3, 5, 6)$ , and  $\Pi_4$  the set of cycles with structure  $(4, 5, 5)$ . The distribution of these elements among maximal subgroups of  $S_{14}$  is shown in Table 2. In Table 2, if the entry in the row indexed by maximal subgroup  $M_i$  and column indexed by cycle structure  $(j)$  is “ $n_m$ ”, then a subgroup isomorphic to  $M_i$  contains  $n$  elements with cycle structure  $(j)$ , and each element with cycle structure  $(j)$  is contained in  $m$  maximal subgroups isomorphic to  $M_i$ . If the entry in the row indexed by maximal subgroup  $M_i$  and the column indexed by cycle

isomorphism type	(14)	(3, 5, 6)	(4, 5, 5)
$A_{14}$	0	0	0
$S_7 \text{ wr } S_2$	3628800, $P$	0	0
$S_2 \text{ wr } S_7$	46080, $P$	0	0
$S_1 \times S_{13}$	0	0	0
$S_2 \times S_{12}$	0	0	0
$S_3 \times S_{11}$	0	2661120, $P$	0
$S_4 \times S_{10}$	0	0	435456, $P$
$S_5 \times S_9$	0	483840, $P$	435456 <sub>2</sub>
$S_6 \times S_8$	0	322560, $P$	0
$\text{PGL}_2(13)$	468 <sub>3</sub>	0	0

**Table 2.** Elements of a given cycle structure in  $S_{14}$  in each maximal subgroup of a given isomorphism type.

structure  $(j)$  is “ $n, P$ ”, then a subgroup isomorphic to  $M_i$  contains  $n$  elements with cycle structure  $(j)$ , and the elements with cycle structure  $(j)$  are partitioned among the maximal subgroups isomorphic to  $M_i$ .

Let  $C'_{14}$  be the set of all subgroups isomorphic to one of  $S_7 \text{ wr } S_2$ ,  $S_3 \times S_{11}$ , or  $S_4 \times S_{10}$ . By showing that the set  $C'_{14}$  is a minimal cover of the elements of  $\Pi$ , we will show that  $C_{14}$  is also a minimal cover of  $S_{14}$ .

**Lemma 2.3.** *Any minimal cover of  $\Pi$  contains all subgroups isomorphic to  $S_7 \text{ wr } S_2$ .*

*Proof.* Let  $\mathcal{B}$  be a minimal cover of  $S_{14}$ . Any cover of  $S_{14}$  must contain some mix of subgroups conjugate to  $S_7 \text{ wr } S_2$ ,  $S_2 \text{ wr } S_7$ , or  $\text{PGL}_2(13)$  to cover the elements of  $\Pi_0$ . Examining Table 2, if  $M$  is a maximal subgroup of  $S_{14}$  and  $M \cap \Pi_0 \neq \emptyset$ , then  $M \cap \Pi = M \cap \Pi_0$ . Hence any minimal cover of the elements of  $\Pi$  must contain a minimal cover of the elements of  $\Pi_0$ , which is precisely all subgroups isomorphic to  $S_7 \text{ wr } S_2$ .  $\square$

Lemmas 2.2 and 2.3 show that it suffices to restrict our attention to subgroups isomorphic to one of  $S_3 \times S_{11}$ ,  $S_4 \times S_{10}$ ,  $S_5 \times S_9$ , or  $S_6 \times S_8$  covering elements of  $\Pi_3 \cup \Pi_4$  when determining a minimal cover of the permutations in  $\Pi$ . We define  $H_1 := \text{Sym}(\{1, 2, 3\}) \times \text{Sym}(\{4, \dots, 14\})$  and will use this notation henceforth.

**Lemma 2.4.** *If a minimal cover  $\mathcal{B}$  of the elements of  $\Pi$  does not contain a subgroup isomorphic to  $S_3 \times S_{11}$ , then there are at least 11 subgroups isomorphic to  $S_3 \times S_{11}$  not contained in  $\mathcal{B}$ .*

*Proof.* Let  $\mathcal{B}$  be a minimal cover of the elements of  $\Pi$ . Since we know that  $C_{14}$  is a cover of  $\Pi$ , we can compare  $\mathcal{B}$  to  $C_{14}$ . Define  $\mathcal{B}' := \mathcal{B} \setminus C_{14}$  and  $C' := C_{14} \setminus \mathcal{B}$ . This

implies

$$\begin{aligned}\mathcal{B} &= (\mathcal{B} \cap \mathcal{C}_{14}) \cup \mathcal{B}', \\ \mathcal{C}_{14} &= (\mathcal{B} \cap \mathcal{C}_{14}) \cup \mathcal{C}'.\end{aligned}$$

Since  $\mathcal{B}$  is a minimal cover of the elements of  $\Pi$ , we have  $|\mathcal{B}'| \leq |\mathcal{C}'|$ . By Lemmas 2.2 and 2.3,  $\mathcal{B}'$  consists only of subgroups isomorphic to either  $S_5 \times S_9$  or  $S_6 \times S_8$ , and  $\mathcal{C}'$  consists only of subgroups isomorphic to either  $S_3 \times S_{11}$  or  $S_4 \times S_{10}$ . Moreover, we will assume that  $\mathcal{C}'$  consists of  $c_3$  subgroups isomorphic to  $S_3 \times S_{11}$  and  $c_4$  subgroups isomorphic to  $S_4 \times S_{10}$ . This means that

$$|\mathcal{B}'| \leq |\mathcal{C}'| = c_3 + c_4,$$

and we want to show that if  $c_3 \geq 1$ , then  $c_3 \geq 11$ .

Since we are assuming that  $\mathcal{B}$  does not contain a subgroup isomorphic to  $S_3 \times S_{11}$ , without loss of generality we may assume that  $H_1 := \text{Sym}(\{1, 2, 3\}) \times \text{Sym}(\{4, \dots, 14\}) \notin \mathcal{B}$ . This means that the subgroups in  $\mathcal{B}'$  must cover every element with cycle structure  $(3, 5, 6)$  in  $H_1$ . Let  $\{4, \dots, 14\} = A \cup A^c$ , where  $|A| = 5$ . If  $\mathcal{B}$  is a cover of  $\Pi$ , then, for each such set  $A$ , either  $\text{Sym}(A) \times \text{Sym}(A^c \cup \{1, 2, 3\})$  or  $\text{Sym}(A^c) \times \text{Sym}(A \cup \{1, 2, 3\})$  is contained in  $\mathcal{B}'$ . Hence at least  $\binom{11}{5} = 462$  subgroups are contained in  $\mathcal{B}'$ . Let  $\mathcal{B}' = \mathcal{D}_1 \cup \mathcal{D}_2$ , where  $\mathcal{D}_1$  consists of the 462 subgroups needed to cover  $\Pi_3 \cap H_1$ .

We will now bound from above  $c_4$ , the number of groups isomorphic to  $S_4 \times S_{10}$  that are in  $\mathcal{C}_{14}$  but not in  $\mathcal{B}$ . From Table 2, we see that, if  $M_i$  is a maximal subgroup isomorphic to  $S_i \times S_{14-i}$ , then  $\Pi_4 \cap M_6 = \emptyset$  and

$$|\Pi_4 \cap M_4| = |\Pi_4 \cap M_5| = 435456.$$

Furthermore, the elements of  $\Pi_4$  are partitioned among the maximal subgroups isomorphic to  $S_4 \times S_{10}$ . This means that, if there are  $n_4$  total elements with cycle structure  $(4, 5, 5)$  contained in the subgroups of  $\mathcal{B}'$ , then  $\mathcal{B}'$  can cover the elements from at most  $n_4/435456$  subgroups isomorphic to  $S_4 \times S_{10}$ ; in other words,

$$c_4 \leq \frac{n_4}{435456}.$$

To bound  $n_4$  from above, we first observe that  $\mathcal{D}_2$  contains at most  $435456 \cdot |\mathcal{D}_2|$  distinct elements with cycle structure  $(4, 5, 5)$  (in the case when every subgroup of  $\mathcal{D}_2$  is isomorphic to  $S_5 \times S_9$ ). Consider now  $\mathcal{D}_1$ . The subgroups from  $\mathcal{D}_1$  cover the most elements with cycle structure  $(4, 5, 5)$  when each subgroup is isomorphic to  $S_5 \times S_9$ , so we will assume that each subgroup of  $\mathcal{D}_1$  is isomorphic to  $S_5 \times S_9$  to attain an upper bound. Each element with cycle structure  $(4, 5, 5)$  is contained in exactly two subgroups isomorphic to  $S_5 \times S_9$ , and two subgroups  $\text{Sym}(A) \times \text{Sym}(\{1, \dots, 14\} \setminus A)$  and  $\text{Sym}(B) \times \text{Sym}(\{1, \dots, 14\} \setminus B)$  isomorphic to  $S_5 \times S_9$  in  $\mathcal{D}_1$  overlap in these elements precisely when  $A \cap B = \emptyset$ . Since

both  $A$  and  $B$  are subsets of  $\{4, \dots, 14\}$ , and we are assuming that  $\mathcal{D}_1$  contains  $\text{Sym}(A) \times \text{Sym}(\{1, \dots, 14\} \setminus A)$  for every subset  $A$  of  $\{4, \dots, 14\}$  of size 5, each subgroup in  $\mathcal{D}_1$  intersects exactly  $\binom{11-5}{5} = 6$  other subgroups of  $\mathcal{D}_1$  in elements of  $\Pi_4$ . Since each element of  $\Pi_4$  is contained in exactly two subgroups isomorphic to  $S_5 \times S_9$ , there are exactly

$$\frac{1}{2} \binom{11}{5} \binom{6}{5} \cdot 3! \cdot 4! \cdot 4! = 4790016$$

elements of  $\Pi_4$  that are contained in two subgroups of  $\mathcal{D}_1$ . Hence  $\mathcal{D}_1$  contains at most  $435456 \cdot |\mathcal{D}_1| - 4790016$  elements with cycle structure  $(4, 5, 5)$ , which implies

$$c_4 \leq \frac{n_4}{435456} \leq \frac{435456 \cdot |\mathcal{D}_2| + 435456 \cdot |\mathcal{D}_1| - 4790016}{435456} = |\mathcal{D}_2| + |\mathcal{D}_1| - 11 = |\mathcal{B}'| - 11.$$

Therefore,

$$c_3 + c_4 = |\mathcal{C}'| \geq |\mathcal{B}'| \geq 11 + c_4,$$

and so  $c_3 \geq 11$ , as desired.  $\square$

We now further characterize a hypothetical minimal cover  $\mathcal{B}$  of the elements of  $\Pi$ .

**Lemma 2.5.** *Assume that  $H_1 \notin \mathcal{B}$ , and let the subgroup  $H_2 \cong S_3 \times S_{11}$  of  $S_{14}$  stabilize the decomposition  $B_2 \cup (\{1, \dots, 14\} \setminus B_2)$ , where  $|B_2| = 3$ . If  $H_2 \notin \mathcal{B}$ , then  $\{1, 2, 3\} \cap B_2 \neq \emptyset$ .*

*Proof.* Let  $B_2$  indeed be such a set without overlap with  $\{1, 2, 3\}$  — without loss of generality, say it is  $\{4, 5, 6\}$ . The output of `PossibleExtensions`( $[[1, 2, 3], [4, 5, 6]]$ ) in GAP (see Function 7 in the [online supplement](#)) shows that, up to an automorphism,  $\{1, 2, 4\}$  is the only possibility for  $B_3$ , where  $H_3 \cong S_3 \times S_{11}$  stabilizes the decomposition of  $\{1, \dots, 14\}$  into  $B_3$  and  $\{1, \dots, 14\} \setminus B_3$  and  $H_3 \notin \mathcal{B}$ . The output of `PossibleExtensions`( $[[1, 2, 3], [4, 5, 6], [1, 2, 4]]$ ) reveals that no set of four subgroups not in  $\mathcal{B}$  can contain two subgroups whose corresponding 3-sets are disjoint. By [Lemma 2.4](#), there are at least 11 subgroups isomorphic to  $S_3 \times S_{11}$  not in  $\mathcal{B}$ , and so, without loss of generality,  $\{1, 2, 3\} \cap B_2 \neq \emptyset$ .  $\square$

We may now use the program `PossibleExtensions_2` (see Function 8 in the [online supplement](#)), on the presumption that corresponding fixed 3-sets representing groups isomorphic to  $S_3 \times S_{11}$  removed from  $\mathcal{B}$  must intersect.

**Lemma 2.6.** *If a collection  $H_1, \dots, H_k$  is not in  $\mathcal{B}$ , where  $H_i$  stabilizes a decomposition of the set  $\{1, \dots, 14\}$  into  $B_i \cup \{1, \dots, 14\} \setminus B_i$ ,  $|B_i| = 3$ , and  $B_1 = \{1, 2, 3\}$ , then we may assume  $1 \in \bigcap_{i=1}^k B_i$ .*

*Proof.* We observe at the outset that, by [Lemma 2.4](#),  $H_1 \notin \mathcal{B}$  implies that  $k \geq 11$ . Again without loss of generality, we let  $B_2$  be one of  $\{1, 2, 4\}$  or  $\{1, 4, 5\}$ , since  $|B_1 \cap B_2| \in \{1, 2\}$ . We will first examine the case where  $B_2 = \{1, 4, 5\}$ . The output of `PossibleExtensions_2`( $[[1, 2, 3], [1, 4, 5]]$ ) shows that,

without loss of generality, the only possibilities for  $B_3$ , when  $1 \notin B_3$ , are  $\{2, 3, 4\}$  and  $\{2, 4, 6\}$ . The output of `PossibleExtensions_2`( $[[1, 2, 3], [1, 4, 5], [2, 3, 4]]$ ) then shows that if  $B_3 = \{2, 3, 4\}$ , the only possibility for  $B_4$  is  $\{1, 2, 4\}$ , and the output of `PossibleExtensions_2`( $[[1, 2, 3], [1, 4, 5], [2, 3, 4], [1, 2, 4]]$ ) shows there is no possibility for  $B_5$ . Meanwhile, if  $B_3 = \{2, 4, 6\}$ , the output of `PossibleExtensions_2`( $[[1, 2, 3], [1, 4, 5], [2, 4, 6]]$ ) shows that there is no possible  $B_4$  in this case. Therefore, if  $|B_1 \cap B_2| = 1$ , then we may assume that  $1 \in B_i$  for any  $i$ ,  $1 \leq i \leq k$ .

Now let  $B_2 = \{1, 2, 4\}$ ; i.e., let  $B_1 \cap B_2 = \{1, 2\}$ . Then up to symmetry,  $1 \in B_3$  is equivalent to  $2 \in B_3$ ; thus, assuming  $B_3 \cap \{1, 2\} = \emptyset$ , without a loss of generality  $\{3, 4\} \subseteq B_3$  and  $B_3 = \{3, 4, 5\}$ . The output of `PossibleExtensions_2`( $[[1, 2, 3], [1, 2, 4], [3, 4, 5]]$ ) then shows that  $B_4 = \{1, 3, 4\}$ . Finally, we see that the output of `PossibleExtensions_2`( $[[1, 2, 3], [1, 2, 4], [3, 4, 5], [1, 3, 4]]$ ) shows that there is no possible  $B_5$ . Thus, if  $B_1 \cap B_2 = \{1, 2\}$ , then  $B_i \cap \{1, 2\} \neq \emptyset$  for any  $i$ ,  $1 \leq i \leq k$ . Note that this shows that  $B_i \cap B_j \cap B_\ell \neq \emptyset$  for any  $i, j, \ell \in \{1, \dots, k\}$ .

Moreover, if  $B_1 \cap B_2 = \{1, 2\}$  and  $B_1 \cap B_2 \cap B_3 \cap B_4 = \emptyset$ , then without loss of generality we may let  $B_3 \cap \{1, 2\} = \{1\}$  and  $B_4 \cap \{1, 2\} = \{2\}$ . Note that if  $B_3 \cap B_1 = \{1\}$ , we are done, as in the first case above, as well as if  $B_3 \cap B_2 = \{2\}$ . Therefore, to continue, we must assume that  $B_3 = \{1, 3, 4\}$ , and similarly that  $B_4 = \{2, 3, 4\}$ . However, under these assumptions, `PossibleExtensions_2`( $[[1, 2, 3], [1, 2, 4], [1, 3, 4], [2, 3, 4]]$ ) shows that it is impossible to extend the list to a  $B_5$ . Therefore, all the  $B_i$  have nonempty intersection, and without loss of generality,  $1 \in \bigcap_{i=1}^k B_i$ .  $\square$

**Lemma 2.7.**  $\mathcal{B}$  contains all subgroups isomorphic to  $S_3 \times S_{11}$ .

*Proof.* We again observe at the outset that, by [Lemma 2.4](#),  $H_1 \notin \mathcal{B}$  implies  $k \geq 11$ . [Lemma 2.6](#) implies that we may assume each  $B_i$  is of the form  $\{1, x, y\}$ , where  $x, y \in \{2, \dots, 14\}$ . Hence there are at most  $\binom{13}{2} = 78$  subgroups isomorphic to  $S_3 \times S_{11}$  omitted from  $\mathcal{B}$ , meaning that for any potential list, we have that the output of the GAP function `455Shortage`( $[\text{list}]$ ) is at most 78 (see Function 5 in the [online supplement](#)). However, we also have `455Shortage`( $[[1,2,3],[1,4,5]]$ ) =  $\frac{286}{3} > 78$ , implying that any two subgroups  $H_i$  and  $H_j$  not in  $\mathcal{B}$  must have  $|B_i \cap B_j| = 2$ . Without loss of generality we may let  $B_1 = \{1, 2, 3\}$  and  $B_2 = \{1, 2, 4\}$ , and assume that  $2 \notin B_3$ . Then since  $|B_1 \cap B_3| = |B_2 \cap B_3| = 2$ , necessarily  $B_3 = \{1, 3, 4\}$ . However, `455Shortage`( $[[1,2,3],[1,2,4],[1,3,4]]$ ) =  $106 > 78$ , so without loss of generality all  $B_i$  contain  $\{1, 2\}$ , meaning that for all  $i$ , there exists some  $x$  such that  $B_i = \{1, 2, x\}$ . Since there are only 12 such  $x$  possible and `455Shortage`( $[[1,2,3],[1,2,4]]$ ) =  $46 > 12$ , we have a contradiction. Thus, all 364 subgroups isomorphic to  $S_3 \times S_{11}$  are in any minimal cover  $\mathcal{B}$  of  $S_{14}$ .  $\square$

**Theorem 2.8.**  $\mathcal{C}_{14}$  is a minimal cover of  $\Pi$  (and therefore of  $S_{14}$ ), and  $\sigma(S_{14}) = 3096$ .



*Proof.* Since subgroups isomorphic to either  $S_4 \times S_{10}$  or  $S_5 \times S_9$  contain the same number of  $\Pi_4$  elements (those with  $(4, 5, 5)$  cycle structure) — 435456 — the best-case scenario for covering those elements is the number of such elements divided by 435456, namely  $\binom{14}{4} \frac{1}{2} \binom{10}{5} \cdot 3! \cdot 4! \cdot 4! / 435456 = 1001$ . By Lemmas 2.3 and 2.7, we have already established that every other class of subgroups contained in  $C'_{14}$  is shared by  $\mathcal{B}$ . Therefore, any minimal cover of  $\Pi_3 \cup \Pi_4$  must contain at least  $364 + 1001 = 1365$  subgroups, and so any minimal cover of  $\Pi$  (and hence any minimal cover of  $S_{14}$ ) contains at least  $1 + 14 + 1716 + 1365 = 3096$  subgroups. Combined with Lemma 2.1, we have  $\sigma(S_{14}) = 3096$ .  $\square$

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