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Bas Lemmens and Lewis White

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On the complexity of detecting positive eigenvectors of nonlinear cone maps

Bas Lemmens and Lewis White

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In recent work with Lins and Nussbaum, the first author gave an algorithm that can detect the existence of a positive eigenvector for order-preserving homogeneous maps on the standard positive cone. The main goal of this paper is to determine the minimum number of iterations this algorithm requires. It is known that this number is equal to the illumination number of the unit ball B_{v} of the variation norm, $\|x\|_{\mathrm{v}} := \max_i x_i - \min_i x_i$ on $V_0 := \{x \in \mathbb{R}^n : x_n = 0\}$. In this paper we show that the illumination number of B_{v} is equal to $\binom{n}{\lceil n/2 \rceil}$, and hence provide a sharp lower bound for the running time of the algorithm.

1. Introduction

Classical Perron–Frobenius theory concerns the spectral properties of square nonnegative matrices. In recent decades this theory has been extended to a variety of nonlinear maps that preserve a partial ordering induced by a cone (see [Lemmens and Nussbaum 2012] for an up-to-date account).

Of particular interest are order-preserving homogeneous maps $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$, where

$$\mathbb{R}^n_{\geq 0} := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i = 1, \dots, n\}$$

is the *standard positive cone*. Recall that $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is *order-preserving* if $f(x) \leq f(y)$ whenever $x \leq y$ and $x, y \in \mathbb{R}^n_{\geq 0}$. Here, $w \leq z$ if $z - w \in \mathbb{R}^n_{\geq 0}$. Furthermore, f is said to be *homogeneous* if $f(\lambda x) = \lambda f(x)$ for all $\lambda \geq 0$ and $x \in \mathbb{R}^n_{\geq 0}$. Such maps arise in mathematical biology [Nussbaum 1989; Schoen 1986] and in optimal control and game theory [Bewley and Kohlberg 1976; Rosenberg and Sorin 2001].

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It is known [Lemmens and Nussbaum 2012, Corollary 5.4.2] that if $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is a continuous, order-preserving, homogeneous map, then there exists $v \in \mathbb{R}^n_{\geq 0}$ such that

$$f(v) = r(f)v,$$

where

$$r(f) := \lim_{k\to\infty} \|f^k\|^{1/k}_{\mathbb{R}^n_{\geq 0}}$$

is the *cone spectral radius* of f and

$$\|g\|_{\mathbb{R}^n_{\geq 0}} := \sup\{\|g(x)\| : x \in \mathbb{R}^n_{\geq 0} \text{ and } \|x\| \leq 1\}.$$

Thus, as in the case of nonnegative matrices, continuous order-preserving homogeneous maps on $\mathbb{R}^n_{\geq 0}$ have an eigenvector in the cone corresponding to the spectral radius.

In many applications it is important to know if the map has a *positive* eigenvector, i.e., an eigenvector that lies in the interior of $\mathbb{R}^n_{\geq 0}$, that is, $\mathbb{R}^n_{>0} := \{x \in \mathbb{R}^n_{\geq 0} : x_i > 0 \text{ for } i = 1, \dots, n\}$. This appears to be a much more subtle problem. There exists a variety of sufficient conditions in the literature; see [Cavazos-Cadena 2012; Gaubert and Gunawardena 2004; Lemmens and Nussbaum 2012, Chapter 6; Nussbaum 1988]. Recently, Lemmens, Lins and Nussbaum [Lemmens et al. $\geq$ 2019, §5] gave an algorithm that can confirm the existence of a positive eigenvector for continuous, order-preserving, homogeneous maps $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$. The main goal of this paper is to determine the minimum number of iterations this algorithm needs to perform.

2. Preliminaries

Given a set S in a finite-dimensional vector space V we write S° to denote the interior of S, and we write ∂S to denote the boundary of S with respect to the norm topology on V.

It is known that if $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is an order-preserving homogeneous map and there exists $z \in \mathbb{R}^n_{>0}$ such that $f(z) \in \partial\mathbb{R}^n_{\geq 0}$, then $f(\mathbb{R}^n_{>0}) \subset \partial\mathbb{R}^n_{\geq 0}$; see [Lemmens and Nussbaum 2012, Lemma 1.2.2]. Thus to analyse the existence of a positive eigenvector one may as well consider order-preserving homogeneous maps $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$. Moreover, on $\mathbb{R}^n_{>0}$ we have *Hilbert's metric* d_H, which is given by

$$d_H(x, y) := \log\left(\max_i \frac{x_i}{y_i}\right) - \log\left(\min_i \frac{x_i}{y_i}\right) \quad \text{for } x, y \in \mathbb{R}^n_{>0}.$$

Note that d_H is not a genuine metric, as $d_H(\lambda x, \mu x) = 0$ for all $x \in \mathbb{R}^n_{>0}$ and $\lambda, \mu > 0$. In fact, $d_H(x, y) = 0$ if and only if $x = \lambda y$ for some $\lambda > 0$. However, d_H is a metric on the set of rays in $\mathbb{R}^n_{>0}$.

If $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ is order-preserving and homogeneous, then f is nonexpansive under d_H, i.e.,

$$d_H(f(x), f(y)) \le d_H(x, y) \quad \text{for all } x, y \in \mathbb{R}^n_{>0};$$

see for example [Lemmens and Nussbaum 2012, Proposition 2.1.1]. In particular, order-preserving homogeneous maps $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ are continuous on $\mathbb{R}^n_{>0}$. Moreover, if x and y are eigenvectors of $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ with $f(x) = \lambda x$ and $f(y) = \mu y$, then $\lambda = \mu$; see [Lemmens and Nussbaum 2012, Corollary 5.2.2].

In [Lemmens et al. $\ge$ 2019, Theorem 5.1] the following necessary and sufficient conditions were obtained for an order-preserving homogeneous map $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ to have a nonempty set of eigenvectors, $\mathrm{E}(f) := \{x \in \mathbb{R}^n_{>0} : x \text{ eigenvector of } f\}$, which is bounded under Hilbert's metric.

Theorem 2.1. *If $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ is an order-preserving homogeneous map, then $\mathrm{E}(f)$ is nonempty and bounded under d_H if and only if for each nonempty proper subset J of $\{1, \dots, n\}$ there exists $x^J \in \mathbb{R}^n_{>0}$ such that*

$$\max_{j \in J} \frac{f(x^J)_j}{x^J_j} < \min_{j \in J^c} \frac{f(x^J)_j}{x^J_j}. \tag{2-1}$$

Note that the assertion is trivial in the case $n = 1$, as each order-preserving homogeneous map $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ has a nonempty bounded set of eigenvectors. In case $n \ge 2$, Theorem 2.1 yields the following simple algorithm for detecting positive eigenvectors:

Algorithm 2.2. Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be an order-preserving homogeneous map. Repeat the following steps until every nonempty proper subset J of $\{1, \dots, n\}$ has been recorded:

Step 1 Randomly select x, with $x_1 = 1$ and $0 < x_j < 1$ for all $j \in \{2, \dots, n\}$, and compute $f(x)_j / x_j$ for all $j \in \{1, \dots, n\}$.

Step 2 Record all nonempty proper subsets $J \subset \{1, \dots, n\}$ such that inequality (2-1) holds.

So, if this algorithm halts, then f has an eigenvector in $\mathbb{R}^n_{>0}$ and $\mathrm{E}(f)$ is bounded under Hilbert's metric. If $\mathrm{E}(f)$ is empty or unbounded under d_H, then the algorithm does not halt. This can happen even if the map is linear. Consider for example the linear map $x \mapsto Ax$ on $\mathbb{R}^2_{>0}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

which has no eigenvector in $\mathbb{R}^2_{>0}$. At present no algorithm is known that can decide if an order-preserving homogeneous map on $\mathbb{R}^n_{>0}$ has an empty or an unbounded set of eigenvectors. It is also unknown if there is an efficient way to generate the vectors x in Step 1.

Note that a randomly chosen x in Step 1 can eliminate multiple subsets J in Step 2. So, it is natural to ask for the least number of vectors required to fulfil the $2^n - 2$ inequalities in (2-1). This number corresponds to the minimum number of times the algorithm has to perform Steps 1 and 2. In this paper we show that one needs at least

$$\binom{n}{\lceil \frac{1}{2}n \rceil}$$

vectors and this lower bound is sharp. Here $\lceil a \rceil$ is the smallest integer $n \geq a$. Likewise we write $\lfloor a \rfloor$ to denote the largest integer $n \leq a$.

3. Connection with the illumination number

Recall that given a compact convex set C with nonempty interior in V, a vector $v \in V$ *illuminates* $z \in \partial C$ if $z + \lambda v \in C^\circ$ for all $\lambda > 0$ sufficiently small. A set S is said to *illuminate* C if for each $z \in \partial C$ there exists $v \in S$ such that v illuminates z. The minimal size of illuminating set for C is called the *illumination number* of C and is denoted by $i(C)$. There is a long-standing open conjecture which asserts that $i(C) \leq 2^n$ for every compact convex body in an n-dimensional vector space; see [Boltyanski et al. 1997, Chapter VI] for further details. It is easy to show, see for example [Lemmens et al. $\geq$ 2019, Lemma 4.1], that if S illuminates every extreme point of C, then S illuminates C.

To proceed we need to discuss the connection between illumination numbers and Theorem 2.1. Firstly, we note that if we let $\Sigma_0 := \{x \in \mathbb{R}^n_{>0} : x_n = 1\}$, then (Σ_0, d_H) is a metric space. Given an order-preserving homogeneous map $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ we can consider the *normalised* map $g_f : \Sigma_0 \to \Sigma_0$ given by

$$g_f(x) := \frac{f(x)}{f(x)_n} \quad \text{for } x \in \Sigma_0.$$

The map g_f is nonexpansive under d_H on Σ_0. Moreover, $x \in \Sigma_0$ is a fixed point of g_f if and only if x is an eigenvector of f. Thus, if we let $\mathrm{Fix}(g_f) := \{x \in \Sigma_0 : g_f(x) = x\}$, then $\mathrm{Fix}(g_f)$ is nonempty and bounded in (Σ_0, d_H) if and only if $\mathrm{E}(f)$ is nonempty and bounded in $(\mathbb{R}^n_{>0}, d_H)$.

It not hard to verify that the map $\mathrm{Log} : \Sigma_0 \to V_0$ given by

$$\mathrm{Log}(x) := (\log x_1, \ldots, \log x_n) \quad \text{for} \quad x = (x_1, \ldots, x_n) \in \Sigma_0$$

is an isometry from (Σ_0, d_H) onto $(V_0, \|\cdot\|_{\mathrm{v}})$, where $V_0 := \{x \in \mathbb{R}^n : x_n = 0\}$ and

$$\|x\|_{\mathrm{v}} := \max_i x_i - \min_i x_i$$

is the *variation norm*.

It follows that the map $h : V_0 \to V_0$ satisfying $h \circ \mathrm{Log} = \mathrm{Log} \circ g_f$ is nonexpansive under the variation norm, and $\mathrm{Fix}(h)$ is nonempty and bounded in $(V_0, \|\cdot\|_{\mathrm{v}})$ if and only if $\mathrm{Fix}(g_f)$ is nonempty and bounded in (Σ_0, d_H).

In [Lemmens et al. ≥ 2019, Theorem 3.4] the following result concerning fixed point sets of nonexpansive maps on finite-dimensional normed spaces was proved.

Theorem 3.1. *If $h : V \to V$ is a nonexpansive map on a finite-dimensional normed space V, then* $\mathrm{Fix}(h)$ *is nonempty and bounded if and only if there exist $w^1, \dots, w^m \in V$ such that $\{f(w^i) - w^i : i = 1, \dots, m\}$ illuminates the unit ball of V.*

For $n \geq 2$, the unit ball B_{v} of $(V_0, \|\cdot\|_{\mathrm{v}})$ has $2^n - 2$ extreme points, which are given by

$$\mathrm{ext}(B_{\mathrm{v}}) := \{v_+^I : \varnothing \neq I \subseteq \{1, \dots, n-1\}\} \cup \{v_-^I : \varnothing \neq I \subseteq \{1, \dots, n-1\}\}, \quad (3\text{-}1)$$

where $(v_+^I)_i = 1$ if $i \in I$ and 0 otherwise, and $(v_-^I)_i = -1$ if $i \in I$ and 0 otherwise. See [Nussbaum 1994, §2] for details.

In [Lemmens et al. ≥ 2019] the equivalence in Theorem 2.1 was obtained by using Theorem 3.1 and showing that there exists $x^1, \dots, x^m \in \mathbb{R}^n_{>0}$ that fulfil the $2^n - 2$ inequalities in (2-1) if and only if there exist $y^1, \dots, y^m \in V_0$ that illuminate the $2^n - 2$ extreme points of the unit ball B_{v}. Thus, $i(B_{\mathrm{v}})$ provides a sharp lower bound for the number of times one needs to repeat Steps 1 and 2 in Algorithm 2.2. In the next section we show the following result concerning $i(B_{\mathrm{v}})$:

Theorem 3.2. *If B_{v} is the unit ball of $(V_0, \|\cdot\|_{\mathrm{v}})$ and $n \geq 2$, then*

$$i(B_{\mathrm{v}}) = \binom{n}{\lceil \frac{1}{2} n \rceil}.$$

4. Proof of Theorem 3.2

Note that the map $(x_1, \dots, x_n) \in V_0 \mapsto (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ is an isometry from $(V_0, \|\cdot\|_{\mathrm{v}})$ onto $(\mathbb{R}^{n-1}, \|\cdot\|_H)$, where

$$\|x\|_H := \big(\max_i x_i\big) \vee 0 - \big(\min_i x_i\big) \wedge 0.$$

Here $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$. Note also that if B_H is the unit ball in $(\mathbb{R}^{n-1}, \|\cdot\|_H)$, then

$$\mathrm{ext}(B_H) = \big(\{0, 1\}^{n-1} \cup \{0, -1\}^{n-1}\big) \setminus \{(0, \dots, 0)\}$$

and

$$i(B_H) = i(B_{\mathrm{v}}).$$

For notational simplicity we work with B_H instead of B_{v}.

The two subsets

$$E_+ := \{0,1\}^{n-1} \setminus \{(0,\dots,0)\} \quad \text{and} \quad E_- := \{0,-1\}^{n-1} \setminus \{(0,\dots,0)\}$$

of $\mathrm{ext}(B_H)$ play a key role in the argument. On $\mathrm{ext}(B_H)$ we have the usual partial ordering $x \le y$ if $y - x \in \mathbb{R}^{n-1}_{\ge 0}$, which gives rise to two finite partially ordered sets $(E_+, \le)$ and $(E_-, \le)$.

Recall that subset $\mathcal{A}$ of a partially ordered set $(P, \preceq)$ is called an *antichain* if $x, y \in \mathcal{A}$ and $x \preceq y$ implies $x = y$. A *chain* $\mathcal{C}$ in $(P, \preceq)$ is a totally ordered subset if for each $x, y \in \mathcal{C}$ we have that either $x \preceq y$ or $y \preceq x$. The *length* of a chain $\mathcal{C}$ is the number of distinct elements in $\mathcal{C}$.

Lemma 4.1. *Let $\mathcal{A}$ be an antichain in $(E_+, \le)$ or in $(E_-, \le)$. If $x \ne y$ in $\mathcal{A}$ are illuminated by v and w, respectively, then $v \ne w$.*

Proof. Suppose that $\mathcal{A}$ is an antichain in $(E_+, \le)$ and $x \ne y$ are in $\mathcal{A}$. Then there exist $i \ne j$ such that $0 = x_i < y_i = 1$ and $0 = y_j < x_j = 1$. Now suppose by way of contradiction that z illuminates x and y. So, $\|x + \lambda z\|_H < 1$ and $\|y + \lambda z\|_H < 1$ for all $\lambda > 0$ sufficiently small. Suppose first that $z_i \le z_j$. Then for $\lambda > 0$ small,

$$1 + \lambda z_j = x_j + \lambda z_j \le \|x + \lambda z\|_H < 1,$$

and hence $z_j < 0$. So, $z_i \le z_j < 0$. But then

$$1 + \lambda(z_j - z_i) = x_j + \lambda z_j - \lambda z_i \le \|x + \lambda z\|_H < 1,$$

which is impossible. On the other hand, if $z_j \le z_i$, then $1 + \lambda z_i \le \|y + \lambda z\|_H < 1$, so that $z_j \le z_i < 0$. But then

$$1 + \lambda(z_i - z_j) = y_i + \lambda z_i - \lambda z_j \le \|y + \lambda z\|_H < 1,$$

which again is impossible. Thus, z cannot illuminate both x and y.

The argument for the case where $\mathcal{A}$ is an antichain in $(E_-, \le)$ is similar. □

Lemma 4.2. *If $x, y \in \mathrm{ext}(B_H)$ are such that $x_i = 1$ and $y_i = -1$ for some i, then one needs two distinct vectors to illuminate x and y.*

Proof. Suppose w illuminates x and y. Then $1 + \lambda w_i = x_i + \lambda w_i \le \|x + \lambda w\|_H < 1$ for all $\lambda > 0$ sufficiently small, and hence $w_i < 0$. But also $1 - \lambda w_i = -(y_i + \lambda w_i) \le \|y + \lambda w\|_H < 1$ for all $\lambda > 0$ sufficiently small. This implies that $w_i > 0$, which is impossible. Thus, one needs at least two vectors to illuminate x and y. □

Corollary 4.3. *If B_H is the unit ball of $(\mathbb{R}^{n-1}, \|\cdot\|_H)$ and $n \ge 2$, then*

$$i(B_H) \ge \binom{n}{\lceil \frac{1}{2} n \rceil}.$$

Proof. For $1 \le k, m \le n-1$ define the antichains $\mathcal{A}_+(k) := \{x \in E_+ : \sum_i x_i = k\}$ and $\mathcal{A}_-(m) := \{x \in E_- : \sum_i x_i = -m\}$. If $n > 1$ is odd, then we can take $k := \frac{1}{2}(n-1)$ and $m := \frac{1}{2}(n+1)$ and conclude from Lemmas 4.1 and 4.2 that we need at least

$$\binom{n-1}{\frac{1}{2}(n-1)} + \binom{n-1}{\frac{1}{2}(n+1)} = \binom{n}{\lceil \frac{1}{2}n \rceil}$$

distinct vectors to illuminate the extreme points in $\mathcal{A}_+(k) \cup \mathcal{A}_-(m)$, as for each $x \in \mathcal{A}_+(k)$ and $y \in \mathcal{A}_-(m)$ there exists an i such that $x_i = 1$ and $y_i = -1$.

Likewise if $n > 1$ is even, we can take $k = m = \lceil \frac{1}{2}(n-1) \rceil$, and deduce from Lemmas 4.1 and 4.2 that we need at least

$$\binom{n-1}{\lceil \frac{1}{2}(n-1) \rceil} + \binom{n-1}{\lceil \frac{1}{2}(n-1) \rceil} = \binom{n-1}{\lfloor \frac{1}{2}(n-1) \rfloor} + \binom{n-1}{\lceil \frac{1}{2}(n-1) \rceil} = \binom{n}{\frac{1}{2}n}$$

distinct vectors to illuminate the extreme points in $\mathcal{A}_+(k) \cup \mathcal{A}_-(m)$. □

Lemma 4.4. *If $\mathcal{C}$ is a chain in $(E_+, \le)$ or in $(E_-, \le)$, then there exists w that illuminates each element of $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be a chain in $(E_+, \le)$ or in $(E_-, \le)$. We call a chain $c_1 \le c_2 \le \ldots \le c_m$ in $(E_+, \le)$ or in $(E_-, \le)$ maximal if it has length $n-1$. The chain $\mathcal{C}$ is contained in a maximal chain. As each coordinate permutation is an isometry of $(\mathbb{R}^{n-1}, \|\cdot\|_H)$ and the map $x \mapsto -x$ is an isometry of $(\mathbb{R}^{n-1}, \|\cdot\|_H)$, we may assume without loss of generality that $\mathcal{C}$ is contained in the maximal chain,

$$\mathcal{C}^* : (1, 0, 0, \ldots, 0) \le (1, 1, 0, \ldots, 0) \le \cdots \le (1, 1, \ldots, 1, 0) \le (1, 1, 1, \ldots, 1).$$

Let $w \in \mathbb{R}^{n-1}$ be such that $w_1 < w_2 < \cdots < w_{n-1} < 0$. Now if x is the k-th element in the maximal chain and $k < n-1$, then for all $\lambda > 0$ sufficiently small

$$\|x + \lambda w\|_H = \big(\max_i x_i + \lambda w_i\big) \vee 0 - \big(\min_i x_i + \lambda w_i\big) \wedge 0 = 1 + \lambda w_k - \lambda w_{k+1} < 1.$$

On the other hand, if $x = (1, 1, \ldots, 1)$, then clearly $\|x + \lambda w\|_H = 1 + \lambda w_{n-1} < 1$ for all $\lambda > 0$ small. Thus w illuminates each element of $\mathcal{C}^*$ and we are done. □

To proceed we need to recall a few classical results in the combinatorics of finite partially ordered sets; see [Jukna 2001, §9.1 and 9.2]. Firstly, we recall Dilworth's theorem, which says that if the maximum size of an antichain in a finite partially ordered set $(P, \preceq)$ is r, then P can be partitioned into r disjoint chains. In the case where the partially ordered set is $(\{0, 1\}^d, \le)$, one can combine this result with Sperner's theorem, which says that the maximum size of an antichain in $(\{0, 1\}^d, \le)$ is $\binom{d}{\lceil d/2 \rceil}$. Thus, $(\{0, 1\}^d, \le)$ can be partitioned into $\binom{d}{\lceil d/2 \rceil}$ disjoint chains.

To obtain our result we need some more detailed information about the partitions. In particular, we need a result by De Bruijn, Tengbergen, and Kruyswijk [de Bruijn et al. 1951] concerning symmetric chains; see also [Jukna 2001, Theorem 9.3]. A chain $x^1 \leq \cdots \leq x^k$ in $(\{0,1\}^d, \leq)$ is said to be *symmetric* if

(a) $\left(\sum_{j=1}^d x_j^m\right) + 1 = \sum_{j=1}^d x_j^{m+1}$ for all $1 \leq m < k$, i.e., x^{m+1} is an immediate successor of x^m, and

(b) $\sum_{j=1}^d x_j^k = d - \sum_{j=1}^d x_j^1$.

Theorem 4.5 [de Bruijn et al. 1951]. *The poset* $(\{0,1\}^d, \leq)$ *can be partitioned into* $\binom{d}{\lceil d/2 \rceil}$ *disjoint symmetric chains.*

Let us now prove the main result of the paper.

Proof of Theorem 3.2. First recall that by Corollary 4.3 it suffices to show that $i(B_H) \leq \binom{n}{\lceil n/2 \rceil}$, as $i(B_{\mathrm{v}}) = i(B_H)$. In other words, we only need to show that $\mathrm{ext}(B_H)$ can be illuminated by $\binom{n}{\lceil n/2 \rceil}$ vectors.

There are two cases to consider: $n \geq 2$ even, and $n \geq 2$ odd.

Let us first consider the case where $n \geq 2$ is even. By Dilworth's theorem and Sperner's theorem we know that the partially ordered set $(\{0,1\}^{n-1}, \leq)$ can be partitioned into $\binom{n-1}{\lceil (n-1)/2 \rceil}$ disjoint chains. This implies that each of the partially ordered sets $(E_+, \leq)$ and $(E_-, \leq)$ can be partitioned into $\binom{n-1}{\lceil (n-1)/2 \rceil}$ disjoint chains. It now follows from Lemma 4.4 that we need at most

$$\binom{n-1}{\lceil \frac{1}{2}(n-1) \rceil} + \binom{n-1}{\lceil \frac{1}{2}(n-1) \rceil} = \binom{n-1}{\lfloor \frac{1}{2}(n-1) \rfloor} + \binom{n-1}{\lceil \frac{1}{2}(n-1) \rceil} = \binom{n}{\frac{1}{2}n}$$

distinct vectors to illuminate $\mathrm{ext}(B_H)$. This implies that $i(B_{\mathrm{v}}) = i(B_H) \leq \binom{n}{n/2}$.

Now suppose $n \geq 2$ is odd. By Theorem 4.5 we know that $(\{0,1\}^{n-1}, \leq)$ can be partitioned into $\binom{n-1}{(n-1)/2}$ disjoint symmetric chains.

Let us consider such a symmetric chain decomposition, and let

$$\mathcal{A}_k := \left\{x \in \{0,1\}^{n-1} : \textstyle\sum_i x_i = k\right\},$$

which is an antichain of size $\binom{n-1}{k}$. Each element of $\mathcal{A}_{(n+1)/2}$ is contained in a distinct symmetric chain, and each of these chains contains an $x \in \{0,1\}^{n-1}$ with $\sum_i x_i = \frac{1}{2}(n-1)$. Thus, the symmetric chain decomposition of $(\{0,1\}^{n-1}, \leq)$ consists of

$$\binom{n-1}{\frac{1}{2}(n+1)}$$

chains containing a vector x with $\sum_i x_i = \frac{1}{2}(n+1)$, and

$$\binom{n-1}{\frac{1}{2}(n-1)} - \binom{n-1}{\frac{1}{2}(n+1)}$$

chains consisting of a single vector x with $\sum_i x_i = \frac{1}{2}(n-1)$.

By deleting $(0, 0, \dots, 0)$ from $\{0, 1\}^{n-1}$ we obtain a partition of $(E_+, \leq)$ into disjoint chains. Let $\mathcal{S}$ be the set of vectors $x \in E_+$ which form a singleton chain and $\sum_i x_i = \frac{1}{2}(n-1)$. So,

$$|\mathcal{S}| = \binom{n-1}{\frac{1}{2}(n-1)} - \binom{n-1}{\frac{1}{2}(n+1)}.$$

Now pair each $x \in E_+$ with $x' \in E_-$, where $x'_i = 0$ if $x_i = 1$, and $x'_i = -1$ if $x_i = 0$. In this way we obtain a partition of $(E_-, \leq)$ into disjoint chains with $|\mathcal{S}|$ chains consisting of a single vector. In other words, for each $x \in \mathcal{S}$ we have that $x' \in E_-$ forms a singleton chain in the chain decomposition of $(E_-, \leq)$.

We know from Lemma 4.4 that we can illuminate the $\binom{n-1}{(n+1)/2}$ chains in $(E_+, \leq)$ containing a vector x with $\sum_i x_i = \frac{1}{2}(n+1)$ using $\binom{n-1}{(n+1)/2}$ vectors. Likewise, we can illuminate the corresponding $\binom{n-1}{(n+1)/2}$ chains in $(E_-, \leq)$ with $\binom{n-1}{(n+1)/2}$ vectors. So, it remains to illuminate the singleton chains in $(E_+, \leq)$ and $(E_-, \leq)$.

Note that if we can illuminate each pair $\{x, x'\}$, with $x \in \mathcal{S}$ and x' the corresponding vector in E_-, by a single vector, then we need at most

$$2\binom{n-1}{\frac{1}{2}(n+1)} + \binom{n-1}{\frac{1}{2}(n-1)} - \binom{n-1}{\frac{1}{2}(n+1)} = \binom{n-1}{\frac{1}{2}(n-1)} + \binom{n-1}{\frac{1}{2}(n+1)} = \binom{n}{\lceil \frac{1}{2}n \rceil}$$

vectors to illuminate $\mathrm{ext}(B_H)$, and hence $i(B_{\mathrm{v}}) = i(B_H) \leq \binom{n}{\lceil n/2 \rceil}$ if $n \geq 2$ is odd.

To see how this can be done we consider such a pair $\{x, x'\}$ with $x \in \mathcal{S}$ and let $I := \{i : x_i = 1\}$ and $J := \{i : x_i = 0\}$. So, $I = \{i : x'_i = 0\}$ and $J = \{i : x'_i = -1\}$. Now let $w \in \mathbb{R}^{n-1}$ be such that $w_i < 0$ for all $i \in I$ and $w_i > 0$ for all $i \in J$. Then for all $\lambda > 0$ sufficiently small,

$$\|x + \lambda w\|_H = \max_{i \in I}(1 + \lambda w_i) - 0 < 1$$

and

$$\|x' + \lambda w\|_H = 0 - \min_{i \in J}(-1 + \lambda w_i) < 1.$$

This shows that w illuminates x and x', which completes the proof. □

References

[Bewley and Kohlberg 1976] T. Bewley and E. Kohlberg, "The asymptotic theory of stochastic games", *Math. Oper. Res.* **1**:3 (1976), 197–208. MR Zbl

[Boltyanski et al. 1997] V. Boltyanski, H. Martini, and P. S. Soltan, *Excursions into combinatorial geometry*, Springer, 1997. MR Zbl

[de Bruijn et al. 1951] N. G. de Bruijn, C. van Ebbenhorst Tengbergen, and D. Kruyswijk, "On the set of divisors of a number", *Nieuw Arch. Wiskunde* (2) **23** (1951), 191–193. MR Zbl

[Cavazos-Cadena 2012] R. Cavazos-Cadena, "Equivalence of communication and projective boundedness properties for monotone and homogeneous functions", *Nonlinear Anal.* **75**:2 (2012), 775–785. MR Zbl

[Gaubert and Gunawardena 2004] S. Gaubert and J. Gunawardena, "The Perron–Frobenius theorem for homogeneous, monotone functions", *Trans. Amer. Math. Soc.* **356**:12 (2004), 4931–4950. MR Zbl

[Jukna 2001] S. Jukna, *Extremal combinatorics*, Springer, 2001. MR Zbl

[Lemmens and Nussbaum 2012] B. Lemmens and R. Nussbaum, *Nonlinear Perron–Frobenius theory*, Cambridge Tracts in Mathematics **189**, Cambridge University Press, 2012. MR Zbl

[Lemmens et al. $\geq$ 2019] B. Lemmens, B. Lins, and R. Nussbaum, "Detecting fixed points of nonexpansive maps by illuminating the unit ball", preprint. To appear in *Israel J. Math.* arXiv

[Nussbaum 1988] R. D. Nussbaum, *Hilbert's projective metric and iterated nonlinear maps*, Mem. Amer. Math. Soc. **391**, Amer. Math. Soc., Providence, RI, 1988. MR Zbl

[Nussbaum 1989] R. D. Nussbaum, *Iterated nonlinear maps and Hilbert's projective metric, II*, Mem. Amer. Math. Soc. **401**, Amer. Math. Soc., Providence, RI, 1989. MR Zbl

[Nussbaum 1994] R. D. Nussbaum, "Finsler structures for the part metric and Hilbert's projective metric and applications to ordinary differential equations", *Differential Integral Equations* **7**:5-6 (1994), 1649–1707. MR Zbl

[Rosenberg and Sorin 2001] D. Rosenberg and S. Sorin, "An operator approach to zero-sum repeated games", *Israel J. Math.* **121** (2001), 221–246. MR Zbl

[Schoen 1986] R. Schoen, "The two-sex multiethnic stable population model", *Theoret. Population Biol.* **29**:3 (1986), 343–364. MR Zbl

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b.lemmens@kent.ac.uk
School of Mathematics, Statistics & Actuarial Science, University of Kent, Canterbury, United Kingdom

lcw32@kent.ac.uk
School of Mathematics, Statistics & Actuarial Science, University of Kent, Canterbury, United Kingdom

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