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Lights Out for graphs related to one another by constructions

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The Lights Out problem on graphs, in which each vertex of the graph is in one of two states ("on" or "off"), has been investigated from a variety of perspectives over the last several decades, including parity domination, cellular automata, and harmonic functions on graphs. We consider a variant of the Lights Out problem in which the possible states for each vertex are indexed by the integers modulo k. We examine the space of "null patterns" (i.e., harmonic functions) on graphs, and use this as a way to prove theorems about Lights Out on graphs that are related to one another by two main constructions.

1. Introduction

In the classical version of the Lights Out puzzle, each vertex of a finite graph is either "on" or "off". By "pressing" a vertex, the player toggles the state of that vertex and all adjacent vertices. The goal is to turn off the lights by pressing the correct sequence of vertices. While the winnable configurations on any particular graph can be characterized using ordinary linear algebra over $\mathbb{Z}_2$, see [Anderson and Feil 1998], this puzzle has deep connections to various areas of combinatorics, including parity domination [Amin and Slater 1996; Amin et al. 2002], cellular automata [Sutner 1990], and harmonic functions on graphs [Zaidenberg 2008].

The generalized Lights Out puzzle can be described as follows. Throughout this paper, the term *graph* will mean a finite graph without multiple edges or loops. Let $G = (V, E)$ be a graph with vertex set V and edge set E, and let k be a prime number. A *state* on G is a function $\boldsymbol{s} : V \to \mathbb{Z}_k$. By fixing an ordering on V, we may regard a state $\boldsymbol{s}$ as a column vector in $\mathbb{Z}_k^n$, where $n = |V|$. We will denote the zero state by $\vec{0}$. For any vertex $v \in V$, we define the *closed neighborhood* of v as

$$X(v) = \{v\} \cup \{u \in V : (u, v) \in E\}.$$

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Given $v \in V$, there is an associated state $\boldsymbol{m}_v$ defined by

$$\boldsymbol{m}_v(u) = \begin{cases} 1 & u \in X(v), \\ 0 & u \notin X(v). \end{cases}$$

We think of the states $\boldsymbol{m}_v$ as "moves" in the Lights Out puzzle. Adding state $\boldsymbol{m}_v$ to state $\boldsymbol{s}$ in $\mathbb{Z}_k^n$ corresponds to the action of "pressing" vertex v. That is, pressing vertex v increments the state of v and every vertex adjacent to v by 1 in $\mathbb{Z}_k$. The goal in the puzzle is to convert an initial state $\boldsymbol{s}$ into the zero state by adding a sequence of states of the form $\boldsymbol{m}_v$. For other recent work on this generalized Lights Out problem, see [Edwards et al. 2010; Giffen and Parker 2013; Gravier et al. 2003; Hunziker et al. 2004; Zaidenberg 2009].

It is immediately apparent that the ordering of the vertices in the solution sequence is unimportant; we only need to keep track of the number of times each vertex is pressed. Therefore, a *pattern* on G is a function $\boldsymbol{p} : V \to \mathbb{Z}_k$ where we interpret $\boldsymbol{p}(v)$ as the number of times vertex v is pressed. Let $V = \{v_1, \ldots, v_n\}$. Given an initial state $\boldsymbol{s}$, we call $\boldsymbol{p}$ a *winning pattern* for $\boldsymbol{s}$ if

$$\boldsymbol{s} + \sum_{i=1}^{n} \boldsymbol{p}(v_i)\boldsymbol{m}_{v_i} = \vec{0}.$$

One goal is to determine which initial states on G have associated winning patterns.

Much of our study involves rephrasing the Lights Out puzzle in terms of linear algebra, which is introduced for the basic $\mathbb{Z}_2$ puzzle on grids in [Anderson and Feil 1998]. Let $A = A(G)$ be the adjacency matrix of G based on the ordering $V = \{v_1, \ldots, v_n\}$. The matrix $N = N(G) = A(G) + I_n$ is called the *neighborhood matrix* of G. We use $\mathrm{CS}_k(N)$, $\mathrm{RS}_k(N)$, and $\mathrm{NS}_k(N)$ to denote the column space, row space, and null space of N over $\mathbb{Z}_k$, respectively. The numbers $\mathrm{rank}(N)$ and $\mathrm{null}(N)$ will always be computed over $\mathbb{Z}_k$, and $\mathrm{null}(N) + \mathrm{rank}(N) = n$.

We note that, for any vertex $v_i \in V$, the vector $\boldsymbol{m}_{v_i}$ is the same as the i-th column of N. Therefore, for any pattern $\boldsymbol{p}$ on G, we have

$$\sum_{i=1}^{n} \boldsymbol{p}(v_i)\boldsymbol{m}_{v_i} = N\boldsymbol{p},$$

where $\boldsymbol{p}$ is considered as a column vector in $\mathbb{Z}_k^n$. Thus, $\boldsymbol{p}$ is a winning pattern for $\boldsymbol{s}$ if and only if $N\boldsymbol{p} = -\boldsymbol{s}$, and $\boldsymbol{s}$ has a winning pattern if and only if $\boldsymbol{s} \in \mathrm{CS}_k(N)$. A state $\boldsymbol{s}$ on G will therefore be called *winnable* if and only if $\boldsymbol{s} \in \mathrm{CS}_k(N)$. Applying a pattern in $\mathrm{NS}_k(N)$ to any initial configuration leaves the configuration unchanged. For this reason, we will refer to elements of $\mathrm{NS}_k(N)$ as *null patterns* on G. Since N is a symmetric matrix, it follows that $\mathrm{CS}_k(N)$ can be identified with $\mathrm{RS}_k(N)$, the orthogonal complement of $\mathrm{NS}_k(N)$. A graph G will be called *always winnable* over $\mathbb{Z}_k$ if every initial configuration is winnable. A graph is always winnable over $\mathbb{Z}_k$ if and only if $\operatorname{rank}(N) = n$, or, equivalently, $\det(N) \neq 0 \pmod k$.

Our overall goal is to study winnable states by determining null patterns for various families of graphs. To do this, we develop tools which tell us what happens to these spaces when graphs are combined with one another. In Section 2, we define two main operations on graphs, the vertex join and the edge join. Given graphs $G_1, \dots, G_m$ and a chosen vertex $v_i \in V(G_i)$ for each i, the vertex join H is the disjoint union of $G_1, \dots, G_m$ with all of the vertices v_i identified to a single vertex. Theorem 2.14 shows how the space of null patterns on H is related to the spaces of null patterns on $G_1, \dots, G_m$. Given graphs G_1 and G_2 with chosen vertices $v \in V(G_1)$ and $w \in V(G_2)$ we can form the edge join L of G_1 and G_2 by simply adding edge (v, w) to the disjoint union of G_1 and G_2. Theorem 2.18 shows how the space of null patterns on L relates to the spaces of null patterns on G_1 and G_2. Section 3 explores applications of these results.

2. Graph constructions and main results

We now introduce two main constructions, the vertex join and the edge join. Each of these constructions gives a way of producing a new graph out of two or more existing graphs. We will describe how the set of null patterns for the newly formed graph is related to the sets of null patterns for the component graphs.

Both the vertex join construction and the edge join construction rely on choosing a vertex v in each of the graphs G being joined as a location at which to join. The characterization of null patterns on the joined graph is related not only to the space of null patterns on G but also to the space of null patterns on the graph $G - v$, a graph formed by removing vertex v and all incident edges from G. We introduce the notion of null-difference to describe how the nullity of $N(G)$ relates to the nullity of $N(G - v)$.

Definition 2.1. The *null-difference* at a vertex v in a graph G will be defined by

$$\operatorname{nd}_G(v) = \operatorname{null}(N(G - v)) - \operatorname{null}(N(G)).$$

The null-difference at a vertex v may depend on the prime k. Indeed, we will show in Section 3 that this is the case for cycles. Since k is considered to be a fixed prime, we will use the notation $\operatorname{nd}_G(v)$ without reference to k.

Proposition 2.2. *Let G be a graph. For all $v \in V(G)$, we have* $\operatorname{nd}_G(v) \in \{-1, 0, 1\}$.

Proof. Let $n = |V(G)|$. The matrix $N(G - v)$ is formed by deleting the row and column of $N(G)$ corresponding to v. Let $p = \operatorname{null}(N(G))$, which means that $\operatorname{rank}(N(G)) = n - p$. Let N_1 be the matrix formed by deleting the column of $N(G)$ corresponding to v. Then either (a) $\operatorname{rank}(N_1) = \operatorname{rank}(N(G)) = n - p$ and

$\mathrm{null}(N_1) = p - 1$ or (b) $\mathrm{rank}(N_1) = \mathrm{rank}(N(G)) - 1 = n - p - 1$ and $\mathrm{null}(N_1) = p$. The matrix $N(G - v)$ is obtained by deleting the row of N_1 corresponding to v. This will cause the nullity of N_1 to stay the same or go up by 1. Since, as stated above, $\mathrm{null}(N_1) \in \{p - 1, p\}$, this implies $\mathrm{null}(N(G - v)) \in \{p - 1, p, p + 1\}$. Because $\mathrm{nd}_G(v) = \mathrm{null}(N(G - v)) - p$, this proves the result. □

The three possible values for $\mathrm{nd}_G(v)$ each tell us something very specific about Lights Out winnability and null patterns in relation to the vertex v. The next series of results will look at the cases where $\mathrm{nd}_G(v)$ equals -1, 0, and 1 separately and explain what information is determined in each case.

In order to do this, we first establish some notation based on a graph G and a chosen vertex $v \in V(G)$. Let $\boldsymbol{e}_v$ be the state on G such that $\boldsymbol{e}_v(v) = 1$ and $\boldsymbol{e}_v(w) = 0$ if $w \neq v$. Let f_v be the $\mathbb{Z}_k$-linear transformation which extends a pattern on $G - v$ to a pattern on G that is zero at v. Let r_v be the $\mathbb{Z}_k$-linear transformation that restricts a pattern on G to a pattern on $G - v$. Because we are interested in determining which null patterns on $G - v$ extend to G, the following lemma will be useful.

Lemma 2.3. *Let G be a graph, and let $v \in V(G)$. Suppose $\boldsymbol{p}$ is a null pattern on $G - v$. The pattern $f_v(\boldsymbol{p})$ is either a null pattern on G or a winning pattern on G for the state $\lambda\boldsymbol{e}_v$ for some $\lambda \in \mathbb{Z}_k^*$.*

Proof. The facts that $\boldsymbol{p}$ is null on $G - v$ and that $f_v(\boldsymbol{p})(v) = 0$ imply that $N(G) f_v(\boldsymbol{p})$ is zero except possibly in the position corresponding to v. Thus, $N(G) f_v(\boldsymbol{p}) = \mu\boldsymbol{e}_v$ for some $\mu \in \mathbb{Z}_k$. If $\mu = 0$, then $f_v(\boldsymbol{p})$ is a null pattern on G. If $\mu \neq 0$, then $f_v(\boldsymbol{p})$ is a winning pattern for $\lambda\boldsymbol{e}_v$, where $\lambda = -\mu \pmod k$. □

We first consider the case in which $\mathrm{nd}_G(v) = -1$.

Proposition 2.4. *Let G be a graph, and let $v \in V(G)$. The following are equivalent*:

(1) $\mathrm{nd}_G(v) = -1$.

(2) *The state $\boldsymbol{e}_v$ is not winnable on G.*

(3) *There exists $\boldsymbol{p} \in \mathrm{NS}_k(N(G))$ with $\boldsymbol{p}(v) \neq 0$.*

(4) *The function f_v restricts to an injective linear transformation from the space $\mathrm{NS}_k(N(G - v))$ to $\mathrm{NS}_k(N(G))$, and the restriction of $f_v : \mathrm{NS}_k(N(G - v)) \to \mathrm{NS}_k(N(G))$ has 1-dimensional cokernel.*

Proof. (1) ⇒ (3): We prove the contrapositive. If $\boldsymbol{p}(v) = 0$ for all $\boldsymbol{p} \in \mathrm{NS}_k(N(G))$ then every null pattern on G restricts to a null pattern on $G - v$. This would imply

$$\mathrm{null}(N(G - v)) \geq \mathrm{null}(N(G)),$$

giving $\mathrm{nd}_G(v) \in \{0, 1\}$.

(2) $\Leftrightarrow$ (3): This equivalence follows immediately from the facts that the winnable states on G are precisely the elements of $\mathrm{CS}_k(N(G))$, that $\mathrm{CS}_k(N(G))$ is the orthogonal complement of $\mathrm{NS}_k(N(G))$, and that $\boldsymbol{p}(v) = \boldsymbol{p} \cdot \boldsymbol{e}_v$.

(2), (3) $\Rightarrow$ (4): Suppose that $\boldsymbol{e}_v$ is not winnable on G. This implies that $\lambda\boldsymbol{e}_v$ is not winnable on G for all $\lambda \in \mathbb{Z}_k^*$. By Lemma 2.3, if $\boldsymbol{p}$ is a null pattern on $G - v$, then $f_v(\boldsymbol{p})$ is a null pattern on G. This ensures that the restriction $f_v : \mathrm{NS}_k(N(G-v)) \to \mathrm{NS}_k(N(G))$ is well-defined. Clearly, f_v is injective. Now by (3), there exists a null pattern $\boldsymbol{q}$ on G such that $\boldsymbol{q}(v) \neq 0$, so f_v cannot be surjective. Hence, by Proposition 2.2, the restriction of f_v has 1-dimensional cokernel.

(4) $\Rightarrow$ (1): This is immediate from the definition of $\mathrm{nd}_G(v)$. □

Corollary 2.5. *Let G be a graph, and let $v \in V(G)$. Then $\mathrm{nd}_G(v) \in \{0, 1\}$ if and only if $\boldsymbol{p}(v) = 0$ for every $\boldsymbol{p} \in \mathrm{NS}_k(N(G))$.*

Proof. This follows directly from the equivalence of (1) and (3) in Proposition 2.4. □

Proposition 2.4 also gives the following characterization of always winnable graphs.

Corollary 2.6. *Let G be a graph. Then G is always winnable over $\mathbb{Z}_k$ if and only if $\mathrm{nd}_G(v) \in \{0, 1\}$ for all $v \in V(G)$.*

Proof. If G is always winnable, we have $\mathrm{null}(N(G)) = 0$. Therefore, for all $v \in V(G)$, we have $\mathrm{null}(N(G - v)) \geq \mathrm{null}(N(G))$. This implies $\mathrm{nd}_G(v) \in \{0, 1\}$.

Conversely, if $\mathrm{nd}_G(v) \in \{0, 1\}$ for all $v \in V(G)$, then by Proposition 2.4, the state $\boldsymbol{e}_v$ is winnable on G for all $v \in V(G)$. This implies G is always winnable. □

Because it will be useful later, we also include the following consequence of Proposition 2.4.

Corollary 2.7. *Let G be a graph, and let $v \in V(G)$. If $\mathrm{nd}_G(v) = -1$, then there exists $\boldsymbol{q} \in \mathrm{NS}_k(N(G))$ such that $\boldsymbol{q}(v) = 1$.*

Proof. Since $\mathrm{nd}_G(v) = -1$, the equivalence of (1) and (3) in Proposition 2.4 gives a null pattern $\boldsymbol{p}$ on G with $\boldsymbol{p}(v) \neq 0$. Then $\boldsymbol{q} = \boldsymbol{p}(v)^{-1}\boldsymbol{p}$ is a null pattern on G with $\boldsymbol{q}(v) = 1$. □

Next, we consider the case in which v is a vertex in G with $\mathrm{nd}_G(v) = 0$.

Proposition 2.8. *Let G be a graph and let $v \in V(G)$. The following are equivalent:*

(1) $\mathrm{nd}_G(v) = 0$.

(2) *For all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda\boldsymbol{e}_v$ is winnable on G, and any winning pattern $\boldsymbol{p}$ for $\lambda\boldsymbol{e}_v$ satisfies $\boldsymbol{p}(v) \neq 0$.*

(3) *The functions r_v and f_v restrict to give bijective linear transformations between $\mathrm{NS}_k(N(G))$ and $\mathrm{NS}_k(N(G - v))$, and these restrictions are inverses of one another.*

Proof. (1) ⇒ (2): Suppose that $\mathrm{nd}_G(v) = 0$. By Corollary 2.5, $\boldsymbol{q}(v) = 0$ for every $\boldsymbol{q} \in \mathrm{NS}_k(N(G))$. This implies that $\lambda \boldsymbol{e}_v \perp \boldsymbol{q}$ for all $\boldsymbol{q} \in \mathrm{NS}_k(N(G))$. Since the space of winnable states is the orthogonal complement of the space of null patterns, $\lambda \boldsymbol{e}_v$ is winnable for all $\lambda \in \mathbb{Z}_k^*$.

Now suppose that $\boldsymbol{p}$ is a winning pattern for $\lambda \boldsymbol{e}_v$ for some $\lambda \in \mathbb{Z}_k^*$ with $\boldsymbol{p}(v) = 0$. It follows that $r_v(\boldsymbol{p})$ is null, but $\boldsymbol{p}$ is not null on G. For every $\boldsymbol{q} \in \mathrm{NS}_k(N(G))$, $\boldsymbol{q}(v) = 0$ and thus $r_v(\boldsymbol{q})$ is always a null pattern on $G - v$. Notice that $r_v(\boldsymbol{p})$ must be distinct from $r_v(\boldsymbol{q})$ for all $\boldsymbol{q} \in \mathrm{NS}_k(N(G))$, since the outcome upon applying f_v to these patterns is different. This implies $\mathrm{null}(N(G - v)) > \mathrm{null}(N(G))$, contradicting (1). Thus $\boldsymbol{p}(v) \neq 0$.

(2) ⇒ (3): Suppose (2) is true. Since a pattern $\boldsymbol{q} \in \mathrm{NS}_k(N(G))$ must be orthogonal to every winnable pattern, $\boldsymbol{q}(v) = 0$ in all null patterns $\boldsymbol{q}$ on G. If any $\boldsymbol{q} \in \mathrm{NS}_k(N(G))$ is restricted to $G - v$, the result is a null pattern on $G - v$, and therefore, r_v restricts to give a well-defined function from $\mathrm{NS}_k(N(G))$ to $\mathrm{NS}_k(N(G - v))$.

Clearly, r_v is injective. Let $\boldsymbol{r} \in \mathrm{NS}_k(N(G - v))$. Lemma 2.3 implies that $f_v(\boldsymbol{r})$ is either a null pattern on G or a pattern that wins $\lambda \boldsymbol{e}_v$ for some $\lambda \in \mathbb{Z}_k^*$. The latter is impossible by (2), since $f_v(\boldsymbol{r})(v) = 0$. Thus, $f_v(\boldsymbol{r}) \in \mathrm{NS}_k(N(G))$. This implies that the restriction of r_v is an isomorphism from $\mathrm{NS}_k(N(G))$ to $\mathrm{NS}_k(N(G - v))$ with inverse given by f_v.

(3) ⇒ (1): This is immediate from the definition of $\mathrm{nd}_G(v)$. □

Finally, we consider the case in which v is a vertex in G with $\mathrm{nd}_G(v) = 1$. We will make use of the following fact from linear algebra.

Remark 2.9. Let G be a graph, and let $\boldsymbol{s}$ be a state on G. If $\boldsymbol{p}$ is a winning pattern for $\boldsymbol{s}$, then the full set of winning patterns for $\boldsymbol{s}$ is precisely $\boldsymbol{p} + \mathrm{NS}_k(N(G))$. Indeed, since $\boldsymbol{p}$ is a solution to $N(G)\mathbf{x} = -\boldsymbol{s}$, the full solution set to $N(G)\mathbf{x} = -\boldsymbol{s}$ is the coset of $\mathrm{NS}_k(N(G))$ determined by $\boldsymbol{p}$.

Proposition 2.10. *Let G be a graph, and let $v \in V(G)$. Then the following are equivalent*:

(1) $\mathrm{nd}_G(v) = 1$.

(2) *For all $\lambda \in \mathbb{Z}_k^*$, the state $\lambda \boldsymbol{e}_v$ is winnable on G, and any winning pattern $\boldsymbol{p}$ for $\lambda \boldsymbol{e}_v$ satisfies $\boldsymbol{p}(v) = 0$.*

(3) *The function r_v induces an injective linear transformation from $\mathrm{NS}_k(N(G))$ to $\mathrm{NS}_k(N(G - v))$, and the restriction $r_v : \mathrm{NS}_k(N(G)) \to \mathrm{NS}_k(N(G - v))$ has 1-dimensional cokernel.*

Proof. (1) ⇒ (2): Suppose that $\mathrm{nd}_G(v) = 1$. By Corollary 2.5, every null pattern on G is zero at v. Therefore, as in the first part of the proof of Proposition 2.8, the state $\lambda \boldsymbol{e}_v$ is winnable on G for all $\lambda \in \mathbb{Z}_k^*$. Suppose there exists $\mu \in \mathbb{Z}_k^*$ and a winning pattern $\boldsymbol{p}$ for $\mu\boldsymbol{e}_v$ with $\boldsymbol{p}(v) \neq 0$. By Remark 2.9, the set of all winning patterns for $\mu\boldsymbol{e}_v$ is $\boldsymbol{p} + \mathrm{NS}_k(N(G))$. From this, we see that for any $\lambda \in \mathbb{Z}_k^*$, the set of all winning patterns for $\lambda\boldsymbol{e}_v$ is $\lambda\mu^{-1}\boldsymbol{p} + \mathrm{NS}_k(N(G))$ The facts that $\boldsymbol{p}(v) \neq 0$ and that every element of $\mathrm{NS}_k(N(G))$ is zero at v shows that every element of $\lambda\mu^{-1}\boldsymbol{p} + \mathrm{NS}_k(N(G))$ (i.e., every winning pattern for $\lambda\boldsymbol{e}_v$) is nonzero at v. By Proposition 2.8, this would imply $\mathrm{nd}_G(v) = 0$, contradicting (1). Therefore, any winning pattern $\boldsymbol{p}$ for $\mu\boldsymbol{e}_v$ satisfies $\boldsymbol{p}(v) = 0$.

(2) $\Rightarrow$ (3): Assume (2) is true. Since a null pattern on G has to be perpendicular to every winnable pattern, $\boldsymbol{q}(v) = 0$ for all null patterns $\boldsymbol{q}$ on G. Therefore, a null pattern on G restricted to $G - v$ is still null. Thus, the restriction of r_v gives a well-defined linear transformation $\mathrm{NS}_k(N(G)) \to \mathrm{NS}_k(N(G - v))$.

Clearly, the restriction of r_v is injective. If the restriction of r_v were also surjective, then $\mathrm{nd}_G(v) = 0$, and this contradicts (2) by Proposition 2.8. The cokernel of the restriction of r_v to $\mathrm{NS}_k(N(G))$ is therefore 1-dimensional by Proposition 2.2.

(3) $\Rightarrow$ (1): This is immediate from the definition of $\mathrm{nd}_G(v)$. □

Later in this section, we will formalize the notions of the vertex join of several graphs and the edge join of two graphs mentioned in the Introduction. Our main theorems explain how to determine the dimension of the space of null patterns for the newly formed graph in terms of the dimensions of the spaces of null patterns for all of the graphs being joined together. We do this by using the null-differences at each vertex of the component graphs where the joining will take place to determine the null-difference of the resulting vertex or vertices in the joined graph.

For vertices in the component graphs with null-difference 0, the null-difference itself does not convey sufficient information to determine the behavior of the resulting vertex in the joined graph. We therefore introduce an extension of the null-difference for each vertex v such that $\mathrm{nd}_G(v) = 0$.

Definition 2.11. Let G be a graph and suppose $v \in V(G)$ with $\mathrm{nd}_G(v) = 0$. By Proposition 2.8, the state $\boldsymbol{e}_v$ has a winning pattern $\boldsymbol{q}$, and $\boldsymbol{q}(v) \in \mathbb{Z}_k^*$. Let

$$\lambda_G(v) = -\boldsymbol{q}(v)^{-1} \in \mathbb{Z}_k^*.$$

Corollary 2.5 and Remark 2.9 combine to show that $\boldsymbol{q}(v)$ (and hence $\lambda_G(v)$) is independent of the winning pattern $\boldsymbol{q}$ chosen. For vertices of null-difference 0, we will write $\mathrm{nd}_G(v) = 0(\lambda)$ to indicate that $\mathrm{nd}_G(v) = 0$ and $\lambda_G(v) = \lambda$.

We include the following as an alternate way to view the numbers $\lambda_G(v)$, because it will be helpful when we prove Theorem 2.14.

Lemma 2.12. *Let G be a graph and suppose $v \in V(G)$ with* $\mathrm{nd}_G(v) = 0$. *There exists a pattern $\boldsymbol{p}$ on G such that $\boldsymbol{p}(v) = 1$ and $\boldsymbol{p}$ wins $\mu\boldsymbol{e}_v$ for some $\mu \in \mathbb{Z}_k^*$. For any such $\boldsymbol{p}$, we have $\mu = -\lambda_G(v)$.*

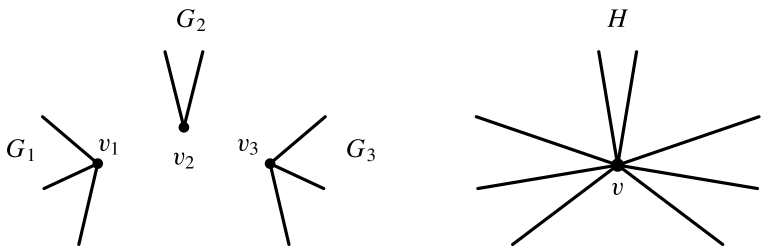


Figure 1. The vertex join H of the graphs G_1, G_2, and G_3 at the vertices v_1, v_2 and v_3.

Proof. By Proposition 2.8, there exists a pattern $\boldsymbol{r}$ on G such that $\boldsymbol{r}(v) \neq 0$ and $\boldsymbol{r}$ wins $\lambda \boldsymbol{e}_v$ for some $\lambda \in \mathbb{Z}_k^*$. Then $\boldsymbol{p} = \boldsymbol{r}(v)^{-1}\boldsymbol{r}$ satisfies $\boldsymbol{p}(v) = 1$, and $\boldsymbol{p}$ wins $\mu \boldsymbol{e}_v$ for $\mu = \boldsymbol{r}(v)^{-1}\lambda$.

Since $\boldsymbol{p}$ wins $\mu \boldsymbol{e}_v$, we conclude that $\boldsymbol{q} = \mu^{-1}\boldsymbol{p}$ wins $\boldsymbol{e}_v$. Then

$$\lambda_G(v) = -\boldsymbol{q}(v)^{-1} = -((\mu^{-1}\boldsymbol{p})(v))^{-1} = -\mu \boldsymbol{p}(v)^{-1} = -\mu. \qquad \square$$

In the remainder of this section, we define the operations of vertex join and edge join, and give theorems that determine the dimensions of the null spaces of the resulting graphs if the null-differences on the vertices of the component graphs are known.

Definition 2.13 (vertex join). Let $m \in \mathbb{Z}$ with $m \geq 2$. For $1 \leq i \leq m$, let G_i be a graph with $v_i \in V(G_i)$. The graph $H = \mathrm{VJ}(\{G_i, v_i\})$ is defined by starting with the disjoint union $\bigcup G_i$ and identifying $\{v_1, v_2, \ldots, v_m\}$ to a single vertex v. The graph H is called the *vertex join* of the graphs $G_1, \ldots, G_m$ at the chosen vertices v_i; see Figure 1.

Theorem 2.14. *Let G_i be graphs for $1 \leq i \leq m$, and let $v_i \in V(G_i)$. Let $H = \mathrm{VJ}(\{G_i, v_i\})$:*

(1) $\mathrm{nd}_H(v) = 1$ *if and only if* $\mathrm{nd}_{G_i}(v_i) = 1$ *for at least one i.*

(2) $\mathrm{nd}_H(v) \in \{0, -1\}$ *if and only if* $\mathrm{nd}_{G_i}(v_i) \in \{0, -1\}$ *for all i. Moreover, in this case,* $\mathrm{nd}_H(v) = -1$ *if and only if* $\sum \lambda_{G_i}(v_i) = m - 1 \pmod k$, *where the sum is taken over all vertices v_i such that* $\mathrm{nd}_{G_i}(v_i) = 0$.

(3) *In the case that* $\mathrm{nd}_H(v) = 0$, *we have*

$$\lambda_H(v) = 1 - m + \sum \lambda_{G_i}(v_i) \pmod k,$$

where, again, the sum is taken over all vertices v_i such that $\mathrm{nd}_{G_i}(v_i) = 0$.

Once $\mathrm{nd}_H(v)$ *is known,* $\mathrm{null}(N(H))$ *can be computed as*

$$\mathrm{null}(N(H)) = -\,\mathrm{nd}_H(v) + \sum_{i=1}^{m} (\mathrm{null}(N(G_i)) + \mathrm{nd}_{G_i}(v_i)).$$

Proof. (1, $\Rightarrow$) Suppose that $\mathrm{nd}_{G_j}(v_j) = 1$ for some j. By Proposition 2.10, there is a pattern $\boldsymbol{p}$ on G_j that wins $\boldsymbol{e}_{v_j}$ with $\boldsymbol{p}(v_j) = 0$. Extend $\boldsymbol{p}$ to H such that $\boldsymbol{p}(w) = 0$ for all vertices $w \in V(H)$ not originally coming from G_j. Then $\boldsymbol{p}$ is a winning pattern on H for $\boldsymbol{e}_v$ with $\boldsymbol{p}(v) = 0$. It now follows from Proposition 2.4 that $\mathrm{nd}_H(v) \neq -1$, and then Corollary 2.5 implies that every null pattern on H is zero at v. As in the first paragraph of the proof of Proposition 2.10, Remark 2.9 now implies that, for every $\mu \in \mathbb{Z}_k^*$, the full set of winning patterns for a state of the form $\mu \boldsymbol{e}_v$ is $\mu \boldsymbol{p} + \mathrm{NS}_k(N(H))$. Thus, for every $\mu \in \mathbb{Z}_k^*$, every winning pattern $\boldsymbol{q}$ on H for $\mu \boldsymbol{e}_v$ also satisfies $\boldsymbol{q}(v) = 0$. By Proposition 2.10, we have $\mathrm{nd}_H(v) = 1$.

(1,$\Leftarrow$) Suppose that $\mathrm{nd}_H(v) = 1$. For the purposes of contradiction, suppose that $\mathrm{nd}_{G_i}(v_i) \in \{0, -1\}$ for all i. Proposition 2.10 implies that $\boldsymbol{e}_v$ is winnable on H, and winning patterns $\boldsymbol{p}$ for $\boldsymbol{e}_v$ on H satisfy $\boldsymbol{p}(v) = 0$. If $\boldsymbol{p}$ is any such pattern, then for each i, $\boldsymbol{p}$ restricts to a null pattern on $G_i - v_i$. By Lemma 2.3, for each i, $\boldsymbol{p}$ restricts to a pattern on G_i that is either null or wins $\mu \boldsymbol{e}_{v_i}$ for some $\mu \in \mathbb{Z}_k^*$. We claim that the latter of these two possibilities cannot happen. If $\mathrm{nd}_{G_i}(v_i) = -1$, Proposition 2.4 implies that, for every $\mu \in \mathbb{Z}_k^*$, the state $\mu \boldsymbol{e}_{v_i}$ is not winnable on G_i. If $\mathrm{nd}_{G_i}(v_i) = 0$, Proposition 2.8 implies that, for all $\mu \in \mathbb{Z}_k^*$, the state $\mu \boldsymbol{e}_{v_i}$ is not winnable on G_i using a pattern with $\boldsymbol{p}(v_i) = 0$. Thus, $\boldsymbol{p}$ restricts to a null pattern on G_i for all i. However, this shows that $\boldsymbol{p} \in \mathrm{NS}_k(N(H))$, contradicting the fact that $\boldsymbol{p}$ was chosen as a winning pattern for $\boldsymbol{e}_v$ on H. This shows that $\mathrm{nd}_{G_i}(v_i) = 1$ for at least one i.

(2) The first biconditional statement in (2) follows immediately from (1) and Proposition 2.2. We prove the second biconditional statement in (2).

(2, second statement $\Rightarrow$) Suppose that $\mathrm{nd}_H(v) = -1$. By Corollary 2.7, there exists a null pattern $\boldsymbol{p}$ on H with $\boldsymbol{p}(v) = 1$. Let $\boldsymbol{p}_i$ be the pattern on G_i given by the restriction of $\boldsymbol{p}$.

For all i such that $\mathrm{nd}_{G_i}(v_i) = -1$, Proposition 2.4 implies that nonzero multiples of $\boldsymbol{e}_{v_i}$ are not winnable on G_i. Therefore, $\boldsymbol{p}_i$ is a null pattern on G_i by Lemma 2.3.

For all i such that $\mathrm{nd}_{G_i}(v_i) = 0$, Corollary 2.5 implies that every null pattern on G_i is zero at v_i. This shows that $\boldsymbol{p}_i$ is not a null pattern on G_i. Lemma 2.3 now shows that $\boldsymbol{p}_i$ is a winning pattern for $\mu \boldsymbol{e}_{v_i}$ for some $\mu \in \mathbb{Z}_k^*$. Lemma 2.12 now implies $\mu = -\lambda_{G_i}(v_i)$. Thus, if $\mathrm{nd}_{G_i}(v_i) = 0$, then $\boldsymbol{p}_i$ is a winning pattern on G_i for $-\lambda_{G_i}(v_i)\boldsymbol{e}_{v_i}$.

Since $\boldsymbol{p}_i(v_i) = 1$ for all i, the contribution of $\boldsymbol{p}$ from all vertices in $G_i - v_i$ to the state at v_i must be equal to $\lambda_{G_i}(v_i) - 1$ if $\mathrm{nd}_{G_i}(v_i) = 0$, and equal to -1 if $\mathrm{nd}_{G_i}(v_i) = -1$. Because $\boldsymbol{p} \in \mathrm{NS}_k(N(H))$, we must have

$$1 + \sum_{\{i : \mathrm{nd}_{G_i}(v_i) = 0\}} [(\lambda_{G_i}(v_i) - 1)] + \sum_{\{i : \mathrm{nd}_{G_i}(v_i) = -1\}} (-1) = 0 \pmod{k},$$

which implies

$$\sum_{\{i : \mathrm{nd}_{G_i}(v_i) = 0\}} \lambda_{G_i}(v_i) = m - 1 \pmod{k}.$$

(2, second statement $\Leftarrow$) Suppose that $\sum_{\{i:\mathrm{nd}_{G_i}(v_i)=0\}} \lambda_{G_i}(v_i) = m-1 \pmod{k}$. To show that $\mathrm{nd}_H(v) = -1$, we construct a null pattern $\boldsymbol{p}$ on H that has $\boldsymbol{p}(v) = 1$.

For each i such that $\mathrm{nd}_{G_i}(v_i) = -1$, Corollary 2.7 implies that there exists a null pattern $\boldsymbol{p}_i$ on G_i with $\boldsymbol{p}_i(v_i) = 1$. For each i such that $\mathrm{nd}_{G_i}(v_i) = 0$, Lemma 2.12 implies that there exists a pattern $\boldsymbol{p}_i$ on G_i that is a winning pattern for $-\lambda_{G_i}(v_i)\boldsymbol{e}_{v_i}$ with $\boldsymbol{p}_i(v_i) = 1$.

Since all of the patterns $\boldsymbol{p}_i$ have $\boldsymbol{p}_i(v_i) = 1$, they glue together to form a pattern $\boldsymbol{p}$ on H, which we will show is null. By construction, $\boldsymbol{p}$ is a winning pattern for $\mu\boldsymbol{e}_v$ on H for some $\mu \in \mathbb{Z}_k$. We need to show that $\mu = 0$. By adding up all of the contributions from the different G_i, we see that

$$-\mu = 1 + \sum_{\{i:\mathrm{nd}_{G_i}(v_i)=0\}} [(\lambda_{G_i}(v_i) - 1)] + \sum_{\{i:\mathrm{nd}_{G_i}(v_i)=-1\}} (-1) = 1 + (m-1) - m = 0 \pmod{k}.$$

Thus, $\boldsymbol{p} \in \mathrm{NS}_k(N(H))$. Since $\boldsymbol{p}(v) \neq 0$, we have $\mathrm{nd}_H(v) = -1$ by Proposition 2.4.

(3) Suppose that $\mathrm{nd}_H(v) = 0$. By (2), we must have $\mathrm{nd}_{G_i}(v_i) \in \{0, -1\}$ for all i and

$$\sum_{\{i:\mathrm{nd}_{G_i}(v_i)=0\}} \lambda_{G_i}(v_i) \neq m - 1 \pmod{k}.$$

If $\mathrm{nd}_{G_i}(v_i) = -1$, then Corollary 2.7 shows that there is a null pattern $\boldsymbol{p}_i$ on G_i with $\boldsymbol{p}_i(v_i) = 1$. If $\mathrm{nd}_{G_i}(v_i) = 0$, then Lemma 2.12 implies that there is a pattern $\boldsymbol{p}_i$ on G_i such that $\boldsymbol{p}_i(v_i) = 1$ and $\boldsymbol{p}_i$ wins $-\lambda_{G_i}(v_i)\boldsymbol{e}_{v_i}$. Gluing these patterns together gives a pattern $\boldsymbol{p}$ on H with $\boldsymbol{p}(v) = 1$. Again by Lemma 2.12, $\boldsymbol{p}$ is a winning pattern on H for $-\lambda_H(v)\boldsymbol{e}_v$. Adding up the contributions from all of the patterns $\boldsymbol{p}_i$ being glued together gives

$$\begin{aligned}\lambda_H(v) &= 1 + \sum_{\{i:\mathrm{nd}_{G_i}(v_i)=0\}} [(\lambda_{G_i}(v_i) - 1)] + \sum_{\{i:\mathrm{nd}_{G_i}(v_i)=-1\}} (-1) \\ &= 1 - m + \sum_{\{i:\mathrm{nd}_{G_i}(v_i)=0\}} \lambda_{G_i}(v_i).\end{aligned}$$ □

Corollary 2.15. *Consider two always winnable graphs G_1 and G_2 over $\mathbb{Z}_k$, and let*

$$H = \mathrm{VJ}(\{G_1, v_2\}, \{G_2, v_2\}).$$

Then H is always winnable if and only if $\mathrm{nd}_H(v) = \mathrm{nd}_{G_1}(v_1) + \mathrm{nd}_{G_2}(v_2)$. *This can happen in only the following two ways*:

(1) *One of the* $\mathrm{nd}_{G_i}(v_i)$ *is* 1 *and the other is* 0, *in which case* $\mathrm{nd}_H(v) = 1$.

(2) $\mathrm{nd}_{G_1}(v_1) = 0(\lambda)$ *and* $\mathrm{nd}_{G_2}(v_2) = 0(\mu)$ *with* $\lambda + \mu \neq 1 \pmod{k}$, *in which case* $\mathrm{nd}_H(v) = 0(\lambda + \mu - 1)$.

Proof. This is immediate from the $m = 2$ case of Theorem 2.14. □

One application of Theorem 2.14 is to determine the dimension of the space of null patterns (and hence, the space of winnable patterns) when P_2 is attached to a graph by identifying one of the vertices of P_2 with a chosen vertex of the graph.

Corollary 2.16. *Let G_1 be a graph and let $v \in V(G_1)$. Let P_2 be a path with two vertices v' and w'. Let*

$$G_1' = \mathrm{VJ}(\{G_1, v\}, \{P_2, v'\}).$$

Let $d = \mathrm{null}(N(G_1))$. Then $\mathrm{null}(N(G_1'))$ is given by the following table:

$\mathrm{nd}_{G_1}(v)$	$\mathrm{null}(N(G_1'))$	$\mathrm{nd}_{G_1'}(v)$	$\mathrm{nd}_{G_1'}(w')$
1	d	1	$0(1)$
-1	$d-1$	$0(-1)$	1
$0(\lambda), \lambda \neq 1$	d	$0(\lambda - 1)$	$0(1-\lambda^{-1})$
$0(1)$	$d+1$	-1	-1

Proof. For all k we have

$$\mathrm{nd}_{P_2}(v') = \mathrm{nd}_{P_2}(w') = -1.$$

In forming G_1', there are three main cases to consider depending on whether $\mathrm{nd}_{G_1}(v)$ is 1, $0(\lambda)$, or -1. For ease of notation, we will refer to the identified vertex $v' = v$ of G_1' as v.

In every case, the graph $G_1' - v$ is the disjoint union of $G_1 - v$ and a single vertex w'. A null pattern $\boldsymbol{p}$ on $G_1' - v$ must restrict to a null pattern on $G_1 - v$ and have $\boldsymbol{p}(w') = 0$. Therefore,

$$\mathrm{null}(N(G_1' - v)) = \mathrm{null}(N(G_1 - v)).$$

Case 1: $\mathrm{nd}_{G_1}(v) = 1 \Rightarrow [\mathrm{nd}_{G_1'}(v) = 1$ and $\mathrm{nd}_{G_1'}(w') = 0(1)]$.

Suppose $\mathrm{nd}_{G_1}(v) = 1$. Then by Theorem 2.14, $\mathrm{nd}_{G_1'}(v) = 1$, showing that

$$\begin{aligned}\mathrm{null}(N(G_1')) &= \mathrm{null}(N(G_1' - v)) - 1\\ &= \mathrm{null}(N(G_1 - v)) - 1\\ &= \mathrm{null}(N(G_1)) = d.\end{aligned}$$

Then $\mathrm{nd}_{G_1'}(w') = 0$. To win $\boldsymbol{e}_{w'}$ on G_1', we press w' exactly $k-1$ times, relying on the fact that the pattern $(k-1)\boldsymbol{e}_v$ can be won on G_1 without pressing v (by Proposition 2.10). Therefore, $\lambda_{G_1'}(w') = -(k-1)^{-1} = 1$, showing that $\mathrm{nd}_{G_1'}(w') = 0(1)$.

Case 2: $\mathrm{nd}_{G_1}(v) = -1 \Rightarrow [\mathrm{nd}_{G_1'}(v) = 0(-1)$ and $\mathrm{nd}_{G_1'}(w') = 1]$.

Suppose $\mathrm{nd}_{G_1}(v) = -1$. Then by Theorem 2.14, $\mathrm{nd}_{G_1'}(v) = 0(-1)$, showing that

$$\begin{aligned}\mathrm{null}(N(G_1')) &= \mathrm{null}(N(G_1' - v)) \\ &= \mathrm{null}(N(G_1 - v)) \\ &= \mathrm{null}(N(G_1)) - 1 = d - 1.\end{aligned}$$

Then $\mathrm{nd}_{G_1'}(w') = 1$.

Case 3a: $\mathrm{nd}_{G_1}(v) = 0(\lambda)$, where $(\lambda \neq 1) \Rightarrow [\mathrm{nd}_{G_1'}(v) = 0(\lambda - 1)$ and $\mathrm{nd}_{G_1'}(w') = 0(1 - \lambda^{-1})]$.

Suppose $\mathrm{nd}_{G_1}(v) = 0(\lambda)$, where $\lambda \neq 1$. By Theorem 2.14, $\mathrm{nd}_{G_1'}(v) = 0(\lambda - 1)$, showing that

$$\begin{aligned}\mathrm{null}(N(G_1')) &= \mathrm{null}(N(G_1' - v)) \\ &= \mathrm{null}(N(G_1 - v)) \\ &= \mathrm{null}(N(G_1)) = d.\end{aligned}$$

Then $\mathrm{nd}_{G_1'}(w') = 0$. We know from Proposition 2.8 that $\boldsymbol{e}_{w'}$ is winnable on G_1'. Let $\boldsymbol{p}$ be a pattern on G_1' that wins $\boldsymbol{e}_{w'}$, and suppose $\boldsymbol{p}(w') = t$. Then $\boldsymbol{p}$, when restricted to G_1, gives a pattern on G_1 that wins $t\boldsymbol{e}_v$ with $\boldsymbol{p}(v) = -t - 1$. Since v is pressed $-\lambda^{-1}$ times in winning $\boldsymbol{e}_v$ on G_1, it follows that v is pressed $-t\lambda^{-1}$ times in winning $t\boldsymbol{e}_v$ on G_1. Thus $-t\lambda^{-1} = -t - 1$. Solving for t gives $t = (\lambda^{-1} - 1)^{-1}$. This implies

$$\lambda_{G_1'}(w') = -t^{-1} = 1 - \lambda^{-1}.$$

Thus $\mathrm{nd}_{G_1'}(w') = 0(1 - \lambda^{-1})$.

Case 3b: $\mathrm{nd}_{G_1}(v) = 0(1) \Rightarrow [\mathrm{nd}_{G_1'}(v) = -1$ and $\mathrm{nd}_{G_1'}(w') = -1]$.

Suppose $\mathrm{nd}_{G_1}(v) = 0(1)$. Then by Theorem 2.14, $\mathrm{nd}_{G_1'}(v) = -1$, showing that

$$\begin{aligned}\mathrm{null}(N(G_1')) &= \mathrm{null}(N(G_1' - v)) + 1 \\ &= \mathrm{null}(N(G_1 - v)) + 1 \\ &= \mathrm{null}(N(G_1)) + 1 = d + 1.\end{aligned}$$

Then $\mathrm{nd}_{G_1'}(w') = -1$. □

As another application of these results, we determine the dimension of the space of null patterns for joining two graphs via a new edge.

Definition 2.17 (edge join). Let G_1 and G_2 be graphs with $v \in V(G_1)$ and $w \in V(G_2)$. Let $H = \mathrm{EJ}(\{G_1, v\}, \{G_2, w\})$ be the graph with $V(H) = V(G_1) \cup V(G_2)$ and $E(H) = E(G_1) \cup E(G_2) \cup \{(v, w)\}$. The graph H will be called the *edge join* of G_1 and G_2 at v and w; see Figure 2.

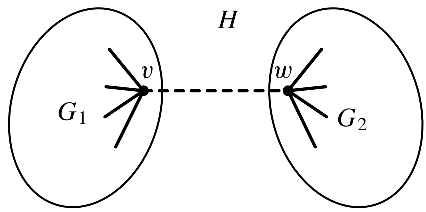


Figure 2. The edge join H of the graphs G_1 and G_2.

Theorem 2.18. *Let G_1 and G_2 be graphs with $v \in V(G_1)$ and $w \in V(G_2)$. Let $d_i = \operatorname{null}(N(G_i))$, and let $H = \operatorname{EJ}(\{G_1, v\}, \{G_2, w\})$. Then $\operatorname{null}(N(H))$ is given by the following table*:

$\operatorname{nd}_{G_1}(v)$	$\operatorname{nd}_{G_2}(w)$	$\operatorname{null}(N(H))$
1	*any*	$d_1 + d_2$
-1	*any*	$d_1 + d_2 + \operatorname{nd}_{G_2}(w) - 1$
$0(\lambda)$	$0(\mu)$	$d_1 + d_2,\ \mu \neq \lambda^{-1}$
$0(\lambda)$	$0(\mu)$	$d_1 + d_2 + 1,\ \mu = \lambda^{-1}$

Cases not covered above can be handled by symmetry.

Proof. Let P_2 be a path with two vertices, v' and w'. We construct H in two steps. Let

$$G_1' = \operatorname{VJ}(\{G_1, v\}, \{P_2, v'\}) \quad \text{and} \quad H = \operatorname{VJ}(\{G_1', w'\}, \{G_2, w\}),$$

where G_1, G_2, and P_2 are as in Figure 3.

We use Corollary 2.16 and then Theorem 2.14 to find $\operatorname{null}(N(H))$. To ease notation, we will refer to the identified vertex $v = v'$ in G_1' as simply v, and similarly we will refer to the vertices $v = v'$ and $w = w'$ of H as v and w, respectively.

Case 1: Suppose $\operatorname{nd}_{G_1}(v) = 1$. Corollary 2.16 implies $\operatorname{nd}_{G_1'}(w') = 0(1)$. If $\operatorname{nd}_{G_2}(w) = 1$, Theorem 2.14 shows that $\operatorname{nd}_H(w) = 1$. If $\operatorname{nd}_{G_2}(w) = -1$, Theorem 2.14 shows that $\operatorname{nd}_H(w) = -1$. If $\operatorname{nd}_{G_2}(w) = 0(\lambda)$, Theorem 2.14 shows that $\operatorname{nd}_H(w) = 0$,

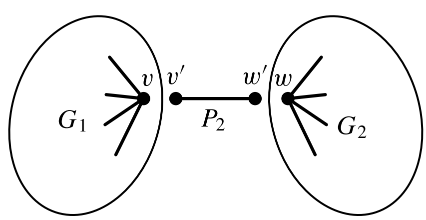


Figure 3. The graphs G_1 and G_2 and the path P_2 used in the proof of Theorem 2.18.

since $\lambda + 1 \neq 1$. Therefore, since $\text{nd}_H(w) = l_{G_2}(w)$ in every case,

$$\begin{aligned}\text{null}(N(H)) &= \text{null}(N(H - w)) - \text{nd}_H(w)\\ &= \text{null}(N(G_1)) + \text{null}(N(G_2 - w)) - \text{nd}_H(w)\\ &= \text{null}(N(G_1)) + \text{null}(N(G_2)) + \text{nd}_{G_2}(w) - \text{nd}_H(w)\\ &= d_1 + d_2.\end{aligned}$$

Case 2: Suppose $\text{nd}_{G_1}(v) = -1$ and $\text{nd}_{G_2}(w) \in \{0, -1\}$. Corollary 2.16 implies $\text{nd}_{G_1'}(w') = 1$. Theorem 2.14 implies $\text{nd}_H(w) = 1$. Therefore,

$$\begin{aligned}\text{null}(N(H)) &= \text{null}(N(H - w)) - 1\\ &= \text{null}(N(G_1)) + \text{null}(N(G_2 - w)) - 1\\ &= \text{null}(N(G_1)) + \text{null}(N(G_2)) + \text{nd}_{G_2}(w) - 1\\ &= d_1 + d_2 + \text{nd}_{G_2}(w) - 1.\end{aligned}$$

Case 3: Suppose $\text{nd}_{G_1}(v) = 0(\lambda)$ and $\text{nd}_{G_2}(w) = 0(\mu)$. Using Corollary 2.16, we find that $\text{nd}_{G_1'}(w') = -1$ if $\lambda = 1$ and $\text{nd}_{G_1'}(w') = 0(1 - \lambda^{-1})$ if $\lambda \neq 1$. In the case that $\lambda = 1$, Theorem 2.14 gives $\text{nd}_H(w) = 0$ when $\mu \neq 1$ and $\text{nd}_H(w) = -1$ when $\mu = 1$. In the case that $\lambda \neq 1$, Theorem 2.14 gives $\text{nd}_H(w) = 0$ when $\mu \neq \lambda^{-1}$ and $\text{nd}_H(w) = -1$ when $\mu = \lambda^{-1}$. In terms of computing dimensions, we then have two possibilities: either $\mu = \lambda^{-1}$, in which case $\text{null}(N(H)) = d_1 + d_2 + 1$, or $\mu \neq \lambda^{-1}$, in which case $\text{null}(N(H)) = d_1 + d_2$. □

Corollary 2.19. *Consider two always winnable graphs G_1 and G_2 over $\mathbb{Z}_k$, and let $H = \text{EJ}(\{G_1, v\}, \{G_2, w\})$. Then H is always winnable if and only if one of the following occurs*:

(1) *Either* $\text{nd}_{G_1}(v) = 1$ *or* $\text{nd}_{G_2}(w) = 1$, *or both.*

(2) $\text{nd}_{G_1}(v) = 0(\lambda)$ *and* $\text{nd}_{G_2}(w) = 0(\mu)$ *with* $\mu \neq \lambda^{-1} \pmod k$.

Proof. This is immediate from Theorem 2.18. We note that part (1) gives a different proof of [Edwards et al. 2010, Corollary 2.11]. □

One useful application of Theorem 2.14 is the idea of graph reduction, i.e., removing a set of vertices, along with all incident edges, from a graph without changing the dimension of the null space of the neighborhood matrix.

Corollary 2.20. (1) *Let G_1 and G_2 be graphs with $v_i \in V(G_i)$, and let $H = \text{VJ}(\{G_1, v_1\}, \{G_2, v_2\})$. Suppose G_2 is always winnable and* $\text{nd}_{G_2}(v_2) = 1$. *Then*

$$\text{null}(N(H)) = \text{null}(N(G_1 - v_1)).$$

(2) *Let H be a graph that has a degree-1 vertex x adjacent to a degree-2 vertex w. Let v be the vertex of H other than x that is adjacent to w. Then*

$$\mathrm{null}(N(H)) = \mathrm{null}(N(H - \{v, w, x\})).$$

Proof. (1) Let v be the vertex of H corresponding to the identification $v_1 = v_2$. By Theorem 2.14, $\mathrm{nd}_H(v) = 1$. Therefore,

$$\begin{aligned}\mathrm{null}(N(H)) &= \mathrm{null}(N(H - v)) - \mathrm{nd}_H(v) \\ &= \mathrm{null}(N(G_1 - v_1)) + \mathrm{null}(N(G_2 - v_2)) - \mathrm{nd}_H(v) \\ &= \mathrm{null}(N(G_1 - v_1)) + \mathrm{null}(N(G_2)) + \mathrm{nd}_{G_2}(v_2) - \mathrm{nd}_H(v) \\ &= \mathrm{null}(N(G_1 - v_1)),\end{aligned}$$

where the last equality is true since $\mathrm{null}(N(G_2)) = 0$ and $\mathrm{nd}_{G_2}(v_2) = \mathrm{nd}_H(v) = 1$.

(2) This comes from part (1) applied to $G_1 = H - \{w, x\}$ and $G_2 = P_3$, a path with three vertices $\{v', w, x\}$ where $\deg(w) = 2$. For all k, P_3 is always winnable, and $\mathrm{nd}_{P_3}(v') = 1$. □

Corollary 2.21. *Let H be a graph and $\{v_1, \dots, v_k\} \in V(H)$ such that $\deg(v_i) = 1$ for all i and each v_i is adjacent to the same vertex $x \in V(H)$. Then*

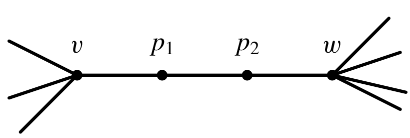
$$\mathrm{null}(N(H)) = \mathrm{null}(N(H - \{v_i : i = 1, \dots, k\})).$$

Proof. We apply Theorem 2.14 to the graphs $\{G = H - \{v_i : i = 1, \dots, k\}, x\}$ and $\{E_i = P_2, x\}$ for $i = 1, \dots, k$, where $V(E_i) = \{x, v_i\}$. We have $\mathrm{nd}_{E_i}(x) = -1$ for all i. By Theorem 2.14, we have

$$\mathrm{nd}_H(x) = \begin{cases} 1 & \text{if } \mathrm{nd}_G(x) = 1, \\ -1 & \text{if } \mathrm{nd}_G(x) = -1, \\ 0 & \text{if } \mathrm{nd}_G(x) = 0(\lambda), \end{cases}$$

where the last two equalities are true since there are $m = k + 1$ graphs involved. Since $\mathrm{nd}_H(x) = \mathrm{nd}_G(x)$ in every case and $\mathrm{null}(N(E_i - \{x\})) = 0$ for all i, we have $\mathrm{null}(N(H)) = \mathrm{null}(N(G))$. □

We close this section with one further result on graphs whose spaces of null patterns are isomorphic. The initial graph H in the following result has distinct vertices v, p_1, p_2, and w in the following configuration:



We show that the dimension of the space of null patterns does not change if vertices p_1 and p_2 are removed, along with incident edges, and then v is identified with w.

Proposition 2.22. *Let H be a graph with distinct vertices $v, p_1, p_2, w \in V(H)$ such that* $\deg(p_1) = \deg(p_2) = 2$, $\{(v, p_1), (p_1, p_2), (p_2, w)\} \subseteq E(H)$, *and* $(v, w) \notin E(H)$. *Let H' be the graph defined by identifying the vertices v and w inside $H - \{p_1, p_2\}$:*

(1) *For every $\boldsymbol{p} \in \mathrm{NS}_k(N(H))$, we have $\boldsymbol{p}(v) = \boldsymbol{p}(w)$.*

(2) *The induced mapping $\mathrm{NS}_k(N(H)) \to \mathrm{NS}_k(N(H'))$ is an isomorphism, and therefore*
$$\mathrm{null}(N(H)) = \mathrm{null}(N(H')).$$

Proof. (1) Let $\boldsymbol{p} \in \mathrm{NS}_k(N(H))$. Then
$$\boldsymbol{p}(v) + \boldsymbol{p}(p_1) + \boldsymbol{p}(p_2) = 0 \pmod k,$$
$$\boldsymbol{p}(p_1) + \boldsymbol{p}(p_2) + \boldsymbol{p}(w) = 0 \pmod k.$$
This shows that $\boldsymbol{p}(v) = \boldsymbol{p}(w)$ as elements of $\mathbb{Z}_k$.

(2) Let $\boldsymbol{p} \in \mathrm{NS}_k(N(H))$. Since $\boldsymbol{p}(v) = \boldsymbol{p}(w)$, $\boldsymbol{p}$ naturally induces a pattern on H' which we will denote by $\boldsymbol{p}'$. We will now show that $\boldsymbol{p}' \in \mathrm{NS}_k(N(H'))$. Let $t = \sum_{u \in X(v)\setminus\{v, p_1\}} \boldsymbol{p}(u)$ and $s = \sum_{u \in X(w)\setminus\{w, p_2\}} \boldsymbol{p}(u)$.

Since $\boldsymbol{p}$ is null at v and w, we have
$$\boldsymbol{p}(v) + \boldsymbol{p}(p_1) + t = 0 \pmod k,$$
$$\boldsymbol{p}(p_2) + \boldsymbol{p}(w) + s = 0 \pmod k.$$
When combined with the equations in part (1), this implies
$$t = \boldsymbol{p}(p_2) \pmod k,$$
$$s = \boldsymbol{p}(p_1) \pmod k.$$
Clearly $\boldsymbol{p}'$ is null on H' except possibly at v', the vertex created by the identification of v with w. We have
$$\sum_{u \in X(v')} \boldsymbol{p}'(u) = \boldsymbol{p}'(v') + s + t = \boldsymbol{p}(v) + \boldsymbol{p}(p_1) + \boldsymbol{p}(p_2) = 0 \pmod k.$$
Hence, $\boldsymbol{p}' \in \mathrm{NS}_k(N(H'))$, and $\boldsymbol{p} \mapsto \boldsymbol{p}'$ gives a linear transformation from $\mathrm{NS}_k(N(H))$ to $\mathrm{NS}_k(N(H'))$. To see that this linear transformation is bijective, notice that any null pattern $\boldsymbol{q}'$ on H' can be extended uniquely to a null pattern $\boldsymbol{q}$ on H as follows:

- $\boldsymbol{q}$ is identical to $\boldsymbol{q}'$ away from $\{v, p_1, p_2, w\}$.
- $\boldsymbol{q}(v) = \boldsymbol{q}(w) = \boldsymbol{q}'(v')$.
- $\boldsymbol{q}(p_1) = -\sum_{u \in X(v)\setminus\{p_1\}} \boldsymbol{q}'(u)$.
- $\boldsymbol{q}(p_2) = -\sum_{u \in X(w)\setminus\{p_2\}} \boldsymbol{q}'(u)$.

The pattern $\boldsymbol{q}$ is null by construction on vertices of H not in $\{p_1, p_2\}$. To see that $\boldsymbol{q}$ is null at p_1, note that

$$\begin{aligned}\boldsymbol{q}(v)+\boldsymbol{q}(p_1)+\boldsymbol{q}(p_2) &= \boldsymbol{q}'(v') - \sum_{u\in X(v)\setminus\{p_1\}} \boldsymbol{q}(u) - \sum_{u\in X(w)\setminus\{p_2\}} \boldsymbol{q}(u)\\ &= \boldsymbol{q}'(v') - \boldsymbol{q}'(v') - \sum_{u\in X(v')} \boldsymbol{q}'(u)\\ &= 0 \pmod k,\end{aligned}$$

where the last equality is true because $\boldsymbol{q}'$ is null on H'. Now $\boldsymbol{q}$ is also null at p_2 since $\boldsymbol{q}(v) = \boldsymbol{q}(w)$. □

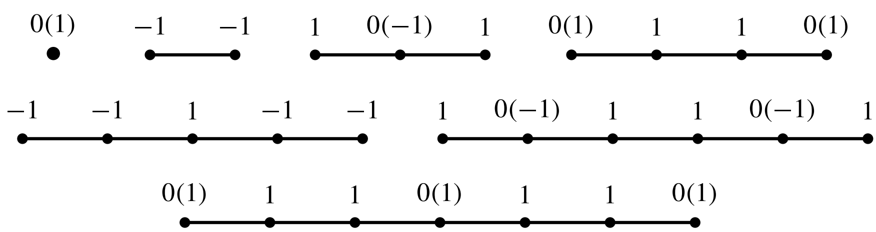
3. Some examples and applications

The following information for paths, cycles, complete graphs, and complete bipartite graphs can be obtained directly, but also follows from [Giffen and Parker 2013, Theorem 4.4], which gives the result in terms of winnable states.

Paths. For P_n, a path with n vertices, we have

$$\operatorname{null}(N(P_n)) = \begin{cases} 0 & \text{if } n \neq 2 \pmod 3,\\ 1 & \text{if } n = 2 \pmod 3.\end{cases}$$

This shows that the null-differences of the vertices of P_n follow the pattern below:



When $n = 2 \pmod 3$, a basis for $\mathrm{NS}_k(N(P_n))$ is given by a pattern of the form

$$(1, -1, 0, 1, -1, 0, \ldots, 1, -1),$$

where the vertices are listed in the order that they are connected along the path.

Cycles. Let C_n be the n-cycle. Then

$$\operatorname{null}(N(C_n)) = \begin{cases} 0 & \text{if } 3\nmid n \text{ and } k \neq 3,\\ 1 & \text{if } 3\nmid n \text{ and } k = 3,\\ 2 & \text{if } 3 \mid n.\end{cases}$$

If $v \in V(C_n)$, then

$$\mathrm{nd}_{C_n}(v) = \begin{cases} 0(3) & \text{if } n = 1 \pmod 3 \text{ and } k \neq 3, \\ 0(-3) & \text{if } n = 2 \pmod 3 \text{ and } k \neq 3, \\ -1 & \text{if } 3 \mid n \text{ or } k = 3. \end{cases}$$

If 3 is not a divisor of n and $k = 3$, a basis for $\mathrm{NS}_k(N(C_n))$ is given by the pattern $\boldsymbol{p}$ such that $\boldsymbol{p}(v) = 1$ for every vertex $v \in V(C_n)$. If $3 \mid n$, a basis for $\mathrm{NS}_k(N(C_n))$ is given by

$$\{(1, -1, 0, 1, -1, 0, \ldots, 1, -1, 0), (0, 1, -1, 0, 1, -1, \ldots, 0, 1, -1)\},$$

where the vertices are listed in the order given by proceeding around the cycle.

Complete graphs. For the complete graph on n vertices K_n, we have $\mathrm{null}(N(K_n)) = n-1$, and $\mathrm{nd}_{K_n}(v) = -1$ for all $v \in V(K_n)$. Choose a vertex $v \in V(K_n)$. A basis of $\mathrm{NS}_k(N(K_n))$ is given by the set of patterns of the form $\boldsymbol{p}_w$, where $w \in V(K_n) \setminus \{v\}$, $\boldsymbol{p}_w(v) = 1$, $\boldsymbol{p}_w(w) = -1$, and $\boldsymbol{p}_w(u) = 0$ if $u \in V(K_n) \setminus \{v, w\}$.

Complete bipartite graphs. Let $K_{m,n}$ be the complete bipartite graph on m and n vertices. We will refer to the set of m vertices as the "left-hand" vertices and the set of n vertices as the "right-hand" vertices. We have

$$\mathrm{null}(N(K_{m,n})) = \begin{cases} 0 & \text{if } k \nmid (mn-1), \\ 1 & \text{if } k \mid (mn-1). \end{cases}$$

When $k \mid (mn-1)$, a basis of $\mathrm{NS}_k(N(K_{m,n}))$ is given by the pattern $\boldsymbol{p}$ which has value n at all left-hand vertices and value -1 at all right-hand vertices.

If $k \mid (mn-1)$, then k can divide neither $(m-1)n-1$ nor $m(n-1)-1$, and therefore, $\mathrm{nd}_{K_{m,n}}(v) = -1$ for all $v \in V(K_{m,n})$. If $k \nmid (mn-1)$, then k may divide neither, one, or both of $(m-1)n-1$ and $m(n-1)-1$. We summarize the possibilities in Table 1.

$k \mid (mn-1)$	$k \mid ((m-1)n-1)$	$k \mid (m(n-1)-1)$	left nd	right nd
yes	no	no	-1	-1
no	no	no	$0(\lambda_L)$	$0(\lambda_R)$
no	no	yes	$0(\lambda_L)$	1
no	yes	no	1	$0(\lambda_R)$
no	yes	yes	1	1

Table 1. Summary of possibilities for complete bipartite graphs. Here "left/right nd" means the null-difference on the left/right-hand vertices, $\lambda_L = (mn-1)(mn-n-1)^{-1}$ and $\lambda_R = (mn-1)(mn-m-1)^{-1}$.

Generalized star graphs. We give an application to generalized star graphs, called "spider graphs" in [Edwards et al. 2010]. A *generalized star* is a connected graph of the form $G = \mathrm{VJ}(\{P_{n_i}, v_i\})$, where $n_i \geq 2$ are integers, P_{n_i} is a path with n_i vertices, and v_i is a degree-1 vertex of P_{n_i}. The vertex $v \in V(G)$ is called the *center* of G, and $G - v$ is a disjoint union of the paths $P_{n_i - 1}$. To avoid trivial cases, we assume $\deg(v) > 2$. Every vertex of G other than v has degree 1 or 2.

The following result gives a more general version of [Edwards et al. 2010, Theorem 3.4].

Proposition 3.1. *Let $G = \mathrm{VJ}(\{P_{n_i}, v_i\})$ be a generalized star as defined above, where $\{n_1, n_2, \ldots, n_m\}$ is a set of integers with $n_i \geq 2$ and $m \geq 3$. For $j \in \{0, 1, 2\}$, let p_j be the number of n_i such that $n_i = j \pmod 3$. Then*

$$\mathrm{nd}_G(v) = \begin{cases} -1 & \text{if } p_0 = 0 \text{ and } k \mid (p_2 - 1), \\ 0(1 - p_2) & \text{if } p_0 = 0 \text{ and } k \nmid (p_2 - 1), \\ 1 & \text{if } p_0 \neq 0. \end{cases}$$

This implies

$$\mathrm{null}(N(G)) = \begin{cases} 1 & \text{if } p_0 = 0 \text{ and } k \mid (p_2 - 1), \\ 0 & \text{if } p_0 = 0 \text{ and } k \nmid (p_2 - 1), \\ p_0 - 1 & \text{if } p_0 \neq 0. \end{cases}$$

In particular, G is always winnable over $\mathbb{Z}_k$ if and only if either $p_0 = 1$ or both $p_0 = 0$ and $k \nmid (p_2 - 1)$.

Proof. This follows immediately from Theorem 2.14 and the characterization of paths given above. □

Star graphs. Let S_n be the star with $n \geq 3$ edges. By Proposition 3.1 or the results on complete bipartite graphs given above, we see that $\mathrm{null}(N(S_n)) = 1$ if $k \mid (n-1)$ and $\mathrm{null}(N(S_n)) = 0$ otherwise. This implies that for $v \in V(S_n)$ such that $\deg(v) = 1$, we have

$$\mathrm{nd}_{S_n}(v) = \begin{cases} -1 & \text{if } k \mid (n-1), \\ 1 & \text{if } k \mid (n-2), \\ 0((n-1)(n-2)^{-1}) & \text{otherwise.} \end{cases}$$

Proposition 3.2. *Let $G = \mathrm{VJ}(\{S_{n_i}, v_i\})$, where $\{n_1, n_2, \ldots, n_m\}$ is a set of integers with $n_i \geq 2$, $m \geq 2$ and $\deg_{S_{n_i}}(v_i) = 1$. Let v be the vertex of G created by the identification of the vertices v_i. Let p_2 be the number of the n_i such that $n_i = 2 \pmod k$. Then*

$$\mathrm{nd}_G(v) = \begin{cases} -1 & \text{if } p_2 = 0 \text{ and } \sum_{i=1}^m (n_i - 1)(n_i - 2)^{-1} = m - 1 \pmod k, \\ 0 & \text{if } p_2 = 0 \text{ and } \sum_{i=1}^m (n_i - 1)(n_i - 2)^{-1} \neq m - 1 \pmod k, \\ 1 & \text{if } p_2 \neq 0, \end{cases}$$

which implies

$$\operatorname{null}(N(G)) = \begin{cases} 1 & \text{if } p_2 = 0 \text{ and } \sum_{i=1}^{m}(n_i-1)(n_i-2)^{-1} = m-1 \pmod{k}, \\ 0 & \text{if } p_2 = 0 \text{ and } \sum_{i=1}^{m}(n_i-1)(n_i-2)^{-1} \neq m-1 \pmod{k}, \\ p_2-1 & \text{if } p_2 \neq 0. \end{cases}$$

In particular, G is always winnable over $\mathbb{Z}_k$ if and only if either $p_2 = 1$ or both $p_2 = 0$ and

$$\sum_{i=1}^{m}(n_i-1)(n_i-2)^{-1} \neq m-1 \pmod{k}.$$

Proof. This follows immediately from Theorem 2.14 using the properties of stars given after Proposition 3.1 and the properties of P_2 and P_3 (to handle the case where n_i might be equal to 2 for some values of i). □

We have a similar result for cycles.

Proposition 3.3. *Let $G = \mathrm{VJ}(\{C_{n_i}, v_i\})$, where $\{n_1, n_2, \ldots, n_m\}$ is a set of integers with $n_i \geq 3$, $m \geq 2$, and $v_i \in V(C_{n_i})$. Let v be the vertex of G created by the identification of the vertices v_i. For $j \in \{0, 1, 2\}$, let p_j be the number of n_i that are congruent to j modulo 3. Then*

$$\operatorname{nd}_G(v) = \begin{cases} -1 & \text{if } 3(p_1 - p_2) = m-1 \pmod{k}, \\ 0 & \text{if } 3(p_1 - p_2) \neq m-1 \pmod{k}, \end{cases}$$

which implies

$$\operatorname{null}(N(G)) = \begin{cases} p_0 + 1 & \text{if } 3(p_1 - p_2) = m-1 \pmod{k}, \\ p_0 & \text{if } 3(p_1 - p_2) \neq m-1 \pmod{k}. \end{cases}$$

In particular, G is always winnable over $\mathbb{Z}_k$ if and only if $p_0 = 0$ and $3(p_1 - p_2) \neq m-1 \pmod{k}$.

Proof. This follows from Theorem 2.14 and the characterization of cycles given above. □

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