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Antiderivatives and linear differential equations using matrices

Yotsanan Meemark and Songpon Sriwongsa





# Antiderivatives and linear differential equations using matrices

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We show how to find the closed-form solutions for antiderivatives of  $x^n e^{ax} \sin bx$  and  $x^n e^{ax} \cos bx$  for all  $n \in \mathbb{N}_0$  and  $a, b \in \mathbb{R}$  with  $a^2 + b^2 \neq 0$  by using an idea of Rogers, who suggested using the inverse of the matrix for the differential operator. Additionally, we use the matrix to illustrate the method to find the particular solution for a nonhomogeneous linear differential equation with constant coefficients and forcing terms involving  $x^n e^{ax} \sin bx$  or  $x^n e^{ax} \cos bx$ .

## 1. Matrix inversion

The concepts of basis and matrix for a linear transformation relative to bases are fundamental in linear algebra. Rogers [1997] suggested an application of the inverse of the matrix for the differential operator on  $C^\infty(\mathbb{R})$  relative to a given basis  $\mathcal{B}$  to obtain antiderivatives of functions in  $\mathcal{B}$ . This idea was used with Chebyshev's polynomials and some binomial identities to get a formula for integrating the power of cosines [Meemark and Leela-apiradee 2011]. Also, the integrals of powers of sine and tangent were obtained by Matlak et al. [2014]. This idea provides a useful application of linear algebra to calculus.

Let  $n$  be a nonnegative integer and  $\mu = a + bi$  a nonzero complex number. In this work, we apply the idea of Rogers with the complex approach to find the antiderivatives of  $x^n e^{ax} \sin bx$  and  $x^n e^{ax} \cos bx$  for all  $n \in \mathbb{N}_0$  and  $a, b \in \mathbb{R}$  with  $a^2 + b^2 \neq 0$ . More precisely,  $x^n e^{\mu x} = x^n e^{ax} \cos bx + ix^n e^{ax} \sin bx$ . The linearity of the integral operator and comparing the real and imaginary parts yield the desired integrals.

Consider the set of linearly independent functions

$$\mathcal{B}_n = \{e^{\mu x}, x e^{\mu x}, \dots, x^n e^{\mu x}\}.$$

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Let  $V$  be the space with the basis  $\mathcal{B}_n$  and  $\mathcal{D} : V \rightarrow V$  be the linear operator defined by  $\mathcal{D}(f) = f'$  for all  $f \in V$ . Since  $V$  contains no nonzero constant function,  $\mathcal{D} : V \rightarrow V$  is invertible. Note that for  $j \in \{0, 1, 2, \dots, n\}$ , we have

$$\mathcal{D}(x^j e^{\mu x}) = \mu x^j e^{\mu x} + j x^{j-1} e^{\mu x}.$$

This yields the following theorem.

**Theorem 1.** *The matrix for  $\mathcal{D}$  relative to the basis  $\mathcal{B}_n$  is*

$$D_n = [\mathcal{D}]_{\mathcal{B}_n} = \begin{bmatrix} \mu & 1 & & & \\ & \mu & 2 & & \\ & & \mu & \ddots & \\ & & & \ddots & n \\ & & & & \mu \end{bmatrix}.$$

According to Rogers' technique [1997], we shall use the inverse of  $D_n$  to find the general formula for  $\int x^n e^{\mu x} dx$ . From the above theorem,  $D_n$  is invertible and  $D_n^{-1}$  is the upper triangular matrix given by

$$D_n^{-1} = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,n} \\ & c_{1,1} & \cdots & c_{1,n} \\ & & \ddots & \vdots \\ & & & c_{n,n} \end{bmatrix}.$$

Identifying  $\int x^n e^{\mu x} dx$  with the value  $D_n^{-1}(x^n e^{\mu x}) \in V$ , we get

$$\int x^n e^{\mu x} dx = \sum_{j=0}^n c_{j,n} x^j e^{\mu x},$$

where the  $c_{j,n}$ ,  $j \in \{0, 1, \dots, n\}$ , satisfy the system of equations

$$\begin{aligned} \mu c_{0,n} + c_{1,n} &= 0, \\ \mu c_{1,n} + 2c_{2,n} &= 0, \\ &\vdots \\ \mu c_{n-1,n} + n c_{n,n} &= 0, \\ \mu c_{n,n} &= 1, \end{aligned}$$

because the product of  $D_n$  and  $D_n^{-1}$  is the identity matrix. Clearly,  $c_{n,n} = 1/\mu$ . The back-substitution yields

$$c_{j,n} = c_{n-(n-j),n} = \left(\frac{-n}{\mu}\right) \left(\frac{-(n-1)}{\mu}\right) \cdots \left(\frac{-(j-1)}{\mu}\right) \left(\frac{1}{\mu}\right) = \binom{n!}{j!} \left(\frac{(-1)^{n-j}}{\mu^{n-j+1}}\right)$$

for all  $j \in \{0, 1, \dots, n-1\}$ . Hence, we have shown:

**Theorem 2.** For each  $j \in \{0, 1, \dots, n\}$ , we have

$$c_{j,n} = \binom{n!}{j!} \left( \frac{(-1)^{n-j}}{\mu^{n-j+1}} \right).$$

Note that the integration by parts provides the recursion

$$\int x^n e^{\mu x} dx = \frac{1}{\mu} x^n e^{\mu x} - \frac{n}{\mu} \int x^{n-1} e^{\mu x} dx.$$

It follows that the algorithm presented in Theorem 2, requiring only the last column of  $D_n^{-1}$ , is more efficient than integration by parts, which requires the computation of the entire matrix  $D_n^{-1}$ .

### 2. Applications

We use the result from Theorem 2 to find the closed-form of  $\int x^n e^{ax} \sin bx dx$  and  $\int x^n e^{ax} \cos bx dx$ . Moreover, we also use the basis introduced in the above section to find the particular solution for a nonhomogeneous linear differential equation with constant coefficients and forcing terms involving  $x^n e^{ax} \sin bx$  or  $x^n e^{ax} \cos bx$ .

For real  $\mu$ , the general form of  $\int x^n e^{\mu x} dx$  derived in Theorem 2 is the final form. Now, we assume that  $\mu = a + ib$  with  $b \neq 0$ ; the rectangular form of  $\int x^n e^{\mu x} dx$  still remains to be computed. First, we express  $\int x^n e^{\mu x} dx = (p_n(x) - iq_n(x))e^{\mu x}$  for some polynomials  $p_n(x)$  and  $q_n(x)$  of degree  $n$  in  $\mathbb{R}[x]$ . Let  $\varrho = |\mu|$  and  $\varphi = \arg(\mu)$ . Then we have

$$\frac{1}{\mu} = \frac{1}{\varrho} e^{-i\varphi} \quad \text{and} \quad \frac{1}{\mu^{n-j+1}} = \frac{1}{\varrho^{n-j+1}} e^{-i\varphi(n-j+1)};$$

hence

$$c_{j,n} = (-1)^{n-j} \binom{n!}{j!} (s_{n-j+1} - it_{n-j+1}),$$

where

$$s_m = \frac{1}{\varrho^m} \cos m\varphi \quad \text{and} \quad t_m = \frac{1}{\varrho^m} \sin m\varphi \quad \text{for } m \in \mathbb{N}.$$

Since

$$\int x^n e^{\mu x} dx = \sum_{j=0}^n c_{j,k} x^j e^{\mu x} = (p_n(x) - iq_n(x))e^{\mu x},$$

by comparing the real and imaginary parts, we have

$$p_n(x) = \sum_{j=0}^n (-1)^{n-j} \binom{n!}{j!} s_{n-j+1} x^j \quad \text{and} \quad q_n(x) = \sum_{j=0}^n (-1)^{n-j} \binom{n!}{j!} t_{n-j+1} x^j.$$

Moreover,

$$\begin{aligned} \int x^n e^{\mu x} dx &= (p_n(x) - iq_n(x))e^{\mu x} = (p_n(x) - iq_n(x))[e^{ax}(\cos bx + i \sin bx)] \\ &= e^{ax}[p_n(x) \cos bx + q_n(x) \sin bx] - i e^{ax}[q_n(x) \cos bx - p_n(x) \sin bx] \end{aligned}$$

and

$$\int x^n e^{\mu x} dx = \int x^n e^{ax} \cos bx dx + i \int x^n e^{ax} \sin bx dx.$$

In conclusion, we obtain the antiderivatives of  $x^n e^{ax} \sin bx$  and  $x^n e^{ax} \cos bx$ .

**Theorem 3.** For  $n \in \mathbb{N} \cup \{0\}$  and  $a, b \in \mathbb{R}$  with  $a^2 + b^2 \neq 0$ ,

$$\int x^n e^{ax} \sin bx dx = -e^{ax}[q_n(x) \cos bx - p_n(x) \sin bx] + C,$$

$$\int x^n e^{ax} \cos bx dx = e^{ax}[p_n(x) \cos bx + q_n(x) \sin bx] + C,$$

where  $p_n(x)$  and  $q_n(x)$  are polynomials of degree  $n$  computed above.

Finally, we remark that to apply the idea of Rogers [1997] and obtain the same results, one may use the basis

$$\mathcal{C}_n = \{e^{ax} \sin bx, e^{ax} \cos bx, x e^{ax} \sin bx, x e^{ax} \cos bx, x^2 e^{ax} \sin bx, x^2 e^{ax} \cos bx, \dots, x^n e^{ax} \sin bx, x^n e^{ax} \cos bx\}$$

instead of  $\mathcal{B}_n$  introduced above. But then the matrix for the differential operator relative to  $\mathcal{C}_n$  has the block matrix form

$$D = \begin{bmatrix} A & I_2 & & & \\ & A & 2I_2 & & \\ & & A & \ddots & \\ & & & \ddots & nI_2 \\ & & & & A \end{bmatrix},$$

where

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and  $I_2$  is the  $2 \times 2$  identity matrix, and the computation for the matrix  $D^{-1}$  is tedious. The use of the complex approach and the basis  $\mathcal{B}_n$  reduce the complexity of the computation. Moreover, our approach can be used to find the particular solution for a nonhomogeneous linear differential equation with constant coefficients and forcing terms involving  $x^n e^{ax} \sin bx$  or  $x^n e^{ax} \cos bx$  as follows.

Recall from Theorem 1 that the matrix for the differential operator relative to the basis  $\mathcal{B}_n$  is

$$D_n = \begin{bmatrix} \mu & 1 & & & \\ & \mu & 2 & & \\ & & \mu & \ddots & \\ & & & \ddots & n \\ & & & & \mu \end{bmatrix}.$$

It is immediate from the linearity of the differential operator that it suffices to find the particular solution of the equation

$$a_k y^{(k)} + \dots + a_0 y = x^n e^{\mu x} = (x^n e^{ax} \cos bx) + i(x^n e^{ax} \sin bx),$$

denoted by  $y_p$ . Note that  $[x^n e^{\mu x}]_{D_n} = (0, \dots, 0, 1)^T$ . Let  $L = a_k D^k + \dots + a_0 I$ . We shall find a solution of  $L[y_p]_{D_n} = (0, \dots, 0, 1)^T$ . Then we get that  $y_1 = \text{Re } y_p$  and  $y_2 = \text{Im } y_p$  are the particular solutions for the equations  $a_k y^{(k)} + \dots + a_0 y = x^n e^{ax} \cos bx$  and  $a_k y^{(k)} + \dots + a_0 y = x^n e^{ax} \sin bx$ , respectively.

**Example.** Consider the equations  $y'' - 3y' + 2y = x e^x \sin x$  and  $y'' - 3y' + 2y = x e^x \cos x$ . As per the set-up above,

$$\mu = 1 + i, \quad L = \begin{bmatrix} \mu^2 - 3\mu + 2 & 2\mu - 3 \\ 0 & \mu^2 - 3\mu + 2 \end{bmatrix},$$

and so the solution  $[y_p]_{D_1}$  of  $L[y_p]_{D_1} = (0, \dots, 0, 1)^T$  is

$$\left( -\frac{2\mu - 3}{(\mu^2 - 3\mu + 2)^2}, \frac{1}{\mu^2 - 3\mu + 2} \right)^T.$$

Then

$$y_p = -\frac{2\mu - 3}{(\mu^2 - 3\mu + 2)^2} e^{\mu x} + \frac{1}{\mu^2 - 3\mu + 2} x e^{\mu x}.$$

Hence, the particular solution of the first equation is

$$y_1 = \text{Im } y_p = e^x \left( \left(-1 - \frac{1}{2}x\right) \sin x - \left(\frac{1}{2} - \frac{1}{2}x\right) \cos x \right),$$

and the particular solution of the second equation is

$$y_2 = \text{Re } y_p = e^x \left( \left(-1 - \frac{1}{2}x\right) \cos x + \left(\frac{1}{2} - \frac{1}{2}x\right) \sin x \right).$$

### 3. Acknowledgments

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
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# involve

2019 vol. 12 no. 1

Optimal transportation with constant constraint	1
WYATT BOYER, BRYAN BROWN, ALYSSA LOVING AND SARAH TAMMEN	
Fair choice sequences	13
WILLIAM J. KEITH AND SEAN GRINDATTI	
Intersecting geodesics and centrality in graphs	31
EMILY CARTER, BRYAN EK, DANIELLE GONZALEZ, RIGOBERTO FLÓREZ AND DARREN A. NARAYAN	
The length spectrum of the sub-Riemannian three-sphere	45
DAVID KLAPHECK AND MICHAEL VANVALKENBURGH	
Statistics for fixed points of the self-power map	63
MATTHEW FRIEDRICHSEN AND JOSHUA HOLDEN	
Analytical solution of a one-dimensional thermistor problem with Robin boundary condition	79
VOLODYMYR HRYNKIV AND ALICE TURCHANINOVA	
On the covering number of $S_{14}$	89
RYAN OPPENHEIM AND ERIC SWARTZ	
Upper and lower bounds on the speed of a one-dimensional excited random walk	97
ERIN MADDEN, BRIAN KIDD, OWEN LEVIN, JONATHON PETERSON, JACOB SMITH AND KEVIN M. STANGL	
Classifying linear operators over the octonions	117
ALEX PUTNAM AND TEVIAN DRAY	
Spectrum of the Kohn Laplacian on the Rossi sphere	125
TAWFIK ABBAS, MADELYNE M. BROWN, RAVIKUMAR RAMASAMI AND YUNUS E. ZEYTUNCU	
On the complexity of detecting positive eigenvectors of nonlinear cone maps	141
BAS LEMMENS AND LEWIS WHITE	
Antiderivatives and linear differential equations using matrices	151
YOTSANAN MEEMARK AND SONGPON SRIWONGSA	
Patterns in colored circular permutations	157
DANIEL GRAY, CHARLES LANNING AND HUA WANG	
Solutions of boundary value problems at resonance with periodic and antiperiodic boundary conditions	171
ALDO E. GARCIA AND JEFFREY T. NEUGEBAUER	

