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Martin Bright and Ivo Kok

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Failure of strong approximation on an affine cone

Martin Bright and Ivo Kok

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We use the Brauer–Manin obstruction to strong approximation on a punctured affine cone to explain why some mod p solutions to a homogeneous Diophantine equation of degree 2 cannot be lifted to coprime integer solutions.

1. Introduction

Let $Y \subset \mathbb{P}^3_{\mathbb{Q}}$ be the quadric surface defined by the equation

$$X_0^2 + 47X_1^2 = 103X_2^2 + (17 \times 47 \times 103)X_3^2. \tag{1}$$

One can easily check that Y is everywhere locally soluble, and so has rational points. Being a quadric surface, Y satisfies weak approximation. In particular, if we fix a prime p, then any smooth point on the reduction of Y at p lifts to a rational point of Y. Given that a point on the reduction of Y is given by $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathbb{F}_p^4$ satisfying (1), and a point of $Y(\mathbb{Q})$ can be given by coprime integers $(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4$ satisfying (1), one might be tempted to think that every $\mathbb{F}_p$-solution $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ can be lifted to a coprime integer solution (x_0, x_1, x_2, x_3).

However, at the end of [Bright 2011], it was remarked that Y has the following interesting feature: if $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is a solution to (1) over $\mathbb{F}_{17}$, then at most half of the nonzero scalar multiples of $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathbb{F}_{17}^4$ can be lifted to coprime 4-tuples $(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4$ defining a point of Y. That observation was a by-product of the calculation of the Brauer–Manin obstruction to rational points on a diagonal quartic surface related to Y. In this note we will interpret the observation as a failure of strong approximation on the punctured affine cone over Y, and will show that this failure is itself due to a Brauer–Manin obstruction.

The same phenomenon has been observed by Lindqvist [2017] in the case of the quadric surface $X_0^2 - pqX_1^2 - X_2X_3$ for p, q odd primes congruent to 1 modulo 8. We expect that example also to be explained by a Brauer–Manin obstruction.

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Following [Colliot-Thélène and Xu 2013], for a variety X over $\mathbb{Q}$, we define $X(\boldsymbol{A}_{\mathbb{Q}})$ to be the set of adelic points of X, that is, the restricted product of $X(\mathbb{Q}_v)$ for all places v, with respect to the subsets $X(\mathbb{Z}_v)$. (One needs to choose a model of X to make sense of the notation $X(\mathbb{Z}_v)$, but since any two models agree outside a finite set of primes, the resulting definition of $X(\boldsymbol{A}_{\mathbb{Q}})$ does not depend on the choice of model.) Similarly, define $X(\boldsymbol{A}_{\mathbb{Q}}^{\infty})$ to be the set of adelic points of X away from ∞, that is, the restricted product of $X(\mathbb{Q}_v)$ for $v \neq \infty$ with respect to the subsets $X(\mathbb{Z}_v)$. Assuming that X has points over every completion of $\mathbb{Q}$, we say that X satisfies *strong approximation away from* ∞ if the image of the diagonal map $X(\mathbb{Q}) \to X(\boldsymbol{A}_{\mathbb{Q}}^{\infty})$ is dense.

If a variety X does not satisfy strong approximation, this can sometimes be explained by a *Brauer–Manin obstruction*. Define

$$X(\boldsymbol{A}_{\mathbb{Q}})^{\mathrm{Br}} = \left\{(P_v) \in X(\boldsymbol{A}_{\mathbb{Q}}) \;\middle|\; \textstyle\sum_v \operatorname{inv}_v A(P_v) = 0 \text{ for all } A \in \operatorname{Br} X\right\},$$

and define $X(\boldsymbol{A}_{\mathbb{Q}}^{\infty})^{\mathrm{Br}}$ to be the image of $X(\boldsymbol{A}_{\mathbb{Q}})^{\mathrm{Br}}$ under the natural projection map $X(\boldsymbol{A}_{\mathbb{Q}}) \to X(\boldsymbol{A}_{\mathbb{Q}}^{\infty})$. Then $X(\boldsymbol{A}_{\mathbb{Q}}^{\infty})^{\mathrm{Br}}$ is a closed subset of $X(\boldsymbol{A}_{\mathbb{Q}}^{\infty})$ that contains the image of $X(\mathbb{Q})$. If $X(\boldsymbol{A}_{\mathbb{Q}}^{\infty})^{\mathrm{Br}} \neq X(\boldsymbol{A}_{\mathbb{Q}}^{\infty})$, we say that there is a Brauer–Manin obstruction to strong approximation away from ∞ on X.

We now return to the variety Y defined above. Let $X \subset \mathbb{A}^4_{\mathbb{Q}}$ be the punctured affine cone over Y; that is, X is the complement of the point $O = (0, 0, 0, 0)$ in the affine variety defined by (1). There is a natural morphism $\pi : X \to Y$ given by restricting the usual quotient map $\mathbb{A}^4 \setminus \{O\} \to \mathbb{P}^3$, so that X is realised as a $\boldsymbol{G}_{\mathrm{m}}$-torsor over Y. To talk about integral points, we must choose a model: Let $\mathcal{X} \subset \mathbb{A}^4_{\mathbb{Z}}$ be the complement of the section $(0, 0, 0, 0)$ in the scheme defined by (1) over $\mathbb{Z}$. If we let $f \in \mathbb{Z}[X_0, X_1, X_2, X_3]$ be the polynomial defining Y, then the integral points of $\mathcal{X}$ are given by

$$\mathcal{X}(\mathbb{Z}) = \{(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \mid x_0, x_1, x_2, x_3 \text{ coprime}, f(x_0, x_1, x_2, x_3) = 0\}.$$

Theorem 1.1. *The group* $\operatorname{Br} X/\operatorname{Br}\mathbb{Q}$ *has order* 2; *a generator is given by the quaternion algebra* $(17, g)$, *where* $g \in \mathbb{Z}[X_0, X_1, X_2, X_3]$ *is a homogeneous linear form defining the tangent hyperplane to* X *at a rational point* $P \in X(\mathbb{Q})$. *There is a Brauer–Manin obstruction to strong approximation on* X *away from* ∞. *Specifically, for any smooth point* $\widetilde{Q} \in \mathcal{X}(\mathbb{F}_{17})$, *at most half of the scalar multiples of* $\widetilde{Q}$ *lift to integer points of* $\mathcal{X}$.

It is interesting to compare this result with the "easy fibration method" of [Colliot-Thélène and Xu 2013, Proposition 3.1]. We have a fibration $\pi : X \to Y$, and the base Y satisfies strong approximation. However, the fibres are isomorphic to $\boldsymbol{G}_{\mathrm{m}}$, which drastically fails to satisfy strong approximation, so we cannot use that method to conclude anything about strong approximation on X.

2. Quadric surfaces

We now gather some basic facts about quadric surfaces. Any nonsingular quadric surface $Y \subset \mathbb{P}^3$ over a field k of characteristic different from 2 may be defined by an equation of the form $\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} = 0$, where $\boldsymbol{M}$ is an invertible 4×4 matrix with entries in k. We define $\Delta_Y \in k^\times/(k^\times)^2$ to be the class of the determinant of $\boldsymbol{M}$, which is easily seen to be invariant under linear changes of coordinates. If $\bar{k}$ is an algebraic closure of k and $\bar{Y}$ is the base change of Y to $\bar{k}$, then $\operatorname{Pic} \bar{Y}$ is isomorphic to $\mathbb{Z}^2$, generated by the classes of the two families of lines on $\bar{Y}$ [Hartshorne 1977, Example II.6.6.1].

Lemma 2.1. *Let k be a field of characteristic not equal to 2, and let $Y \subset \mathbb{P}^3_k$ be a nonsingular quadric surface. Then the two families of lines on $\bar{Y}$ are defined over the field $k(\sqrt{\Delta_Y})$, and are conjugate to each other.*

Proof. We may assume that the matrix $\boldsymbol{M}$ defining Y is diagonal, with entries p, q, r, s. Following [Eisenbud and Harris 2000, Section IV.3.2], we explicitly compute an open subvariety of the Fano scheme of lines on Y by calculating the conditions for the line through $(1:0:a:b)$ and $(0:1:c:d)$ to lie in Y. The resulting affine piece of the Fano scheme is given by

$$\{p+ra^2+sb^2=0,\ rac+sbd=0,\ q+rc^2+sd^2=0\} \subset \mathbb{A}^4_k = \operatorname{Spec} k[a, b, c, d].$$

This is easily verified to consist of two geometric components, each a plane conic, one contained in the plane $qra = -\sqrt{\Delta_Y} d, qsb = \sqrt{\Delta_Y} c$ and the other in the conjugate plane. □

Lemma 2.2. *Let Y be a nonsingular quadric surface over the finite field $\mathbb{F}_q$, with q odd. Then*

$$\#Y(\mathbb{F}_q) = \begin{cases} q^2+2q+1 & \text{if } \Delta_Y \in (\mathbb{F}_q^\times)^2, \\ q^2+1 & \text{otherwise.} \end{cases}$$

Proof. This can be computed directly, but we recall how to obtain it from the Lefschetz trace formula for étale cohomology. Let ℓ be a prime not equal to p. Let $\bar{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$, let $\bar{Y}$ be the base change of Y to $\bar{\mathbb{F}}_q$, and let $F\colon \bar{Y} \to \bar{Y}$ be the Frobenius morphism. The Lefschetz trace formula states that $\#Y(\mathbb{F}_q)$ can be calculated as

$$\#Y(\mathbb{F}_q) = \sum_{i=0}^{4} (-1)^i \operatorname{Tr}(F^* | \mathrm{H}^i(\bar{Y}, \mathbb{Q}_\ell)).$$

Because Y is smooth and projective, there are isomorphisms of Galois modules $\mathrm{H}^0(\bar{Y}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$ and $\mathrm{H}^4(\bar{Y}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-2)$; see [Milne 1980, VI.11.1]. We have $\bar{Y} \cong \mathbb{P}^1 \times \mathbb{P}^1$. The standard calculation of the cohomology groups of projective space [Milne 1980, VI.5.6], and the Künneth formula [Milne 1980, Corollary VI.8.13], give $\mathrm{H}^i(\bar{Y}, \mathbb{Q}_\ell) = 0$ for i odd, and show that $\mathrm{H}^2(\bar{Y}, \mathbb{Q}_\ell)$ has dimension 2. This reduces the formula to

$$\#Y(\mathbb{F}_q) = q^2 + 1 + \mathrm{Tr}(F^*|\mathrm{H}^2(\bar{Y}, \mathbb{Q}_\ell)).$$

Moreover, the cycle class map (arising from the Kummer sequence) gives a Galois-equivariant injective homomorphism

$$\mathrm{Pic}\,\bar{Y} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{H}^2(\bar{Y}, \mathbb{Q}_\ell(1)),$$

which by counting dimensions must be an isomorphism. If Δ_Y is a square in $\mathbb{F}_q$, then the Galois action is trivial and we obtain (after twisting) $\mathrm{Tr}(F^*|\mathrm{H}^2(\bar{Y}, \mathbb{Q}_\ell)) = 2q$. If Δ_Y is not a square in $\mathbb{F}_q$, then F^* acts on $\mathrm{Pic}\,\bar{Y} \cong \mathbb{Z}^2$ by switching the two factors, so with trace zero. In either case we obtain the claimed number of points. (Note that, in the first case, Y is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so we should not be surprised that it has $(q+1)^2$ points.) □

3. Proof of the theorem

Firstly, we calculate the Brauer group of X; it is convenient to do so in more generality.

Lemma 3.1. *Let k be a field of characteristic zero, let $Y \subset \mathbb{P}^3_k$ be a smooth quadric surface, and let $X \subset \mathbb{A}^4_k$ be the punctured affine cone over Y. If $\Delta_Y \in (k^\times)^2$, then we have* $\mathrm{Br}\,X = \mathrm{Br}\,k$. *Otherwise, suppose that X has a k-rational point P, and let g be a homogeneous linear form defining the tangent hyperplane to X at P. Then* $\mathrm{Br}\,X/\mathrm{Br}\,k$ *has order* 2, *and is generated by the class of the quaternion algebra* (Δ_Y, g). *This class does not depend on the choice of P.*

Proof. Let $\bar{k}$ be an algebraic closure of k, and let $\bar{X}$ and $\bar{Y}$ denote the base changes to $\bar{k}$ of X and Y, respectively. By [Ford 2001, Theorem 2.2], we have $\mathrm{Br}(\bar{X}) \cong \mathrm{Br}(\bar{Y})$; but $\bar{Y}$ is a rational variety, so its Brauer group is trivial. So it remains to compute the algebraic Brauer group of X.

We claim that there are no nonconstant invertible regular functions on X. Indeed, let $C \subset \mathbb{A}^4_k$ be the (nonpunctured) affine cone over Y. Because C is Cohen–Macaulay and $(0, 0, 0, 0)$ is of codimension ≥ 2 in C, we have

$$k[X] = k[C] = k[X_0, X_1, X_2, X_3]/(f),$$

where f is the homogeneous polynomial defining Y. This is a graded ring and its invertible elements must all have degree 0, and so are constant.

The Hochschild–Serre spectral sequence gives an injection $\mathrm{Br}\,X/\mathrm{Br}\,k \to \mathrm{H}^1(k, \mathrm{Pic}\,\bar{X})$. (Here we use $k[X]^\times = k^\times$ and $\mathrm{Br}\,\bar{X} = 0$.) By [Hartshorne 1977, Exercise II.6.3], there is an exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Pic}\,\bar{Y} \xrightarrow{\pi^*} \mathrm{Pic}\,\bar{X} \to 0,$$

where $\pi : X \to Y$ is the natural projection and the first map sends 1 to the class of a hyperplane section of $\bar{Y}$. Using Lemma 2.1 shows that $\operatorname{Pic} \bar{X}$ is isomorphic to $\mathbb{Z}$, with $G = \operatorname{Gal}(k(\sqrt{\Delta_Y})/k)$ acting by -1. The inflation-restriction sequence shows $\mathrm{H}^1(k, \operatorname{Pic} \bar{X}) \cong \mathrm{H}^1(G, \operatorname{Pic} \bar{X})$. If Δ_Y is a square, then this group is trivial, and we conclude that $\operatorname{Br} X / \operatorname{Br} k$ is also trivial. Otherwise $G = \{1, \sigma\}$ has order 2, and we have

$$\mathrm{H}^1(G, \operatorname{Pic} \bar{X}) \cong \widehat{\mathrm{H}}^{-1}(G, \operatorname{Pic} \bar{X}) = \frac{\ker(1+\sigma)}{\operatorname{im}(1-\sigma)} = \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

To conclude, it is sufficient to show that the algebra (Δ_Y, g) is nontrivial in $\operatorname{Br} X / \operatorname{Br} k$. Because the polynomial g also defines the tangent plane to Y at $\pi(P)$, the divisor (g) is equal to $\pi^*(L + L')$, where L is a line passing through $\pi(P)$ and L' is its conjugate. By [Bright 2002, Proposition 4.17], this shows that (Δ_Y, g) is a nontrivial element of order 2 in $\operatorname{Br} X / \operatorname{Br} k$. (The reference works with a smooth projective variety, but the proof generalises easily to any smooth X with $k[X]^\times = k^\times$.) □

We now return to the specific case where X is the punctured affine cone over the quadric surface defined by (1). We will need to be more careful about constant algebras than we have been up to this point. Recall that $\mathcal{X}(\mathbb{Z})$ consists of points $P = (x_0, x_1, x_2, x_3)$ where x_0, x_1, x_2, x_3 are coprime integers satisfying (1). Given such a P, we define the linear form

$$\ell_P = x_0 X_0 + 47 x_1 X_1 - 103 x_2 X_2 - (17 \times 47 \times 103) x_3 X_3 \in \mathbb{Z}[X_0, X_1, X_2, X_3]$$

and the quaternion algebra $A_P = (17, \ell_P) \in \operatorname{Br} X$. Note that the linear form ℓ_P does indeed define the tangent plane to X at P, so Lemma 3.1 shows that A_P represents the unique nontrivial class in $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$. We will now evaluate the Brauer–Manin obstruction associated to A_P.

Lemma 3.2. *Fix $P \in \mathcal{X}(\mathbb{Z})$. Then, for any place v of $\mathbb{Q}$ for which* 17 *is a square in $\mathbb{Q}_v$, we have* $\operatorname{inv}_v A_P(Q) = 0$ *for all $Q \in X(\mathbb{Q}_v)$.*

Proof. The homomorphism $\operatorname{Br} X \to \operatorname{Br} \mathbb{Q}_v$ given by evaluation at Q factors through $\operatorname{Br}(X \times_{\mathbb{Q}} \mathbb{Q}_v)$, but the image of A_P in this group is zero. □

Note that Lemma 3.2 applies in particular to $v = \infty$, $v = 2$, $v = 47$ and $v = 103$.

For the following lemma, let $\mathcal{Y}$ be the model for Y over $\mathbb{Z}$ defined by (1), and extend π to the natural projection $\mathcal{X} \to \mathcal{Y}$.

Lemma 3.3. *Fix $P \in \mathcal{X}(\mathbb{Z})$. Let $p \neq 17$ be a prime such that* 17 *is not a square in $\mathbb{Q}_p$, and let $Q \in \mathcal{X}(\mathbb{Z}_p)$ be such that $\pi(Q) \not\equiv \pi(P) \pmod{p}$. Then* $\operatorname{inv}_p A_P(Q) = 0$.

Proof. If $\ell_P(Q)$ is not divisible by p, then $\ell_P(Q)$ is a norm from the unramified extension $\mathbb{Q}_p(\sqrt{17})/\mathbb{Q}_p$ and therefore we have $\operatorname{inv}_p A_P(Q) = 0$.

Now suppose that $\ell_P(Q)$ is divisible by p. Denote by $\widetilde{Y}$ the base change of $\mathcal{Y}$ to $\mathbb{F}_p$. Let $\widetilde{P}, \widetilde{Q} \in \widetilde{Y}(\mathbb{F}_p)$ be the reductions modulo p of $\pi(P), \pi(Q)$ respectively. The variety $\widetilde{Y}$ is a smooth quadric over $\mathbb{F}_p$, and the tangent space $T_{\widetilde{P}}\widetilde{Y}$ is cut out by the reduction modulo p of the linear form ℓ_P. By Lemma 2.1, the scheme $\widetilde{Y} \cap \{\ell_P = 0\}$ consists of two lines that are conjugate over $\mathbb{F}_p(\sqrt{17})$. Therefore the only point of $\widetilde{Y}(\mathbb{F}_p)$ at which ℓ_P vanishes is $\widetilde{P}$. It follows that $\ell_P(Q)$ can only be divisible by p if $\widetilde{Q}$ coincides with $\widetilde{P}$. $\square$

Lemma 3.4. *Let $P, P' \in \mathcal{X}(\mathbb{Z})$ be two points. Then A_P and $A_{P'}$ lie in the same class in* $\operatorname{Br} X$.

Proof. By Lemma 3.1, we already know that the difference $A = A_P - A_{P'}$ lies in $\operatorname{Br}\mathbb{Q}$. It will be enough to show that $\operatorname{inv}_v A = 0$ for $v \neq 17$, for then the product formula shows $\operatorname{inv}_{17} A = 0$ also, and therefore $A = 0$.

For v for which 17 is a square in $\mathbb{Q}_v$, take Q to be any point of $X(\mathbb{Q}_v)$; then Lemma 3.2 shows $\operatorname{inv}_v A_P(Q) = \operatorname{inv}_v A_{P'}(Q) = 0$ and therefore $\operatorname{inv}_v A = 0$.

For $p \neq 17$ such that 17 is not a square in $\mathbb{Q}_p$, Lemma 2.2 shows that $\widetilde{Y} = \mathcal{Y} \times_{\mathbb{Z}} \mathbb{F}_p$ contains a point $\widetilde{Q}$ that is equal neither to $\pi(P)$ nor to $\pi(P')$ modulo p. Hensel's lemma shows that $\widetilde{Q}$ lifts to a point $Q \in \mathcal{X}(\mathbb{Z}_p)$. Lemma 3.3 shows $\operatorname{inv}_p A_P(Q) = \operatorname{inv}_p A_{P'}(Q) = 0$, so again we have $\operatorname{inv}_p A = 0$, completing the proof. $\square$

Lemma 3.5. *Fix $P \in \mathcal{X}(\mathbb{Z})$. For $p \neq 17$, we have $\operatorname{inv}_v A_P(Q) = 0$ for all $Q \in \mathcal{X}(\mathbb{Z}_p)$.*

Proof. If 17 is a square in $\mathbb{Q}_p$, then this follows from Lemma 3.2. Otherwise, Lemma 2.2 shows that $\widetilde{Y} = \mathcal{Y} \times_{\mathbb{Z}} \mathbb{F}_p$ contains at least two points. Weak approximation on Y then gives a point $P' \in \mathcal{X}(\mathbb{Z})$ such that $\pi(P)$ and $\pi(P')$ are different modulo p. By Lemma 3.4, the algebras A_P and $A_{P'}$ lie in the same class in $\operatorname{Br} X$, so it does not matter which we use to evaluate the invariant. Lemma 3.3 then gives the result. $\square$

It remains to evaluate the invariant at 17. In the following lemma, if $Q = (y_0, y_1, y_2, y_3)$ is a point of $\mathcal{X}$, then λQ denotes the point $(\lambda y_0, \lambda y_1, \lambda y_2, \lambda y_3)$.

Lemma 3.6. *Fix $P \in \mathcal{X}(\mathbb{Z})$ and $Q \in \mathcal{X}(\mathbb{Z}_{17})$. For any $\lambda \in \mathbb{Z}_{17}^\times$, having reduction $\tilde{\lambda} \in \mathbb{F}_{17}^\times$, we have*

$$\operatorname{inv}_{17} A_P(\lambda Q) = \begin{cases} \operatorname{inv}_{17} A_P(Q) & \text{if } \tilde{\lambda} \in (\mathbb{F}_{17}^\times)^2, \\ \operatorname{inv}_{17} A_P(Q) + \frac{1}{2} & \text{otherwise.} \end{cases}$$

Proof. Suppose that $\ell_P(Q)$ is nonzero. Because ℓ_P is homogeneous of degree 1, we have

$$\operatorname{inv}_{17} A_P(\lambda Q) = \operatorname{inv}_{17}(17, \lambda\ell_P(Q)) = \operatorname{inv}_{17} A_P(Q) + \operatorname{inv}_{17}(17, \lambda).$$

But $\operatorname{inv}_{17}(17, \lambda)$ is zero if and only if $\tilde{\lambda}$ is a square in $\mathbb{F}_{17}^\times$.

If $\ell_P(Q)$ is zero, then Lemma 2.1 shows that we have $\pi(P) = \pi(Q)$. Using weak approximation on Y, we can find a point $P' \in \mathcal{X}(\mathbb{Z})$ with $\pi(P') \neq \pi(Q)$, and Lemma 3.4 shows that replacing A_P by A'_P gives the same invariant. $\square$

Note that the only singular points of $\mathcal{X} \times_{\mathbb{Z}} \mathbb{F}_{17}$ are those of the form $(0, 0, 0, a)$, and these do not lift to points of $\mathcal{X}(\mathbb{Z}_{17})$. So the smooth points of $\mathcal{X}(\mathbb{F}_{17})$ are precisely those that lift to $\mathcal{X}(\mathbb{Z}_{17})$.

Putting all these calculations together proves the following. Let $U \subset X(\boldsymbol{A}_{\mathbb{Q}}^{\infty})$ be the open subset defined as

$$U = \prod_{p \neq 17} \mathcal{X}(\mathbb{Z}_p) \times \left\{ Q \in \mathcal{X}(\mathbb{Z}_{17}) \,\middle|\, \operatorname{inv}_{17} A_P(Q) = \tfrac{1}{2} \right\}.$$

Then U is a nonempty open subset that does not meet $X(\boldsymbol{A}_{\mathbb{Q}}^{\infty})^{\mathrm{Br}}$, showing that there is a Brauer–Manin obstruction to strong approximation away from ∞ on X. More specifically, for any smooth point $\widetilde{Q} \in \mathcal{X}(\mathbb{F}_{17})$, half of the scalar multiples of $\widetilde{Q}$ lie in the image of U, showing that they do not lift to integer points of $\mathcal{X}$.

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m.j.bright@math.leidenuniv.nl *Mathematisch Instituut, Universiteit Leiden, Leiden, Netherlands*

ivokok@telfort.nl *Mathematisch Instituut, Universiteit Leiden, Leiden, Netherlands*

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