involve
a journal of mathematics

Curves of constant curvature and torsion in the 3-sphere

Debraj Chakrabarti, Rahul Sahay and Jared Williams

msp

2019 vol. 12, no. 2

INVOLVE 12:2 (2019)
dx.doi.org/10.2140/involve.2019.12.235

Curves of constant curvature and torsion in the 3-sphere

Debraj Chakrabarti, Rahul Sahay and Jared Williams

(Communicated by Colin Adams)

We describe the curves of constant (geodesic) curvature and torsion in the three-dimensional round sphere. These curves are the trajectory of a point whose motion is the superposition of two circular motions in orthogonal planes. The global behavior may be periodic or the curve may be dense in a Clifford torus embedded in the 3-sphere. This behavior is very different from that of helices in three-dimensional Euclidean space, which also have constant curvature and torsion.

1. Introduction

Let $(M, \langle\ ,\ \rangle)$ be a three-dimensional Riemannian manifold, let $I \subseteq \mathbb{R}$ be an open interval, and let $\boldsymbol{\gamma} : I \to M$ be a smooth curve in M, which we assume to be parametrized by the arc length t. It is well known that the local geometry of $\boldsymbol{\gamma}$ is characterized by the *curvature* κ and the *torsion* τ. These are functions defined along $\boldsymbol{\gamma}$ and are the coefficients of the well-known *Frenet–Serret formulas* [Spivak 1975b, pp. 21–23]

$$
\begin{aligned}
\frac{D}{dt}\boldsymbol{T}(t) &= \qquad\qquad \kappa(t)\boldsymbol{N}(t),\\
\frac{D}{dt}\boldsymbol{N}(t) &= -\kappa(t)\boldsymbol{T}(t) \qquad\qquad +\tau(t)\boldsymbol{B}(t),\\
\frac{D}{dt}\boldsymbol{B}(t) &= \qquad\qquad -\tau(t)\boldsymbol{N}(t),
\end{aligned}
\tag{1-1}
$$

where the orthogonal unit vector fields $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, with $\boldsymbol{T} = \boldsymbol{\gamma}'$, along the unit-speed curve $\boldsymbol{\gamma}$, constitute its *Frenet frame* and $\frac{D}{dt}$ denotes covariant differentiation along $\boldsymbol{\gamma}$ with respect to the arc length t. We will assume that each of the functions κ and τ

MSC2010: 53A35.
Keywords: Frenet–Serret equations, constant curvature and torsion, geodesic curvature, helix, 3-sphere, curves in the 3-sphere.
All three authors were partially supported by a grant from the NSF (#1600371). Chakrabarti was partially supported by a grant from the Simons Foundation (#316632) and also by an Early Career internal grant from Central Michigan University.

is either nowhere zero or vanishes identically. Additionally, if κ is identically zero, then τ is also taken to be identically zero. For completeness, we include a proof of the set of equations given in (1-1) in Section 3A below. We make the following definition:

Definition 1.1. Let M be a Riemannian manifold of dimension 3. A curve $\boldsymbol{\gamma} : I \to M$, where $I \subseteq \mathbb{R}$ is an open interval, will be called a *helix* (plural: *helices*) if its curvature κ and torsion τ are nonnegative constants. A helix is *nondegenerate* if κ and τ are both positive, and *degenerate* otherwise. We say that the helix $\boldsymbol{\gamma}$ is *periodic* if there is a $T > 0$ such that $\boldsymbol{\gamma}(t+T) = \boldsymbol{\gamma}(t)$ for each $t \in I$.

We take τ to be nonnegative since we use the nonoriented form of the Frenet–Serret equations (see Section 3). Definition 1.1 is motivated by the example of the Euclidean space $\mathbb{R}^3$, where nondegenerate helices are curves of the form

$$\boldsymbol{\gamma}(t) = \cos(\omega t)\boldsymbol{A} + \sin(\omega t)\boldsymbol{B} + t\boldsymbol{C} + \boldsymbol{D}, \tag{1-2}$$

where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}^3$, $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are nonzero and orthogonal with $|\boldsymbol{A}| = |\boldsymbol{B}|$, and $\omega > 0$. These are elegant curves that are invariant under a one-parameter group of isometries of the ambient space. Note that there are no nondegenerate periodic helices in $\mathbb{R}^3$.

The aim of this paper is to study helices in the three-dimensional round sphere $\mathbb{S}^3$. Thanks to the fact that $\mathbb{S}^3$ is compact, we expect that a nondegenerate helix in $\mathbb{S}^3$ should “come back where it started from” provided we wait long enough, and therefore, there is a possibility that, for a favorable choice of the curvature and torsion, the helix is actually periodic, though locally it is not much different from a helix in $\mathbb{R}^3$. Globally, a nondegenerate helix in $\mathbb{S}^3$ has *two* fundamental angular frequencies, ω_1 and ω_2, as opposed to the single fundamental angular frequency, ω, of the helix given by (1-2) in $\mathbb{R}^3$. A nondegenerate helix in $\mathbb{S}^3$ may be thought of as the trajectory of a particle which performs two superimposed circular motions with frequencies ω_1 and ω_2. These fundamental angular frequencies must satisfy

$$\omega_2 > 1 > \omega_1,$$

a constraint which arises because a curve with nonzero curvature and torsion must lie in the positively curved compact space $\mathbb{S}^3$. Unlike in $\mathbb{R}^3$ where nondegenerate helices are embedded noncompact submanifolds, depending on the fundamental angular frequencies ω_1 and ω_2, a nondegenerate helix in $\mathbb{S}^3$ can either be periodic (when it is a compact embedded submanifold) or be dense in a flat 2-torus contained in $\mathbb{S}^3$ (when the image of the helix is not an embedded submanifold of $\mathbb{S}^3$). This divergence of global behavior from the flat case is the main topic of this paper.

The properties of helices in the 3-sphere and other spaces of constant curvature have been studied before; see, e.g., [Arroyo et al. 2004; Barros 1997]. While some of the results of this paper can be found scattered in these and other references, we use a completely elementary and direct approach to the problem, based on an explicit integration of the Frenet equations, the possibility of which does not seem to be widely known in the context of spheres. Our method uses nothing beyond undergraduate calculus and linear algebra. Of course, the questions considered in this paper can be asked in any number of spatial dimensions, and for other Riemannian manifolds. Here, for simplicity we consider the special case of $\mathbb{S}^3$, which allows us to visualize helices using the stereographic projection of $\mathbb{S}^3$ onto $\mathbb{R}^3$ (see Section 2A below). The method used in this paper will likely generalize to round spheres of any number of dimensions.

2. Main results

We consider $\mathbb{S}^3$ to be embedded in the Euclidean space $\mathbb{R}^4$ in the natural way as the hypersurface $\{x_1^2+x_2^2+x_3^2+x_4^2=1\}$, and endow $\mathbb{S}^3$ with the Riemannian metric induced from $\mathbb{R}^4$. This entails no loss of generality because the metric so induced is the same as the standard round metric of $\mathbb{S}^3$ with constant sectional curvature 1. To state our results concisely, let us introduce the following definition:

Definition 2.1. A smooth curve $\boldsymbol{\gamma}$ in $\mathbb{R}^4$ will be called a *Lissajous curve* if there are numbers $\omega_2 > \omega_1 \geq 0$ and vectors $\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{A}_2, \boldsymbol{B}_2 \in \mathbb{R}^4$ such that, for each t,

$$\boldsymbol{\gamma}(t) = \cos(\omega_1 t)\boldsymbol{A}_1 + \sin(\omega_1 t)\boldsymbol{B}_1 + \cos(\omega_2 t)\boldsymbol{A}_2 + \sin(\omega_2 t)\boldsymbol{B}_2. \tag{2-1}$$

We will call ω_1 and ω_2 the *fundamental angular frequencies* of the curve $\boldsymbol{\gamma}$ and $\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{A}_2, \boldsymbol{B}_2$ the *coefficient vectors* of $\boldsymbol{\gamma}$.

Therefore, a Lissajous curve, in our sense, can be thought of as the trajectory of a point in $\mathbb{R}^4$ which oscillates with frequency ω_1 in the $\boldsymbol{A}_1\boldsymbol{B}_1$-plane and with frequency ω_2 in the $\boldsymbol{A}_2\boldsymbol{B}_2$-plane. Note also that the projection of $\boldsymbol{\gamma}$ on any two-dimensional linear subspace different from the $\boldsymbol{A}_1\boldsymbol{B}_1$- and $\boldsymbol{A}_2\boldsymbol{B}_2$-planes is a planar Lissajous curve in the usual sense of the term [Hasselblatt and Katok 2003, pp. 114–115]. We are now ready to describe helices in $\mathbb{S}^3$:

Theorem 1. *Let $\kappa, \tau \geq 0$ be given numbers where, if $\kappa = 0$, then $\tau = 0$:*

(1) *There exists a helix $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{S}^3$ with constant curvature κ and torsion τ.*

(2) *Such a helix $\boldsymbol{\gamma}$ is a Lissajous curve in the form of* (2-1).

(3) *The fundamental angular frequencies of $\boldsymbol{\gamma}$ are distinct and are given by*

$$\omega_1 = \sqrt{\tfrac{1}{2}(\chi^2 - \sqrt{\chi^4 - 4\tau^2})}, \tag{2-2}$$

$$\omega_2 = \sqrt{\tfrac{1}{2}(\chi^2 + \sqrt{\chi^4 - 4\tau^2})}, \tag{2-3}$$

with

$$\chi^2 = \kappa^2 + \tau^2 + 1. \tag{2-4}$$

(4) *If* $\kappa > 0$, *then the frequencies* ω_1 *and* ω_2 *satisfy*

$$\omega_2 > 1 > \omega_1. \tag{2-5}$$

(5) *If* $\tau \neq 0$ *then the four coefficient vectors* $\boldsymbol{A}_1$, $\boldsymbol{B}_1$, $\boldsymbol{A}_2$, $\boldsymbol{B}_2$ *are orthogonal in* $\mathbb{R}^4$, *and their magnitudes are given by*

$$|\boldsymbol{A}_1|^2 = |\boldsymbol{B}_1|^2 = \frac{1-\omega_2^2}{\omega_1^2-\omega_2^2}, \tag{2-6}$$

$$|\boldsymbol{A}_2|^2 = |\boldsymbol{B}_2|^2 = \frac{1-\omega_1^2}{\omega_2^2-\omega_1^2}. \tag{2-7}$$

(6) *If* $\tau = 0$, *then* $\boldsymbol{\gamma}$ *is a circle given by*

$$\boldsymbol{\gamma}(t) = \boldsymbol{A}_1 + \cos(\omega t)\boldsymbol{A}_2 + \sin(\omega t)\boldsymbol{B}_2,$$

where

$$\omega = \sqrt{\kappa^2+1}.$$

Further, *the coefficient vectors* $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{B}_2$ *are mutually orthogonal and we have*

$$|\boldsymbol{A}_2| = |\boldsymbol{B}_2| = \frac{1}{\omega} \quad \textit{and} \quad |\boldsymbol{A}_1| = \sqrt{1-\frac{1}{\omega^2}}.$$

Several interesting features may be noted here:

(a) The local existence of helices follows from the existence theorem for solutions of systems of ordinary differential equations on manifolds. However, we prove the existence of helices directly by solving the Frenet–Serret equations and obtain an explicit representation of the solution.

(b) When $\kappa, \tau > 0$, the curve $\boldsymbol{\gamma}$ may be thought of as the trajectory of a motion consisting of two superimposed circular motions in perpendicular planes: one at a "slow" frequency $\omega_1 < 1$ and the other at a "fast" frequency $\omega_2 > 1$. This global behavior arises from the fact that the curve $\boldsymbol{\gamma}$ must lie on the compact surface $\mathbb{S}^3$. Observe that there is no such restriction on the angular frequency ω of the Euclidean helix given by (1-2).

(c) When $\kappa = 0$, by definition we have $\tau = 0$. Such a curve is a *geodesic*; i.e., its unit tangent field is autoparallel along the curve. Therefore, geodesics on the sphere $\mathbb{S}^3$ are of the form

$$\boldsymbol{\gamma}(t) = \cos(t)\boldsymbol{A} + \sin(t)\boldsymbol{B},$$

where $|\boldsymbol{A}| = |\boldsymbol{B}| = 1$ and $\boldsymbol{A}$, $\boldsymbol{B}$ are mutually orthogonal. Thus, we recapture the well-known fact that geodesics in $\mathbb{S}^3$ are great circles.

(c) An alternative approach to the study of curves in $\mathbb{S}^3$ is given by the use of the *Hopf map*, which is a smooth map from $\mathbb{S}^3$ to the 2-sphere $\mathbb{S}^2(4)$; see [Pinkall 1985; Arroyo et al. 2004; Barros 1997] for details. Given a curve γ in $\mathbb{S}^2(4)$, its complete lift under the Hopf map is a flat surface in $\mathbb{S}^3$, called the *Hopf cylinder* shaped on γ. It turns out that helices in $\mathbb{S}^3$ can be thought of as geodesics of Hopf tori shaped on circles in $\mathbb{S}^2(4)$, and from this relation more general versions of some of the results in this paper can be deduced; see [Arroyo et al. 2004; Barros 1997].

As already pointed out, our method is based on a direct integration of the Frenet equations, and yields explicit formulas for the helices. For the particular problem of characterizing curves of constant curvature and torsion considered in this paper, our method enjoys several advantages over that based on the Hopf map. First of all, it is explicit and gives very simple formulas describing the helices. It is certainly much simpler and uses nothing beyond elementary calculus and linear algebra. But the most important advantage is that it can be generalized to study curves of functions with constant Frenet curvature in higher-dimensional spheres. Indeed, on the sphere $\mathbb{S}^n$, it is easy to write down an analog of (3-6), or its matrix form (5-2). Clearly, $\boldsymbol{C}$ will be replaced by an $(n+1)\times(n+1)$ matrix which is skew symmetric, tridiagonal and whose entries are the functions of the helix with constant Frenet curvature, and therefore the Frenet equations can again be explicitly integrated. While the method based on the Hopf map is very powerful, it is also specific to $\mathbb{S}^3$ and does not generalize to higher dimensions.

We now turn to the question of uniqueness and periodicity of helices. First, note that if $\boldsymbol{\gamma}$ is a helix in a Riemannian 3-manifold M and $f : M \to M$ is a self-isometry of M, then $f \circ \boldsymbol{\gamma}$ is also a helix in M with the same curvature and torsion as that of $\boldsymbol{\gamma}$. In $\mathbb{R}^3$, the converse holds; i.e., helices with the same curvature and torsion are congruent under an isometry of $\mathbb{R}^3$. This is a special case of the well-known fundamental theorem of curves, see, e.g., [Stoker 1969, pp. 65–67], according to which a Frenet curve is determined up to congruence by its curvature and torsion as functions of arc length. It is easy to see that a similar statement must also hold in $\mathbb{S}^3$, and using the explicit form of helices determined above, we verify this fundamental theorem in this special case below. We also determine when helices are periodic.

Recall that a *Clifford torus* is a Riemannian 2-manifold which is the metric product of two circles. Clearly, a Clifford torus is flat; i.e., its Gaussian curvature vanishes identically. It is well known that there are Clifford tori embedded in the sphere $\mathbb{S}^3$; e.g., for $0 < \lambda < 1$, the surface in $\mathbb{R}^4$ given by

$$\mathcal{C}_\lambda = \{\boldsymbol{X} \in \mathbb{R}^4 : x_1^2 + x_2^2 = \lambda,\ x_3^2 + x_4^2 = 1 - \lambda\}$$

is clearly contained in $\mathbb{S}^3$ and is therefore a Clifford torus in $\mathbb{S}^3$ which is flat in the Riemannian metric induced by the round metric of $\mathbb{S}^3$.

Theorem 2. *Let $\kappa, \tau \geq 0$ be given numbers where, if $\kappa = 0$, then $\tau = 0$:*

(1) *If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two helices in $\mathbb{S}^3$ with the same curvature κ and torsion τ, then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are congruent; i.e., there is a Riemannian isometry $f : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\boldsymbol{\beta} = f \circ \boldsymbol{\alpha}$.*

(2) *A helix $\boldsymbol{\gamma}$ is periodic if and only if the ratio of the angular frequencies*
$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{\chi^2 - \sqrt{\chi^4 - 4\tau^2}}{\chi^2 + \sqrt{\chi^4 - 4\tau^2}}} \tag{2-8}$$
is a rational number, where $\chi^2 = \kappa^2 + \tau^2 + 1$.

(3) *If $\kappa, \tau > 0$, there exists a Clifford torus $\mathbb{T}^2_{\boldsymbol{\gamma}}$ contained in $\mathbb{S}^3$ such that the image of $\boldsymbol{\gamma}$ lies on $\mathbb{T}^2_{\boldsymbol{\gamma}}$.*

(4) *If $\kappa, \tau > 0$ and ω_1/ω_2 is irrational, the image of $\boldsymbol{\gamma}$ is dense in the torus $\mathbb{T}^2_{\boldsymbol{\gamma}}$.*

2A. *Visualization of helices.* One way to visualize $\mathbb{S}^3$ is to use the stereographic projection $\sigma : \mathbb{S}^3 \setminus \{p\} \to \mathbb{R}^3$, where p is a point in $\mathbb{S}^3$ which serves as the pole of the projection. It is well known that σ is conformal; i.e., it preserves angles but not lengths. Using Wolfram Mathematica, we produced visualizations of two helices in $\mathbb{S}^3$ which are shown in Figures 1 and 2. Each of these pictures represents two distinct perspective projections onto $\mathbb{R}^2$ of the stereographic projection of the helix, where p is chosen to not be on the helix. The helix in Figure 1 is nonperiodic and therefore dense in a Clifford torus. The helix in Figure 2 is periodic and therefore an embedded curve in $\mathbb{S}^3$. The hue and brightness of the curves are functions of the fourth coordinate of the curve $\boldsymbol{\gamma}$ in its embedding in $\mathbb{R}^4$.

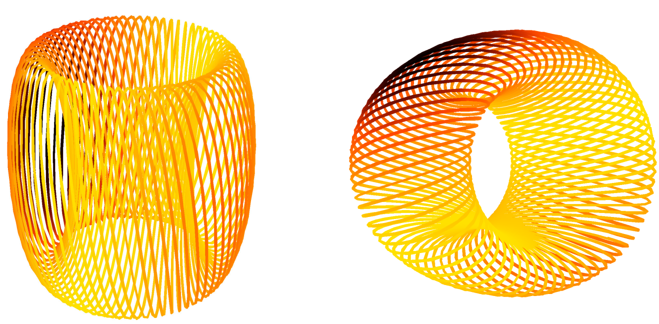

Figure 1. Two views of a dense helix in $\mathbb{S}^3$ with $\kappa = \frac{5\sqrt{3}}{4}$ and $\tau = \frac{\sqrt{29}}{4}$. The corresponding fundamental angular frequencies are then $\omega_1 = \frac{1}{2}$, $\omega_2 = \frac{\sqrt{29}}{2}$, and thus, their ratio is the irrational number $\omega_2/\omega_1 = \sqrt{29}$.

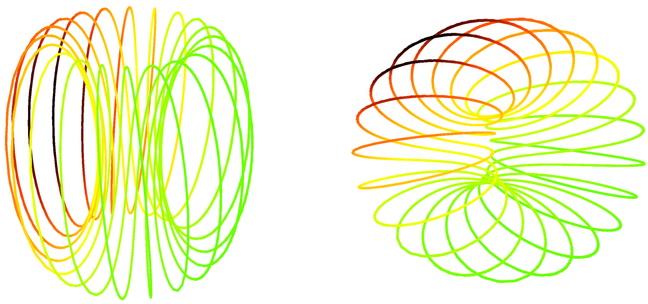

Figure 2. Two views of a periodic helix in $\mathbb{S}^3$ with $\kappa = \frac{\sqrt{15}}{3}$ and $\tau = \frac{5}{12}$. The corresponding fundamental angular frequencies are then $\omega_1 = \frac{1}{4}$, $\omega_2 = \frac{5}{3}$, and thus, their ratio is the rational number $\omega_2/\omega_1 = \frac{20}{3}$.

3. The Frenet–Serret equations

3A. ***The Frenet–Serret equations in a three-dimensional Riemannian manifold.*** Consider a three-dimensional Riemannian manifold $(M, \langle\,,\rangle)$ and an arc-length parametrized curve $\boldsymbol{\gamma} : I \to M$, where $I \subset \mathbb{R}$ is an open interval. Let $\frac{D}{dt}$ represent the covariant derivative of a vector field along a curve (parametrized by t). Let $\boldsymbol{T} = \boldsymbol{\gamma}'$ denote the *unit tangent vector field* of $\boldsymbol{\gamma}$. Since $\langle \boldsymbol{T}(t), \boldsymbol{T}(t)\rangle = 1$ for each t we have

$$0 = \frac{d}{dt}\langle \boldsymbol{T}(t), \boldsymbol{T}(t)\rangle = 2\Big\langle \boldsymbol{T}(t), \frac{D}{dt}\boldsymbol{T}(t)\Big\rangle.$$

Then, the curvature function κ is defined as

$$\kappa(t) = \Big|\frac{D}{dt}\boldsymbol{T}(t)\Big|. \tag{3-1}$$

We will assume that either $\kappa(t) \neq 0$ for all t or that $\kappa \equiv 0$. In the case where κ never vanishes, we define the *normal vector field* to $\boldsymbol{\gamma}$ by

$$\boldsymbol{N}(t) = \frac{1}{\kappa(t)}\frac{D}{dt}\boldsymbol{T}(t).$$

Then $\boldsymbol{N}$ is a unit vector field along $\boldsymbol{\gamma}$ which is always orthogonal to $\boldsymbol{T}$. The definition of $\boldsymbol{N}$ gives the first Frenet–Serret equation

$$\frac{D}{dt}\boldsymbol{T}(t) = \kappa(t)\boldsymbol{N}(t). \tag{3-2}$$

Similarly, since $\langle \boldsymbol{N}(t), \boldsymbol{N}(t)\rangle = 1$ for each $t \in I$,

$$2\Big\langle \boldsymbol{N}(t), \frac{D}{dt}\boldsymbol{N}(t)\Big\rangle = 0,$$

and because $\langle \boldsymbol{T}(t), \boldsymbol{N}(t)\rangle = 0$ for each $t \in I$,

$$\left\langle \frac{D}{dt}\boldsymbol{T}(t), \boldsymbol{N}(t)\right\rangle + \left\langle \boldsymbol{T}(t), \frac{D}{dt}\boldsymbol{N}(t)\right\rangle = \kappa(t) + \left\langle \boldsymbol{T}(t), \frac{D}{dt}\boldsymbol{N}(t)\right\rangle = 0$$

by (3-2). Then,

$$\frac{D}{dt}\boldsymbol{N}(t) = -\kappa(t)\boldsymbol{T}(t) + \text{vector orthogonal to } \boldsymbol{T}(t) \text{ and } \boldsymbol{N}(t).$$

We define a torsion function τ by

$$\tau(t) = \left|\frac{D}{dt}\boldsymbol{N}(t) + \kappa(t)\boldsymbol{T}(t)\right|. \tag{3-3}$$

We will assume that either $\tau \neq 0$ for all t, or $\tau \equiv 0$. If $\tau(t) \neq 0$ for all t, then we set

$$\boldsymbol{B}(t) = \frac{1}{\tau(t)}\left(\frac{D}{dt}\boldsymbol{N}(t) + \kappa(t)\boldsymbol{T}(t)\right)$$

so that $\boldsymbol{B}$ is a unit vector field along $\boldsymbol{\gamma}$ which is orthogonal to $\boldsymbol{T}$ and $\boldsymbol{N}$ for all t. If $\tau \equiv 0$, then we choose $\boldsymbol{B}$ to be an autoparallel vector field along $\boldsymbol{\gamma}$ such that the vectors $\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$ and $\boldsymbol{B}(t)$ form an orthonormal basis of $T_{\boldsymbol{\gamma}(t)}\mathbb{S}^3$. In both cases we have

$$\frac{D}{dt}\boldsymbol{N}(t) = -\kappa(t)\boldsymbol{T}(t) + \tau(t)\boldsymbol{B}(t). \tag{3-4}$$

Finally, since $\langle \boldsymbol{B}(t), \boldsymbol{B}(t)\rangle = 1$ for each $t \in I$,

$$\left\langle \boldsymbol{B}(t), \frac{D}{dt}\boldsymbol{B}(t)\right\rangle = 0,$$

and because $\langle \boldsymbol{N}(t), \boldsymbol{B}(t)\rangle = 0$ for each $t \in I$,

$$\left\langle \frac{D}{dt}\boldsymbol{N}(t), \boldsymbol{B}(t)\right\rangle + \left\langle \boldsymbol{N}(t), \frac{D}{dt}\boldsymbol{B}(t)\right\rangle = \tau(t) + \left\langle \boldsymbol{N}(t), \frac{D}{dt}\boldsymbol{B}(t)\right\rangle$$

by (3-4). Then,

$$\frac{D}{dt}\boldsymbol{B}(t) = -\tau(t)\boldsymbol{N}(t) + \text{vector orthogonal to } \boldsymbol{N}(t) \text{ and } \boldsymbol{B}(t)$$
$$\implies \frac{D}{dt}\boldsymbol{B}(t) = -\tau(t)\boldsymbol{N}(t) + c\boldsymbol{T}(t)$$

since $\boldsymbol{T}(t)$ is orthogonal to $\boldsymbol{N}(t)$ and $\boldsymbol{B}(t)$ for each $t \in I$ by construction. By taking the dot product of both sides with $\boldsymbol{T}(t)$ we have

$$\left\langle \frac{D}{dt}\boldsymbol{B}(t), \boldsymbol{T}(t)\right\rangle = -\tau(t)\boldsymbol{N}(t) + c\boldsymbol{T}(t) = c$$

and by the product rule

$$\left\langle \frac{D}{dt}\boldsymbol{B}(t), \boldsymbol{T}(t)\right\rangle = \frac{d}{dt}\langle \boldsymbol{B}(t), \boldsymbol{T}(t)\rangle - \left\langle \boldsymbol{B}(t), \frac{D}{dt}\boldsymbol{T}(t)\right\rangle = 0 \quad \implies \quad c = 0.$$

Therefore we have our third and final Frenet–Serret equation

$$\frac{D}{dt}\boldsymbol{B}(t) = -\tau(t)\boldsymbol{N}(t). \tag{3-5}$$

Equations (3-2), (3-4) and (3-5) constitute the *Frenet–Serret equations* in a three-dimensional Riemannian manifold and characterize the local geometry of the curve $\boldsymbol{\gamma}$. This concludes the derivation of the Frenet–Serret formulas in the case where $\kappa(t) \neq 0$ for all t.

However, in the case where $\kappa \equiv 0$, we define $\tau \equiv 0$ and choose $\boldsymbol{N}$, $\boldsymbol{B}$ to be autoparallel vector fields along $\boldsymbol{\gamma}$ such that the vectors $\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$, and $\boldsymbol{B}(t)$ form an orthonormal basis of $T_{\boldsymbol{\gamma}(t)}\mathbb{S}^3$. Under this choice, the Frenet–Serret formulas given in (1-1) are again satisfied.

Note that we are not assuming that the manifold M is orientable. In the case where M is in fact oriented (i.e., M is orientable, and one of the two orientations is specified), there is a variant of the Frenet–Serret equations in which one assumes that the frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ is positively oriented. Indeed, one can then take $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$, the cross product defined in the tangent space by the Riemannian metric and the orientation. Then, one must allow the torsion τ to assume negative values. The equations (1-1) continue to hold with this new interpretation. However, in this paper, we use the nonoriented form of the Frenet–Serret equations, where κ and τ are always nonnegative. Geometrically, this means that while considering helices in $\mathbb{S}^3$, we disregard the chirality.

3B. ***The Frenet equations in $\mathbb{S}^3$.*** We begin by specializing the Frenet–Serret equations given in (1-1) to the case of the embedded sphere $\mathbb{S}^3$ in $\mathbb{R}^4$. Let

$$\iota : \mathbb{S}^3 \hookrightarrow \mathbb{R}^4$$

be the natural embedding. Given a curve $\boldsymbol{\gamma} : I \to \mathbb{S}^3$, where $I \subset \mathbb{R}$ is an open interval, we may think of $\boldsymbol{\gamma}$ as a curve in $\mathbb{R}^4$, by identifying $\boldsymbol{\gamma}$ with $\iota \circ \boldsymbol{\gamma}$. Similarly, given a vector field $\boldsymbol{V}$ along the curve $\boldsymbol{\gamma}$ which assigns to each point $t \in I$ a vector $\boldsymbol{V}(t) \in T_{\boldsymbol{\gamma}(t)}\mathbb{S}^3$, we can identify $\boldsymbol{V}$ with the vector field $\iota_*\boldsymbol{V}$ along $\iota \circ \boldsymbol{\gamma}$, which assigns to the point $t \in I$ the vector $\iota_*\boldsymbol{V}(t) \in T_{\iota\circ\boldsymbol{\gamma}(t)}\mathbb{R}^4$. In order to simplify notation, we adopt the following conventions:

(1) Consistently identifying $\mathbb{S}^3$ with the embedded image, we will omit the map ι and its pushforward ι_* from the notation. Thus, we will think of the $\boldsymbol{\gamma}$ in $\mathbb{S}^3$ as a curve in $\mathbb{R}^4$ whose image lies in $\mathbb{S}^3$. Similarly, we will think of a vector field $\boldsymbol{V}$ in $\mathbb{S}^3$ along $\boldsymbol{\gamma}$ as a vector field in $\mathbb{R}^4$ along $\boldsymbol{\gamma}$ such that, for each t, the vector $\boldsymbol{V}(t) \in T_{\boldsymbol{\gamma}(t)}\mathbb{R}^4$ lies in the subspace $T_{\boldsymbol{\gamma}(t)}\mathbb{S}^3$.

(2) We identify the tangent bundle $T\mathbb{R}^4$ with $\mathbb{R}^4 \times \mathbb{R}^4$. Therefore, all vector fields in $\mathbb{R}^4$ may be identified with $\mathbb{R}^4$-valued functions.

(3) Given a vector field $\boldsymbol{V}$ along a curve $\boldsymbol{\gamma}$ in $\mathbb{S}^3$, by the previous two parts, we can identify it with an $\mathbb{R}^4$-valued function. We will let $\boldsymbol{V}'$ denote its derivative in the Euclidean space $\mathbb{R}^4$; i.e., if $\boldsymbol{V}$ is represented using the natural coordinates as

$$\boldsymbol{V}(t) = (v_1(t), v_2(t), v_3(t), v_4(t)),$$

where $v_j : I \to \mathbb{R}^4$ is smooth, $j = 1, \dots, 4$, then

$$\boldsymbol{V}'(t) = (v_1'(t), v_2'(t), v_3'(t), v_4'(t)).$$

Of course, this is nothing but a coordinate expression for the covariant derivative of the vector field $\boldsymbol{V}$ along $\boldsymbol{\gamma}$ with respect to the flat Euclidean metric of $\mathbb{R}^4$.

Proposition 3.1. *Let $\boldsymbol{\gamma} : I \to \mathbb{S}^3$ be a smooth curve in the 3-sphere parametrized by arc length, and let $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$ be its Frenet frame. Using the notational convention explained above, we think of $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$ as functions from I to $\mathbb{R}^4$. Then, these vector-valued functions satisfy the differential equations*

$$\begin{aligned} \boldsymbol{T}'(t) + \boldsymbol{\gamma}(t) &= -\kappa(t)\boldsymbol{N}(t), \\ \boldsymbol{N}'(t) &= -\kappa(t)\boldsymbol{T}(t) + \tau(t)\boldsymbol{B}(t), \\ \boldsymbol{B}'(t) &= -\tau(t)\boldsymbol{N}(t). \end{aligned} \tag{3-6}$$

Remark. These formulas can be derived from a consideration of the Gauss equation satisfied by the second fundamental form of the embedding of $\mathbb{S}^3$ in $\mathbb{R}^4$; see [Spivak 1975b, p. 35]. We give a more elementary proof based on a direct computation of the covariant derivative.

Proof. We begin by recalling the following fact from differential geometry [Spivak 1975a, p. 2]. Let M be an embedded submanifold of $\mathbb{R}^N$, and for each point $x \in M$, let

$$\mathcal{P}_x : T_x\mathbb{R}^N \to T_xM$$

denote the orthogonal projection (where we identify T_xM in the natural way with a subspace of $T_x\mathbb{R}^N = \mathbb{R}^N$). We endow M with the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^N$. Let $\boldsymbol{\gamma} : I \to M$ be a smooth curve in M, where $I \subset \mathbb{R}$ is an open interval, and assume that $\boldsymbol{\gamma}$ is parametrized by arc length. If $\boldsymbol{V}$ is a vector field along $\boldsymbol{\gamma}$, it is well known that the covariant derivative of $\boldsymbol{V}$ is given by

$$\frac{D}{dt}\boldsymbol{V}(t) = \mathcal{P}_{\boldsymbol{\gamma}(t)}(\boldsymbol{V}'(t)) \in T_{\boldsymbol{\gamma}(t)}M.$$

When M is a hypersurface in $\mathbb{R}^N$ and $x \in M$, we may write, for $\boldsymbol{R} \in T_x\mathbb{R}^N \simeq \mathbb{R}^N$,

$$\mathcal{P}_x(\boldsymbol{R}) = \boldsymbol{R} - (\boldsymbol{N}(x) \cdot \boldsymbol{R})\boldsymbol{N}(x),$$

where $\boldsymbol{N}(x)$ denotes a unit vector normal to the hypersurface M at the point x. Consequently we obtain the following formula for differentiating a vector field $\boldsymbol{V}$

along the curve $\boldsymbol{\gamma}$:

$$\frac{D}{dt}\boldsymbol{V}(t) = \boldsymbol{V}'(t) - (\boldsymbol{N}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{V}'(t))\boldsymbol{N}(\boldsymbol{\gamma}(t)).$$

When $M = \mathbb{S}^3$ in $\mathbb{R}^4$, we may take for $\boldsymbol{X} \in \mathbb{S}^3$

$$\boldsymbol{N}(\boldsymbol{X}) = \boldsymbol{X},$$

so that

$$\frac{D}{dt}\boldsymbol{V}(t) = \boldsymbol{V}'(t) - (\boldsymbol{\gamma}(t) \cdot \boldsymbol{V}'(t))\boldsymbol{\gamma}(t). \tag{3-7}$$

We now compute $\boldsymbol{\gamma}(t) \cdot \boldsymbol{V}'(t)$ when $\boldsymbol{V}$ is one of the Frenet frame vector fields $\boldsymbol{T}$, $\boldsymbol{B}$, $\boldsymbol{N}$. Note that the four vectors $\boldsymbol{\gamma}(t)$, $\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$, and $\boldsymbol{B}(t)$ form an orthonormal set in $\mathbb{R}^4$. Observe that for each t we have the equations

$$\begin{aligned}
\boldsymbol{T}'(t) \cdot \boldsymbol{\gamma}(t) &= (\boldsymbol{T} \cdot \boldsymbol{\gamma})'(t) - \boldsymbol{T}(t) \cdot \boldsymbol{\gamma}'(t) = 0 - (\boldsymbol{T}(t) \cdot \boldsymbol{T}(t)) = -1,\\
\boldsymbol{N}'(t) \cdot \boldsymbol{\gamma}(t) &= (\boldsymbol{N} \cdot \boldsymbol{\gamma})'(t) - \boldsymbol{N}(t) \cdot \boldsymbol{\gamma}'(t) = 0 - (\boldsymbol{N}(t) \cdot \boldsymbol{T}(t)) = 0,\\
\boldsymbol{B}'(t) \cdot \boldsymbol{\gamma}(t) &= (\boldsymbol{B} \cdot \boldsymbol{\gamma})'(t) - \boldsymbol{B}(t) \cdot \boldsymbol{\gamma}'(t) = 0 - (\boldsymbol{B}(t) \cdot \boldsymbol{T}(t)) = 0.
\end{aligned}$$

In the first equation, we have used the fact that $\boldsymbol{T} = \boldsymbol{\gamma}'$. Using (3-7) and the above computations, we obtain the following representations of the covariant derivatives of the Frenet frame:

$$\frac{D}{dt}\boldsymbol{T}(t) = \boldsymbol{T}'(t) + \boldsymbol{\gamma}(t), \quad \frac{D}{dt}\boldsymbol{N}(t) = \boldsymbol{N}'(t), \quad \frac{D}{dt}\boldsymbol{B}(t) = \boldsymbol{B}'(t).$$

By combining these with the Frenet–Serret equations (1-1) in a Riemannian 3-manifold, (3-6) follows. ☐

4. Lissajous curves in $\mathbb{S}^3$

We now prove a few results that will be needed to complete the proof of Theorem 1. The following lemma will be required.

Lemma 4.1. *Let $\alpha_0, \alpha_1, \dots, \alpha_N$ be distinct nonnegative real numbers, and suppose that for each $t \geq 0$, we have*

$$\sum_{k=0}^{N} (a_k \sin(\alpha_k t) + b_k \cos(\alpha_k t)) = 0, \tag{4-1}$$

where the coefficients a_k, b_k are complex numbers. Then we have $a_k = b_k = 0$ for each k.

Proof. We can assume without loss of generality that $\alpha_0 = 0$ (simply take $a_0 = b_0 = 0$). For $k = 1, \dots, N$, let us set $\alpha_{-k} = -\alpha_k$. Then, (4-1) takes the form

$$\sum_{k=-N}^{N} c_k e^{i\alpha_k t} = 0, \tag{4-2}$$

where

$$c_k = \begin{cases} \frac{1}{2i}(a_{|k|} + ib_{|k|}), & k > 0, \\ a_0, & k = 0, \\ \frac{1}{2i}(-a_{|k|} + ib_{|k|}), & k < 0. \end{cases}$$

For each $k \geq 0$, it follows that $c_k = c_{-k} = 0$ if and only if $a_k = b_k = 0$.

Fix an integer ℓ, $|\ell| \leq N$, and multiply both sides of (4-2) by $e^{-i\alpha_\ell t}$. Integrating on the interval $[0, T]$ and dividing by T, we see that for each $T \geq 0$ we have

$$\sum_{\substack{k=-N \\ k \neq \ell}}^{N} \frac{c_k}{T} \int_0^T e^{i(\alpha_k - \alpha_\ell)t}\, dt + c_\ell = 0. \tag{4-3}$$

Note, however, that if $k \neq \ell$, we have

$$\left| \int_0^T e^{i(\alpha_k - \alpha_\ell)t}\, dt \right| \leq \left| \frac{e^{i(\alpha_k - \alpha_\ell)T} - 1}{i(\alpha_k - \alpha_\ell)} \right| \leq \frac{2}{|\alpha_k - \alpha_\ell|}.$$

Since for each k, this is bounded independently of T, as $T \to \infty$, each term in the first sum of (4-3) goes to 0, which shows that $c_\ell = 0$. Therefore, $a_\ell = b_\ell = 0$. Since ℓ is arbitrary, the lemma is proved. □

We will also need the following proposition.

Proposition 4.2. *Suppose that the Lissajous curve given by* (2-1) *lies in* $\mathbb{S}^3$:

(a) *If* $\omega_1 \neq 0$ *and* $\boldsymbol{\gamma}$ *has constant speed, then the coefficient vectors of* $\boldsymbol{\gamma}$ *satisfy the following relations*:

(1) $\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{A}_2, \boldsymbol{B}_2$ *are orthogonal.*
(2) $|\boldsymbol{A}_1| = |\boldsymbol{B}_1|$ *and* $|\boldsymbol{A}_2| = |\boldsymbol{B}_2|$.
(3) $|\boldsymbol{A}_1|^2 + |\boldsymbol{A}_2|^2 = 1$.

(b) *If* $\omega_1 = 0$, *then*:

(1) $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{B}_2$ *are orthogonal.*
(2) $|\boldsymbol{A}_2| = |\boldsymbol{B}_2|$.
(3) $|\boldsymbol{A}_1|^2 + |\boldsymbol{A}_2|^2 = 1$.

Proof. For use in the later portions of this proof, we will first compute $|\boldsymbol{\gamma}(t)|^2$. Using (2-1) and the fact that $\boldsymbol{\gamma}$ lies in $\mathbb{S}^3$, for each t we have

$$\begin{aligned}\boldsymbol{\gamma}(t)\cdot\boldsymbol{\gamma}(t) = 1 = {} & |\boldsymbol{A}_1|^2\cos^2(\omega_1 t) + |\boldsymbol{B}_1|^2\sin^2(\omega_1 t) + |\boldsymbol{A}_2|^2\cos^2(\omega_2 t) + |\boldsymbol{B}_2|^2\sin^2(\omega_2 t) \\ & + 2(\boldsymbol{A}_1\cdot\boldsymbol{B}_1)\cos(\omega_1 t)\sin(\omega_1 t) + 2(\boldsymbol{A}_1\cdot\boldsymbol{A}_2)\cos(\omega_1 t)\cos(\omega_2 t) \\ & + 2(\boldsymbol{A}_1\cdot\boldsymbol{B}_2)\cos(\omega_1 t)\sin(\omega_2 t) + 2(\boldsymbol{B}_1\cdot\boldsymbol{A}_2)\sin(\omega_1 t)\cos(\omega_2 t) \\ & + 2(\boldsymbol{B}_1\cdot\boldsymbol{B}_2)\sin(\omega_1 t)\sin(\omega_2 t) + 2(\boldsymbol{A}_2\cdot\boldsymbol{B}_2)\cos(\omega_2 t)\sin(\omega_2 t)\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}(|\boldsymbol{A}_1|^2+|\boldsymbol{B}_1|^2+|\boldsymbol{A}_2|^2+|\boldsymbol{B}_2|^2)\\
&\quad+\tfrac{1}{2}(|\boldsymbol{A}_1|^2-|\boldsymbol{B}_1|^2)\cos(2\omega_1 t)+(\boldsymbol{A}_1\cdot\boldsymbol{B}_1)\sin(2\omega_1 t)\\
&\quad+\tfrac{1}{2}(|\boldsymbol{A}_2|^2-|\boldsymbol{B}_2|^2)\cos(2\omega_2 t)+(\boldsymbol{A}_2\cdot\boldsymbol{B}_2)\sin(2\omega_2 t)\\
&\quad+(\boldsymbol{A}_1\cdot\boldsymbol{A}_2-\boldsymbol{B}_1\cdot\boldsymbol{B}_2)\cos((\omega_1+\omega_2)t)+(\boldsymbol{A}_1\cdot\boldsymbol{B}_2+\boldsymbol{B}_1\cdot\boldsymbol{A}_2)\sin((\omega_1+\omega_2)t)\\
&\quad+(\boldsymbol{B}_1\cdot\boldsymbol{B}_2+\boldsymbol{A}_1\cdot\boldsymbol{A}_2)\cos((\omega_2-\omega_1)t)+(\boldsymbol{A}_1\cdot\boldsymbol{B}_2-\boldsymbol{B}_1\cdot\boldsymbol{A}_2)\sin((\omega_2-\omega_1)t). \qquad (4\text{-}4)
\end{aligned}
$$

First, we prove part (a) of the proposition. Let us begin by assuming that $\omega_2 \neq 3\omega_1$. Since $\omega_2 \neq 3\omega_1$ and $\omega_1 \neq 0$, we see that the five numbers

$$0, \quad 2\omega_1, \quad 2\omega_2, \quad \omega_2+\omega_1, \quad \text{and} \quad \omega_2-\omega_1$$

are all distinct. By Lemma 4.1, each of the coefficients in the expression for $\boldsymbol{\gamma}(t)\cdot\boldsymbol{\gamma}(t)-1$ vanishes, as in the left-hand side of (4-1). Thus, from the coefficients of (4-4), we obtain

$$
\begin{gathered}
\tfrac{1}{2}(|\boldsymbol{A}_1|^2+|\boldsymbol{B}_1|^2+|\boldsymbol{A}_2|^2+|\boldsymbol{B}_2|^2)=1,\\
\tfrac{1}{2}(|\boldsymbol{A}_1|^2-|\boldsymbol{B}_1|^2)=0, \quad \boldsymbol{A}_1\cdot\boldsymbol{B}_1=0,\\
\tfrac{1}{2}(|\boldsymbol{A}_2|^2-|\boldsymbol{B}_2|^2)=0, \quad \boldsymbol{A}_2\cdot\boldsymbol{B}_2=0,\\
\boldsymbol{A}_1\cdot\boldsymbol{A}_2-\boldsymbol{B}_1\cdot\boldsymbol{B}_2=0, \quad \boldsymbol{A}_1\cdot\boldsymbol{B}_2+\boldsymbol{B}_1\cdot\boldsymbol{A}_2=0,\\
\boldsymbol{B}_1\cdot\boldsymbol{B}_2+\boldsymbol{A}_1\cdot\boldsymbol{A}_2=0, \quad \boldsymbol{A}_1\cdot\boldsymbol{B}_2-\boldsymbol{B}_1\cdot\boldsymbol{A}_2=0,
\end{gathered}
$$

which yields

$$|\boldsymbol{A}_1|^2+|\boldsymbol{B}_1|^2+|\boldsymbol{A}_2|^2+|\boldsymbol{B}_2|^2=2, \tag{4-5}$$

$$|\boldsymbol{A}_1|=|\boldsymbol{B}_1|, \tag{4-6}$$

$$|\boldsymbol{A}_2|=|\boldsymbol{B}_2|, \tag{4-7}$$

$$\boldsymbol{A}_1\cdot\boldsymbol{B}_1=\boldsymbol{A}_1\cdot\boldsymbol{A}_2=\boldsymbol{A}_1\cdot\boldsymbol{B}_2=\boldsymbol{B}_1\cdot\boldsymbol{A}_2=\boldsymbol{B}_1\cdot\boldsymbol{B}_2=\boldsymbol{A}_2\cdot\boldsymbol{B}_2=0. \tag{4-8}$$

Equation (4-8) shows that the vectors $\boldsymbol{A}_1$, $\boldsymbol{B}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{B}_2$ are orthogonal, which is conclusion (1) of the proposition. Moreover, (4-6) and (4-7) are precisely conclusion (2). Further, recognize that by using (4-5)–(4-7), we get

$$|\boldsymbol{A}_1|^2+|\boldsymbol{A}_2|^2=1, \tag{4-9}$$

which is conclusion (3).

To complete the proof of part (a) of the proposition, we now consider the case when $\omega_2=3\omega_1$. Let us set $\omega_1=\omega$, and thus $\omega_2=3\omega$. Therefore, we have

$$2\omega_1=2\omega, \quad 2\omega_2=6\omega, \quad \omega_2-\omega_1=2\omega, \quad \text{and} \quad \omega_2+\omega_1=4\omega.$$

Notice, $2\omega_1=\omega_2-\omega_1=2\omega$. Therefore, comparing coefficients in (4-4) and using Lemma 4.1 now gives

$$|\boldsymbol{A}_1|^2+|\boldsymbol{A}_2|^2+|\boldsymbol{B}_1|^2+|\boldsymbol{B}_2|^2=2, \tag{4-10}$$

$$|\boldsymbol{A}_1|^2-|\boldsymbol{B}_1|^2+2(\boldsymbol{A}_1\cdot\boldsymbol{A}_2+\boldsymbol{B}_1\cdot\boldsymbol{B}_2)=0, \tag{4-11}$$

$$|\boldsymbol{A}_2| = |\boldsymbol{B}_2|, \tag{4-12}$$
$$\boldsymbol{A}_1 \cdot \boldsymbol{B}_1 + \boldsymbol{A}_1 \cdot \boldsymbol{B}_2 - \boldsymbol{B}_1 \cdot \boldsymbol{A}_2 = 0, \tag{4-13}$$
$$\boldsymbol{A}_2 \cdot \boldsymbol{B}_2 = 0, \tag{4-14}$$
$$\boldsymbol{A}_1 \cdot \boldsymbol{A}_2 = \boldsymbol{B}_1 \cdot \boldsymbol{B}_2, \tag{4-15}$$
$$\boldsymbol{A}_1 \cdot \boldsymbol{B}_2 = -\boldsymbol{B}_1 \cdot \boldsymbol{A}_2. \tag{4-16}$$

Since $\boldsymbol{\gamma}$ has constant speed, there exists a $C > 0$ such that for all t, we have $|\boldsymbol{\gamma}'(t)| = C$. The relation $\boldsymbol{\gamma}'(t) \cdot \boldsymbol{\gamma}'(t) = C^2$ yields, using (4-4),

$$\begin{aligned}
0 = {} & \big(\tfrac{1}{2}\omega^2|\boldsymbol{B}_1|^2 + 9\omega^2|\boldsymbol{B}_2|^2 + \omega^2|\boldsymbol{A}_1|^2 + 9\omega^2|\boldsymbol{A}_2|^2\big) - C^2 \\
& + \big(\tfrac{1}{2}(\omega^2|\boldsymbol{B}_1|^2 - \omega^2|\boldsymbol{A}_1|^2) + 3\omega^2\boldsymbol{B}_1 \cdot \boldsymbol{B}_2 + 3\omega^2\boldsymbol{A}_1 \cdot \boldsymbol{A}_2\big)\cos(2\omega t) \\
& + 9\omega^2\,\tfrac{1}{2}(|\boldsymbol{B}_2|^2 - |\boldsymbol{A}_2|^2)\cos(6\omega t) \\
& - \big(\omega^2(\boldsymbol{A}_1 \cdot \boldsymbol{B}_1) + 3\omega^2(\boldsymbol{B}_1 \cdot \boldsymbol{A}_2) - 3\omega^2(\boldsymbol{B}_2 \cdot \boldsymbol{A}_1)\big)\sin(2\omega t) \\
& - 9\omega^2(\boldsymbol{A}_2 \cdot \boldsymbol{B}_2)\sin(6\omega t) + 3\omega^2(\boldsymbol{B}_1 \cdot \boldsymbol{B}_2 - \boldsymbol{A}_1 \cdot \boldsymbol{A}_2)\cos(4\omega t) \\
& - 3\omega^2(\boldsymbol{B}_1 \cdot \boldsymbol{A}_2 + \boldsymbol{A}_1 \cdot \boldsymbol{B}_2)\sin(4\omega t).
\end{aligned} \tag{4-17}$$

Using Lemma 4.1, this gives the relations

$$\omega^2(|\boldsymbol{B}_1|^2 + 9|\boldsymbol{B}_2|^2 + |\boldsymbol{A}_1|^2 + 9|\boldsymbol{A}_2|^2) = 2C^2, \tag{4-18}$$
$$|\boldsymbol{B}_1|^2 - |\boldsymbol{A}_1|^2 + 6(\boldsymbol{B}_1 \cdot \boldsymbol{B}_2 + \boldsymbol{A}_1 \cdot \boldsymbol{A}_2) = 0, \tag{4-19}$$
$$|\boldsymbol{B}_2| = |\boldsymbol{A}_2|, \tag{4-20}$$
$$\boldsymbol{A}_1 \cdot \boldsymbol{B}_1 + 3(\boldsymbol{B}_1 \cdot \boldsymbol{A}_2 - \boldsymbol{B}_2 \cdot \boldsymbol{A}_1) = 0, \tag{4-21}$$
$$\boldsymbol{A}_2 \cdot \boldsymbol{B}_2 = 0, \tag{4-22}$$
$$\boldsymbol{B}_1 \cdot \boldsymbol{B}_2 = \boldsymbol{A}_1 \cdot \boldsymbol{A}_2, \tag{4-23}$$
$$\boldsymbol{B}_1 \cdot \boldsymbol{A}_2 = -\boldsymbol{A}_1 \cdot \boldsymbol{B}_2. \tag{4-24}$$

Comparing (4-10)–(4-16) and (4-18)–(4-24), we see that we have obtained three new relations, which are (4-18), (4-19) and (4-21). Combining (4-11) with (4-19) we see that

$$|\boldsymbol{A}_1| = |\boldsymbol{B}_1| \quad \text{and} \quad \boldsymbol{A}_1 \cdot \boldsymbol{A}_2 = -\boldsymbol{B}_1 \cdot \boldsymbol{B}_2.$$

Similarly, (4-13) and (4-21) imply

$$\boldsymbol{A}_1 \cdot \boldsymbol{B}_1 = 0 \quad \text{and} \quad \boldsymbol{A}_1 \cdot \boldsymbol{B}_2 = \boldsymbol{B}_1 \cdot \boldsymbol{A}_2.$$

Combining these last two relations with (4-10)–(4-16), we get that the coefficient vectors $\boldsymbol{A}_1$, $\boldsymbol{B}_1$, $\boldsymbol{A}_2$, $\boldsymbol{B}_2$ are mutually orthogonal, $|\boldsymbol{A}_1| = |\boldsymbol{B}_1|$, and $|\boldsymbol{A}_2| = |\boldsymbol{B}_2|$. Conclusions (1), (2) and (3) follow again.

Now we will prove part (b) of the proposition. We set $\omega_1 = 0$ and $\omega_2 = \omega$ in (4-4). Then in (4-4) we have the three distinct frequencies

$$0 = 2\omega_1, \quad 2\omega_2 = 2\omega, \quad \omega_2 + \omega_1 = \omega_2 - \omega_1 = \omega.$$

Therefore, by Lemma 4.1 we have

$$2|\boldsymbol{A}_1|^2 + |\boldsymbol{A}_2|^2 + |\boldsymbol{B}_2|^2 = 2, \tag{4-25}$$

$$|\boldsymbol{A}_2| = |\boldsymbol{B}_2|, \tag{4-26}$$

$$\boldsymbol{A}_1 \cdot \boldsymbol{A}_2 = 0, \quad \boldsymbol{A}_1 \cdot \boldsymbol{B}_2 = 0 \quad \text{and} \quad \boldsymbol{A}_2 \cdot \boldsymbol{B}_2 = 0.$$

Equations (4-25) and (4-26) are precisely conclusions (1) and (2) of part (b) of the proposition. Additionally, combining (4-25) and (4-26) we have conclusion (3). □

In connection with the proof of part (a) of the above proposition, we note that if $\omega_2 = 3\omega_1$, one can construct Lissajous curves in $\mathbb{S}^3$ (of nonconstant speed) for which the coefficient vectors are not orthogonal.

5. Proof of Theorem 1

(1) Adjoining the relation $\boldsymbol{\gamma}'(t) = \boldsymbol{T}(t)$ to the Frenet–Serret equations (3-6) we obtain the system of equations

$$\begin{aligned}
\boldsymbol{\gamma}'(t) &= \phantom{-\boldsymbol{\gamma}(t)\ -\kappa\boldsymbol{T}(t)} \boldsymbol{T}(t), \\
\boldsymbol{T}'(t) &= -\boldsymbol{\gamma}(t) \phantom{-\kappa\boldsymbol{T}(t)} - \kappa\boldsymbol{N}(t), \\
\boldsymbol{N}'(t) &= \phantom{-\boldsymbol{\gamma}(t)} - \kappa\boldsymbol{T}(t) \phantom{-\kappa\boldsymbol{N}(t)} + \tau\boldsymbol{B}(t), \\
\boldsymbol{B}'(t) &= \phantom{-\boldsymbol{\gamma}(t)\ -\kappa\boldsymbol{T}(t)} - \tau\boldsymbol{N}(t),
\end{aligned} \tag{5-1}$$

where κ and τ are the given constants. We now rewrite these equations in matrix form. Note that the four vectors $\boldsymbol{\gamma}(t)$, $\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$, $\boldsymbol{B}(t)$ form an orthonormal basis of $\mathbb{R}^4$. Let $\boldsymbol{X}(t)$ denote the 4×4 matrix whose rows are these four vectors. Then for each t, the matrix $\boldsymbol{X}(t)$ is orthogonal. When the curvature κ and the torsion τ are constants, the augmented Frenet–Serret equations given by the set of equations in (5-1) in the sphere equation may be written in matrix form as

$$\boldsymbol{X}'(t) = \boldsymbol{C} \cdot \boldsymbol{X}(t), \tag{5-2}$$

where $\boldsymbol{C}$ denotes the skew-symmetric matrix

$$\boldsymbol{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \kappa & 0 \\ 0 & -\kappa & 0 & \tau \\ 0 & 0 & -\tau & 0 \end{bmatrix}. \tag{5-3}$$

From the theory of ordinary differential equations we know that the solution to the constant coefficient system presented in (5-2) exists for all t and is given by

$$\boldsymbol{X}(t) = e^{t\boldsymbol{C}} \cdot \boldsymbol{X}(0). \tag{5-4}$$

This proves part (1) of the theorem.

(2) In order to calculate the matrix exponential $e^{t\boldsymbol{C}}$, we first recognize that since $\boldsymbol{C}$ is skew-symmetric it can be diagonalized and thus written as

$$\boldsymbol{C} = \boldsymbol{P}(i\boldsymbol{D})\boldsymbol{P}^{-1},$$

where $\boldsymbol{D}$ is a diagonal matrix with real entries of the form

$$\boldsymbol{D} = \operatorname{diag}(\omega_1, -\omega_1, \omega_2, -\omega_2), \tag{5-5}$$

where $\omega_1, \omega_2 \geq 0$. This is because the eigenvalues of the real skew-symmetric matrix $\boldsymbol{C}$ are purely imaginary and occur in complex conjugate pairs. Therefore,

$$e^{t\boldsymbol{C}} = \boldsymbol{P}e^{it\boldsymbol{D}}\boldsymbol{P}^{-1}. \tag{5-6}$$

From (5-5) it follows that

$$e^{it\boldsymbol{D}} = \operatorname{diag}(e^{i\omega_1 t}, e^{-i\omega_1 t}, e^{i\omega_2 t}, e^{-i\omega_2 t}).$$

Since $\boldsymbol{\gamma}(t)$ is the first row of the matrix $\boldsymbol{X}(t) = \boldsymbol{P}e^{it\boldsymbol{D}}\boldsymbol{P}^{-1} \cdot \boldsymbol{X}(0)$ it follows that

$$\boldsymbol{\gamma}(t) = \cos(\omega_1 t)\boldsymbol{A}_1 + \sin(\omega_1 t)\boldsymbol{B}_1 + \cos(\omega_2 t)\boldsymbol{A}_2 + \sin(\omega_2 t)\boldsymbol{B}_2,$$

where the coefficient vectors $\boldsymbol{A}_1$, $\boldsymbol{B}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{B}_2$ are constant vectors in $\mathbb{R}^4$. This proves part (2) of the theorem.

(3) The diagonal entries of $i\boldsymbol{D}$, where $\boldsymbol{D}$ is as in (5-5), are the eigenvalues of the matrix $\boldsymbol{C}$ of (5-3). We find them by solving the characteristic equation

$$\det(\boldsymbol{C} - x\boldsymbol{I}) = x^4 + (\kappa^2 + \tau^2 + 1)x^2 + \tau^2 = x^4 + \chi^2 x^2 + \tau^2 = 0, \tag{5-7}$$

with χ as in (2-4). The solutions of the characteristic equation are

$$x = \pm i\omega_1 \quad \text{or} \quad x = \pm i\omega_2,$$

where ω_1, ω_2 are as in (2-2) and (2-3). This proves part (3) of the theorem.

(4) Since $\kappa > 0$, we have

$$\begin{aligned} \chi^4 - 4\tau^2 &= (\kappa^2 + \tau^2 + 1)^2 - 4\tau^2 \\ &= \kappa^4 + (\tau^2 - 1)^2 + 2\kappa^2\tau^2 + 2\kappa^2 \\ &> (\tau^2 - 1)^2. \end{aligned} \tag{5-8}$$

Therefore, from (2-2) and (2-3) we see that $\omega_2 > \omega_1$. By the definition of χ in (2-4) we have,

$$\chi^2 = \kappa^2 + \tau^2 + 1 > 1 + \tau^2. \tag{5-9}$$

Combining (5-8) and (5-9) we have

$$\begin{aligned} \chi^2 + \sqrt{\chi^4 - 4\tau^2} &> (1+\tau^2) + \sqrt{(\tau^2-1)^2} && \text{(5-10)} \\ &= 1 + \tau^2 + |\tau^2 - 1| \\ &= \begin{cases} 2\tau^2, & \tau \geq 1, \\ 2, & \tau < 1. \end{cases} && \text{(5-11)} \end{aligned}$$

Therefore, $\chi^2 + \sqrt{\chi^4 - 4\tau^2} > 2$, and we have

$$\omega_2^2 = \tfrac{1}{2}(\chi^2 + \sqrt{\chi^4 - 4\tau^2}) > 1. \tag{5-12}$$

Then, by making use of (5-11),

$$\omega_1^2 = \frac{\chi^2 - \sqrt{\chi^4 - 4\tau^2}}{2} = \frac{2\tau^2}{\chi^2 + \sqrt{\chi^2 - 4\tau^2}} < \begin{cases} 1, & \tau \geq 1, \\ \tau^2, & \tau < 1. \end{cases} \tag{5-13}$$

Thus,

$$\omega_1 < 1. \tag{5-14}$$

This proves part (4) of the theorem.

(5) Note that $|\boldsymbol{\gamma}(t)| = 1$ for all t and $|\boldsymbol{\gamma}'(t)| = 1$ for all t as well, since $\boldsymbol{\gamma}$ lies in $\mathbb{S}^3$ and has unit speed. Since $\tau \neq 0$ by (2-2), we know that $\omega_1 > 0$. Therefore, by part (a) of Proposition 4.2:

(1) The coefficient vectors $\boldsymbol{A}_1$, $\boldsymbol{B}_1$, $\boldsymbol{A}_2$, $\boldsymbol{B}_2$ are mutually orthogonal, which is one of the conclusions of part (5) of Theorem 1.

(2) $|\boldsymbol{A}_1| = |\boldsymbol{B}_1|$ and $|\boldsymbol{A}_2| = |\boldsymbol{B}_2|$, which is part of (2-6)–(2-7). We will however need to work further to complete the proof of part (5).

(3) We have
$$|\boldsymbol{A}_1|^2 + |\boldsymbol{A}_2|^2 = 1. \tag{5-15}$$

Now, let $\boldsymbol{\alpha}(t) = \boldsymbol{\gamma}'(t)$. Then, $|\boldsymbol{\alpha}(t)| = 1$ (since $\boldsymbol{\gamma}(t)$ is parametrized by arc length) and differentiating (2-1), we see that $\boldsymbol{\alpha}(t)$ may be represented as

$$\begin{aligned} \boldsymbol{\alpha}(t) &= \omega_1 \cos(\omega_1 t)\boldsymbol{B}_1 - \omega_1 \sin(\omega_1 t)\boldsymbol{A}_1 + \omega_2 \cos(\omega_2 t)\boldsymbol{B}_2 - \omega_2 \sin(\omega_2 t)\boldsymbol{A}_2 \\ &= \cos(\omega_1 t)\boldsymbol{P}_1 + \sin(\omega_1 t)\boldsymbol{Q}_1 + \cos(\omega_2 t)\boldsymbol{P}_2 + \sin(\omega_2 t)\boldsymbol{Q}_2, \end{aligned}$$

where $\boldsymbol{P}_1 = \omega_1 \boldsymbol{B}_1$, $\boldsymbol{Q}_1 = -\omega_1 \boldsymbol{A}_1$, $\boldsymbol{P}_2 = \omega_2 \boldsymbol{B}_2$, and $\boldsymbol{Q}_2 = -\omega_2 \boldsymbol{A}_2$. This shows that $\boldsymbol{\alpha}$ is a Lissajous curve in $\mathbb{S}^3$.

Now, we claim that $|\boldsymbol{\alpha}'(t)|$ is constant independently of t. Recall, that

$$\boldsymbol{\alpha}'(t) = \boldsymbol{\gamma}''(t) = \boldsymbol{T}'(t),$$

where $\boldsymbol{T}$ is the tangent vector field in the Frenet frame $(\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B})$. Therefore, by the first equation in (3-6), we have

$$\boldsymbol{\gamma}''(t) = -\kappa \boldsymbol{N}(t) - \boldsymbol{\gamma}(t),$$

which yields,

$$\begin{aligned} \boldsymbol{\alpha}'(t) \cdot \boldsymbol{\alpha}'(t) &= \kappa^2 + 1 + 2\kappa(\boldsymbol{N}(t) \cdot \boldsymbol{\gamma}(t)) \\ &= \kappa^2 + 1, \end{aligned} \tag{5-16}$$

where $\boldsymbol{N}(t) \cdot \boldsymbol{\gamma}(t) = 0$ since $\boldsymbol{N}(t) \in T_{\boldsymbol{\gamma}(t)}\mathbb{S}^3$. We may now apply conclusion (3) of part (a) of Proposition 4.2 to obtain that $|\boldsymbol{P}_1|^2 + |\boldsymbol{P}_2|^2 = 1$, which is equivalent to

$$\omega_1^2|\boldsymbol{A}_1|^2 + \omega_2^2|\boldsymbol{A}_2|^2 = 1. \tag{5-17}$$

Combining (5-15) and (5-17), we get

$$\omega_1^2|\boldsymbol{A}_1|^2 + \omega_2^2(1 - |\boldsymbol{A}_1|^2) = 1$$

and

$$\omega_1^2(1 - |\boldsymbol{A}_2|^2) + \omega_2^2|\boldsymbol{A}_2|^2 = 1.$$

Solving these equations for $|\boldsymbol{A}_1|^2$ and $|\boldsymbol{A}_2|^2$, we obtain (2-6) and (2-7).

(6) If $\tau = 0$, by (2-2), we have $\omega_1 = 0$. We set $\omega_2 = \omega = \sqrt{\kappa^2 + 1}$ by (2-3) and then by parts (1)–(3) of this theorem, proved above, $\boldsymbol{\gamma}$ is given by

$$\boldsymbol{\gamma}(t) = \boldsymbol{A}_1 + \cos(\omega t)\boldsymbol{A}_2 + \sin(\omega t)\boldsymbol{B}_2.$$

Furthermore, by part (b) of Proposition 4.2 we know that $|\boldsymbol{A}_2| = |\boldsymbol{B}_2|$ and that the coefficient vectors $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{B}_2$ are mutually orthogonal. Note that

$$\boldsymbol{\gamma}'(t) = -\omega\sin(\omega t)\boldsymbol{A}_2 + \omega\cos(\omega t)\boldsymbol{B}_2.$$

Since $\boldsymbol{\gamma}'(t)$ has unit speed, we have

$$\boldsymbol{\gamma}'(t) \cdot \boldsymbol{\gamma}'(t) = 1 = \omega^2|\boldsymbol{A}_2|^2\sin^2(\omega t) + \omega^2|\boldsymbol{B}_2|^2\cos^2(\omega t) = \omega^2|\boldsymbol{A}_2|^2,$$

which implies,

$$|\boldsymbol{A}_2| = \frac{1}{\omega}. \tag{5-18}$$

Combining (5-18) with conclusion (3) of part (b) of Proposition 4.2, we get

$$|\boldsymbol{A}_1| = \sqrt{1 - \frac{1}{\omega^2}}. \tag{5-19}$$

6. Proof of Theorem 2

(1) Suppose that we have two helices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $\mathbb{S}^3$ with the same curvature $\kappa \geq 0$ and torsion $\tau \geq 0$. Then, by part (3) of Theorem 1, the fundamental angular frequencies ω_1 and ω_2 of these two curves are the same. Thus, the curves are represented as

$$\boldsymbol{\alpha}(t) = \cos(\omega_1 t)\boldsymbol{A}_1 + \sin(\omega_1 t)\boldsymbol{B}_1 + \cos(\omega_2 t)\boldsymbol{A}_2 + \sin(\omega_2 t)\boldsymbol{B}_2,$$
$$\boldsymbol{\beta}(t) = \cos(\omega_1 t)\boldsymbol{C}_1 + \sin(\omega_1 t)\boldsymbol{D}_1 + \cos(\omega_2 t)\boldsymbol{C}_2 + \sin(\omega_2 t)\boldsymbol{D}_2.$$

If $\tau \neq 0$, by part (5) of Theorem 1, we also know that

$$|\boldsymbol{A}_1| = |\boldsymbol{B}_1| = |\boldsymbol{C}_1| = |\boldsymbol{D}_1| = \sqrt{\frac{1-\omega_2^2}{\omega_1^2-\omega_2^2}},$$
$$[2pt]|\boldsymbol{A}_2| = |\boldsymbol{B}_2| = |\boldsymbol{C}_2| = |\boldsymbol{D}_2| = \sqrt{\frac{1-\omega_1^2}{\omega_2^2-\omega_1^2}},$$

and that the sets of vectors $\{\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{A}_2, \boldsymbol{B}_2\}$ and $\{\boldsymbol{C}_1, \boldsymbol{D}_1, \boldsymbol{C}_2, \boldsymbol{D}_2\}$ are both mutually orthogonal. Therefore, there exists an orthogonal map, $G : \mathbb{R}^4 \to \mathbb{R}^4$ in $O(4)$ such that $G(\boldsymbol{A}_1) = \boldsymbol{C}_1,\ G(\boldsymbol{B}_1) = \boldsymbol{D}_1,\ G(\boldsymbol{A}_2) = \boldsymbol{C}_2$, and $G(\boldsymbol{B}_2) = \boldsymbol{D}_2$. Then $f = G|_{\mathbb{S}^3}$ is an isometry of $\mathbb{S}^3$ and it is clear that $\boldsymbol{\beta} = f \circ \boldsymbol{\alpha}$.

If $\tau = 0$, then by part (6) of Theorem 1, we know that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ take the form

$$\boldsymbol{\alpha}(t) = \boldsymbol{A}_1 + \cos(\omega t)\boldsymbol{A}_2 + \sin(\omega t)\boldsymbol{B}_2,$$
$$\boldsymbol{\beta}(t) = \boldsymbol{C}_1 + \cos(\omega t)\boldsymbol{C}_2 + \sin(\omega t)\boldsymbol{D}_2,$$

where $\omega_1 = 0$ and $\omega = \omega_2 = \sqrt{\kappa^2+1}$ by (2-2) and (2-3). Furthermore, by part (6) of Theorem 1, we know that

$$|\boldsymbol{A}_2| = |\boldsymbol{B}_2| = |\boldsymbol{C}_2| = |\boldsymbol{D}_2| = \frac{1}{\omega}, \quad |\boldsymbol{A}_1| = |\boldsymbol{C}_1| = \sqrt{1-\frac{1}{\omega^2}},$$

and that the sets of vectors $\{\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{B}_2\}$ and $\{\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{D}_2\}$ are both mutually orthogonal. Therefore, there again exists an orthogonal map $G : \mathbb{R}^4 \to \mathbb{R}^4$ in $O(4)$ such that $G(\boldsymbol{A}_1) = \boldsymbol{C}_1,\ G(\boldsymbol{A}_2) = \boldsymbol{C}_2$, and $G(\boldsymbol{B}_2) = \boldsymbol{D}_2$. Then $f = G|_{\mathbb{S}^3}$ is again an isometry of $\mathbb{S}^3$ and it is clear that $\boldsymbol{\beta} = f \circ \boldsymbol{\alpha}$.

(2) Let $\boldsymbol{\gamma}$ be a helix in $\mathbb{S}^3$ which can be written, thanks to Theorem 1, in the form of (2-1):

$$\boldsymbol{\gamma}(t) = \cos(\omega_1 t)\boldsymbol{A}_1 + \sin(\omega_1 t)\boldsymbol{B}_1 + \cos(\omega_2 t)\boldsymbol{A}_2 + \sin(\omega_2 t)\boldsymbol{B}_2.$$

Now suppose that $\boldsymbol{\gamma}$ is periodic with period T. Then, for each $t \in \mathbb{R}$, we have

$$\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(t+T).$$

First, let us assume that $\tau \neq 0$ and consequently, because of (2-2), $\omega_1 \neq 0$. Then, comparing the coefficients of $\boldsymbol{A}_1$ and $\boldsymbol{B}_1$, we obtain,

$$\cos(\omega_1(t+T)) = \cos(\omega_1 t) \quad \text{and} \quad \sin(\omega_1(t+T)) = \sin(\omega_1 t) \tag{6-1}$$

for each $t \in \mathbb{R}$. This shows that there exists a nonzero $m \in \mathbb{Z}$ such that

$$T = \frac{2\pi m}{\omega_1}.$$

Similarly, we compare the coefficients of $\boldsymbol{A}_2$ and $\boldsymbol{B}_2$, to get

$$\cos(\omega_2(t+T)) = \cos(\omega_2 t) \quad \text{and} \quad \sin(\omega_2(t+T)) = \sin(\omega_2 t) \tag{6-2}$$

for each $t \in \mathbb{R}$. This shows that there exists a nonzero $n \in \mathbb{Z}$ such that

$$T = \frac{2\pi n}{\omega_2}.$$

It follows that

$$\frac{\omega_1}{\omega_2} = \frac{m}{n} \in \mathbb{Q}.$$

Now we prove the converse. Suppose that $\omega_1/\omega_2 = m/n \in \mathbb{Q}$. Then, let

$$T = \frac{2\pi m}{\omega_1} = \frac{2\pi n}{\omega_2}.$$

Thus, (6-1) and (6-2) hold. Therefore,

$$\boldsymbol{\gamma}(t+T) = \boldsymbol{\gamma}(t),$$

which proves part (2) of Theorem 2 in the case where $\tau \neq 0$. If, on the other hand, $\tau = 0$ and consequently $\omega_1 = 0$, then $\boldsymbol{\gamma}$ is given by

$$\boldsymbol{\gamma}(t) = \boldsymbol{A}_1 + \cos(\omega_2 t)\boldsymbol{A}_2 + \sin(\omega_2 t)\boldsymbol{B}_2,$$

which is always periodic with a period of

$$T = \frac{2\pi}{\omega_2}.$$

This proves part (2) of the theorem.

(3) Let κ, $\tau > 0$ and let $\boldsymbol{A}_1$, $\boldsymbol{B}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{B}_2$ be the orthonormal basis of $\mathbb{R}^4$ consisting of the unit vectors along the coefficient vectors $\boldsymbol{A}_1$, $\boldsymbol{B}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{B}_2$ of $\boldsymbol{\gamma}$ as given in (2-1). We denote the coordinates of a point $\boldsymbol{X} \in \mathbb{R}^4$ by

$$\boldsymbol{X} = x_1\boldsymbol{A}_1 + x_2\boldsymbol{B}_1 + x_3\boldsymbol{A}_2 + x_4\boldsymbol{B}_2.$$

By parts (2) and (5) of Theorem 1, we have that $\boldsymbol{\gamma}$ is represented in these coordinates by $x_1 = |\boldsymbol{A}_1|\cos(\omega_1 t)$, $x_2 = |\boldsymbol{A}_1|\sin(\omega_1 t)$, $x_3 = |\boldsymbol{A}_2|\cos(\omega_2 t)$, and $x_4 = |\boldsymbol{A}_2|\sin(\omega_2 t)$. Consider the torus in $\mathbb{R}^4$ given by

$$\mathbb{T}^2_{\boldsymbol{\gamma}} = \{\boldsymbol{X} \in \mathbb{R}^4 : x_1^2 + x_2^2 = |\boldsymbol{A}_1|^2,\ x_3^2 + x_4^2 = |\boldsymbol{A}_2|^2\}. \tag{6-3}$$

Clearly $\boldsymbol{\gamma}$ lies on $\mathbb{T}^2_{\boldsymbol{\gamma}}$. It is clear that $\mathbb{T}^2_{\boldsymbol{\gamma}}$ is a flat Clifford torus in $\mathbb{R}^4$, and is contained in $\mathbb{S}^3$, since if $\boldsymbol{X} = (x_1, x_2, x_3, x_4) \in \mathbb{T}^2_{\boldsymbol{\gamma}}$, then

$$|\boldsymbol{X}|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = |\boldsymbol{A}_1|^2 + |\boldsymbol{A}_2|^2 = \frac{1-\omega_2^2}{\omega_1^2-\omega_2^2} + \frac{1-\omega_1^2}{\omega_2^2-\omega_1^2} = 1,$$

where the last equality follows by (2-6) and (2-7).

(4) Note that the helix $\boldsymbol{\gamma}$ is a solution of the differential equations on the torus $\mathbb{T}^2_{\boldsymbol{\gamma}}$

$$\frac{d\theta_1}{dt} = \omega_1 \quad \text{and} \quad \frac{d\theta_2}{dt} = \omega_2,$$

where θ_1, θ_2 are angular coordinates on the circles $x_1^2 + x_2^2 = |\boldsymbol{A}_1|^2$ and $x_3^2 + x_4^2 = |\boldsymbol{A}_2|^2$ respectively. If $\omega_1/\omega_2 \notin \mathbb{Q}$, then a classical result in the theory of dynamical systems [Hasselblatt and Katok 2003, Proposition 4.2.8, p. 113] shows that the image of $\boldsymbol{\gamma}$ is dense in $\mathbb{T}^2_{\boldsymbol{\gamma}}$. Therefore, part (4) of the theorem is proven.

Acknowledgements

The authors gratefully acknowledge the comments and suggestions of the referee, which led to many improvements in the paper.

References

[Arroyo et al. 2004] J. Arroyo, M. Barros, and O. J. Garay, "Models of relativistic particle with curvature and torsion revisited", *Gen. Relativity Gravitation* **36**:6 (2004), 1441–1451. MR Zbl

[Barros 1997] M. Barros, "General helices and a theorem of Lancret", *Proc. Amer. Math. Soc.* **125**:5 (1997), 1503–1509. MR Zbl

[Hasselblatt and Katok 2003] B. Hasselblatt and A. Katok, *A first course in dynamics*, Cambridge Univ. Press, 2003. MR Zbl

[Pinkall 1985] U. Pinkall, "Hopf tori in S^3", *Invent. Math.* **81**:2 (1985), 379–386. MR Zbl

[Spivak 1975a] M. Spivak, *A comprehensive introduction to differential geometry, III*, Publish or Perish, Boston, 1975. MR Zbl

[Spivak 1975b] M. Spivak, *A comprehensive introduction to differential geometry, IV*, Publish or Perish, Boston, 1975. MR Zbl

[Stoker 1969] J. J. Stoker, *Differential geometry*, Pure and Applied Math. **20**, Wiley, New York, 1969. MR Zbl

Received: 2017-06-23 Revised: 2017-10-13 Accepted: 2018-04-22

chakr2d@cmich.edu *Department of Mathematics, Central Michigan University, Mt. Pleasant, MI, United States*

rsahay@berkeley.edu *University of California, Berkeley, CA, United States*

jrwv8w@mail.missouri.edu *Department of Physics and Astronomy, University of Missouri, Columbia, MO, United States*

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2019 is US $195/year for the electronic version, and $260/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2019 vol. 12 no. 2

1944-4176(2019)12:2;1-3