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Antiderivatives and linear differential equations using matrices

Yotsanan Meemark and Songpon Sriwongsa

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We show how to find the closed-form solutions for antiderivatives of $x^n e^{ax} \sin bx$ and $x^n e^{ax} \cos bx$ for all $n \in \mathbb{N}_0$ and $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$ by using an idea of Rogers, who suggested using the inverse of the matrix for the differential operator. Additionally, we use the matrix to illustrate the method to find the particular solution for a nonhomogeneous linear differential equation with constant coefficients and forcing terms involving $x^n e^{ax} \sin bx$ or $x^n e^{ax} \cos bx$.

1. Matrix inversion

The concepts of basis and matrix for a linear transformation relative to bases are fundamental in linear algebra. Rogers [1997] suggested an application of the inverse of the matrix for the differential operator on $C^\infty(\mathbb{R})$ relative to a given basis $\mathcal{B}$ to obtain antiderivatives of functions in $\mathcal{B}$. This idea was used with Chevbyshev's polynomials and some binomial identities to get a formula for integrating the power of cosines [Meemark and Leela-apiradee 2011]. Also, the integrals of powers of sine and tangent were obtained by Matlak et al. [2014]. This idea provides a useful application of linear algebra to calculus.

Let n be a nonnegative integer and $\mu = a + bi$ a nonzero complex number. In this work, we apply the idea of Rogers with the complex approach to find the antiderivatives of $x^n e^{ax} \sin bx$ and $x^n e^{ax} \cos bx$ for all $n \in \mathbb{N}_0$ and $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. More precisely, $x^n e^{\mu x} = x^n e^{ax} \cos bx + i x^n e^{ax} \sin bx$. The linearity of the integral operator and comparing the real and imaginary parts yield the desired integrals.

Consider the set of linearly independent functions

$$\mathcal{B}_n = \{e^{\mu x}, x e^{\mu x}, \dots, x^n e^{\mu x}\}.$$

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Let V be the space with the basis $\mathcal{B}_n$ and $\mathcal{D} : V \to V$ be the linear operator defined by $\mathcal{D}(f) = f'$ for all $f \in V$. Since V contains no nonzero constant function, $\mathcal{D} : V \to V$ is invertible. Note that for $j \in \{0, 1, 2, \dots, n\}$, we have

$$\mathcal{D}(x^j e^{\mu x}) = \mu x^j e^{\mu x} + j x^{j-1} e^{\mu x}.$$

This yields the following theorem.

Theorem 1. *The matrix for $\mathcal{D}$ relative to the basis $\mathcal{B}_n$ is*

$$D_n = [\mathcal{D}]_{\mathcal{B}_n} = \begin{bmatrix} \mu & 1 & & & \\ & \mu & 2 & & \\ & & \mu & \ddots & \\ & & & \ddots & n \\ & & & & \mu \end{bmatrix}.$$

According to Rogers' technique [1997], we shall use the inverse of D_n to find the general formula for $\int x^n e^{\mu x}\, dx$. From the above theorem, D_n is invertible and D_n^{-1} is the upper triangular matrix given by

$$D_n^{-1} = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,n} \\ & c_{1,1} & \cdots & c_{1,n} \\ & & \ddots & \vdots \\ & & & c_{n,n} \end{bmatrix}.$$

Identifying $\int x^n e^{\mu x}\, dx$ with the value $D_n^{-1}(x^n e^{\mu x}) \in V$, we get

$$\int x^n e^{\mu x}\, dx = \sum_{j=0}^{n} c_{j,n} x^j e^{\mu x},$$

where the $c_{j,n},\ j \in \{0, 1, \dots, n\}$, satisfy the system of equations

$$\begin{aligned} \mu c_{0,n} + c_{1,n} &= 0, \\ \mu c_{1,n} + 2c_{2,n} &= 0, \\ &\vdots \\ \mu c_{n-1,n} + n c_{n,n} &= 0, \\ \mu c_{n,n} &= 1, \end{aligned}$$

because the product of D_n and D_n^{-1} is the identity matrix. Clearly, $c_{n,n} = 1/\mu$. The back-substitution yields

$$c_{j,n} = c_{n-(n-j),n} = \left(\frac{-n}{\mu}\right)\left(\frac{-(n-1)}{\mu}\right)\cdots\left(\frac{-(j-1)}{\mu}\right)\left(\frac{1}{\mu}\right) = \left(\frac{n!}{j!}\right)\left(\frac{(-1)^{n-j}}{\mu^{n-j+1}}\right)$$

for all $j \in \{0, 1, \dots, n-1\}$. Hence, we have shown:

Theorem 2. *For each* $j \in \{0, 1, \ldots, n\}$, *we have*

$$c_{j,n} = \left(\frac{n!}{j!}\right)\left(\frac{(-1)^{n-j}}{\mu^{n-j+1}}\right).$$

Note that the integration by parts provides the recursion

$$\int x^n e^{\mu x}\,\mathrm{d}x = \frac{1}{\mu}x^n e^{\mu x} - \frac{n}{\mu}\int x^{n-1}e^{\mu x}\,\mathrm{d}x.$$

It follows that the algorithm presented in Theorem 2, requiring only the last column of D_n^{-1}, is more efficient than integration by parts, which requires the computation of the entire matrix D_n^{-1}.

2. Applications

We use the result from Theorem 2 to find the closed-form of $\int x^n e^{ax}\sin bx\,\mathrm{d}x$ and $\int x^n e^{ax}\cos bx\,\mathrm{d}x$. Moreover, we also use the basis introduced in the above section to find the particular solution for a nonhomogeneous linear differential equation with constant coefficients and forcing terms involving $x^n e^{ax}\sin bx$ or $x^n e^{ax}\cos bx$.

For real μ, the general form of $\int x^n e^{\mu x}\,\mathrm{d}x$ derived in Theorem 2 is the final form. Now, we assume that $\mu = a + ib$ with $b \neq 0$; the rectangular form of $\int x^n e^{\mu x}\,\mathrm{d}x$ still remains to be computed. First, we express $\int x^n e^{\mu x}\,\mathrm{d}x = (p_n(x) - iq_n(x))e^{\mu x}$ for some polynomials $p_n(x)$ and $q_n(x)$ of degree n in $\mathbb{R}[x]$. Let $\varrho = |\mu|$ and $\varphi = \arg(\mu)$. Then we have

$$\frac{1}{\mu} = \frac{1}{\varrho}e^{-i\varphi} \quad \text{and} \quad \frac{1}{\mu^{n-j+1}} = \frac{1}{\varrho^{n-j+1}}e^{-i\varphi(n-j+1)};$$

hence

$$c_{j,n} = (-1)^{n-j}\left(\frac{n!}{j!}\right)(s_{n-j+1} - it_{n-j+1}),$$

where

$$s_m = \frac{1}{\varrho^m}\cos m\varphi \quad \text{and} \quad t_m = \frac{1}{\varrho^m}\sin m\varphi \quad \text{for } m \in \mathbb{N}.$$

Since

$$\int x^n e^{\mu x}\,\mathrm{d}x = \sum_{j=0}^{n} c_{j,k}x^j e^{\mu x} = (p_n(x) - iq_n(x))e^{\mu x},$$

by comparing the real and imaginary parts, we have

$$p_n(x) = \sum_{j=0}^{n}(-1)^{n-j}\left(\frac{n!}{j!}\right)s_{n-j+1}x^j \quad \text{and} \quad q_n(x) = \sum_{j=0}^{n}(-1)^{n-j}\left(\frac{n!}{j!}\right)t_{n-j+1}x^j.$$

Moreover,

$$\begin{aligned}\int x^n e^{\mu x}\,\mathrm{d}x &= (p_n(x) - iq_n(x))e^{\mu x} = (p_n(x) - iq_n(x))[e^{ax}(\cos bx + i\sin bx)]\\ &= e^{ax}[p_n(x)\cos bx + q_n(x)\sin bx] - ie^{ax}[q_n(x)\cos bx - p_n(x)\sin bx]\end{aligned}$$

and

$$\int x^n e^{\mu x}\,\mathrm{d}x = \int x^n e^{ax}\cos bx\,\mathrm{d}x + i\int x^n e^{ax}\sin bx\,\mathrm{d}x.$$

In conclusion, we obtain the antiderivatives of $x^n e^{ax}\sin bx$ and $x^n e^{ax}\cos bx$.

Theorem 3. *For $n \in \mathbb{N}\cup\{0\}$ and $a, b \in \mathbb{R}$ with $a^2+b^2 \neq 0$,*

$$\int x^n e^{ax}\sin bx\,\mathrm{d}x = -e^{ax}[q_n(x)\cos bx - p_n(x)\sin bx] + C,$$

$$\int x^n e^{ax}\cos bx\,\mathrm{d}x = e^{ax}[p_n(x)\cos bx + q_n(x)\sin bx] + C,$$

where $p_n(x)$ and $q_n(x)$ are polynomials of degree n computed above.

Finally, we remark that to apply the idea of Rogers [1997] and obtain the same results, one may use the basis

$$\mathcal{C}_n = \{e^{ax}\sin bx,\ e^{ax}\cos bx,\ xe^{ax}\sin bx,\ xe^{ax}\cos bx,\\ x^2e^{ax}\sin bx,\ x^2e^{ax}\cos bx,\ \dots,\ x^ne^{ax}\sin bx,\ x^ne^{ax}\cos bx\}$$

instead of $\mathcal{B}_n$ introduced above. But then the matrix for the differential operator relative to $\mathcal{C}_n$ has the block matrix form

$$D = \begin{bmatrix} A & I_2 & & & \\ & A & 2I_2 & & \\ & & A & \ddots & \\ & & & \ddots & nI_2 \\ & & & & A \end{bmatrix},$$

where

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and I_2 is the 2×2 identity matrix, and the computation for the matrix D^{-1} is tedious. The use of the complex approach and the basis $\mathcal{B}_n$ reduce the complexity of the computation. Moreover, our approach can be used to find the particular solution for a nonhomogeneous linear differential equation with constant coefficients and forcing terms involving $x^n e^{ax}\sin bx$ or $x^n e^{ax}\cos bx$ as follows.

Recall from Theorem 1 that the matrix for the differential operator relative to the basis $\mathcal{B}_n$ is

$$D_n = \begin{bmatrix} \mu & 1 & & & \\ & \mu & 2 & & \\ & & \mu & \ddots & \\ & & & \ddots & n \\ & & & & \mu \end{bmatrix}.$$

It is immediate from the linearity of the differential operator that it suffices to find the particular solution of the equation

$$a_k y^{(k)} + \cdots + a_0 y = x^n e^{\mu x} = (x^n e^{ax} \cos bx) + i(x^n e^{ax} \sin bx),$$

denoted by y_p. Note that $[x^n e^{\mu x}]_{D_n} = (0, \ldots, 0, 1)^T$. Let $L = a_k D^k + \cdots + a_0 I$. We shall find a solution of $L[y_p]_{D_n} = (0, \ldots, 0, 1)^T$. Then we get that $y_1 = \operatorname{Re} y_p$ and $y_2 = \operatorname{Im} y_p$ are the particular solutions for the equations $a_k y^{(k)} + \cdots + a_0 y = x^n e^{ax} \cos bx$ and $a_k y^{(k)} + \cdots + a_0 y = x^n e^{ax} \sin bx$, respectively.

Example. Consider the equations $y'' - 3y' + 2y = xe^x \sin x$ and $y'' - 3y' + 2y = xe^x \cos x$. As per the set-up above,

$$\mu = 1 + i, \quad L = \begin{bmatrix} \mu^2 - 3\mu + 2 & 2\mu - 3 \\ 0 & \mu^2 - 3\mu + 2 \end{bmatrix},$$

and so the solution $[y_p]_{D_1}$ of $L[y_p]_{D_1} = (0, \ldots, 0, 1)^T$ is

$$\left(-\frac{2\mu - 3}{(\mu^2 - 3\mu + 2)^2}, \frac{1}{\mu^2 - 3\mu + 2} \right)^T.$$

Then

$$y_p = -\frac{2\mu - 3}{(\mu^2 - 3\mu + 2)^2} e^{\mu x} + \frac{1}{\mu^2 - 3\mu + 2} x e^{\mu x}.$$

Hence, the particular solution of the first equation is

$$y_1 = \operatorname{Im} y_p = e^x \left(\left(-1 - \tfrac{1}{2}x \right) \sin x - \left(\tfrac{1}{2} - \tfrac{1}{2}x \right) \cos x \right),$$

and the particular solution of the second equation is

$$y_2 = \operatorname{Re} y_p = e^x \left(\left(-1 - \tfrac{1}{2}x \right) \cos x + \left(\tfrac{1}{2} - \tfrac{1}{2}x \right) \sin x \right).$$

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