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Quantum metrics from traces on full matrix algebras

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We prove that, in the sense of the Gromov–Hausdorff propinquity, certain natural quantum metrics on the algebras of $(n \times n)$-matrices are separated by a positive distance when n is not prime.

1. Introduction and background

Motivated by high energy physics, M. A. Rieffel [1998; 2004] developed the notion of a "noncommutative" or "quantum" compact metric space, and initiated the study of topologies over classes of such quantum metric spaces. The convergence is studied by use of a distance on the classes of these spaces. F. Latrémolière [2015; 2016a] introduced the quantum Gromov–Hausdorff propinquity, see also [Rieffel 2016], as a noncommutative analogue of the Gromov–Hausdorff distance [Burago et al. 2001], adapted to the theory of C*-algebras.

Quantum metric spaces are built by adding some quantum metric to unital C*-algebras [Rieffel 2004, pp. 1–65]; see [Latrémolière 2016b] for a survey with many examples and references. C*-algebras are certain norm-closed algebras of bounded linear operators on Hilbert spaces, up to an appropriate notion of *-isomorphism [Davidson 1996, Theorem I.9.12]. Thus, it is natural to look to various classes of C*-algebras and study them from the viewpoint of quantum compact metric spaces and the Gromov–Hausdorff propinquity. The first author and F. Latrémolière did just that in [Aguilar and Latrémolière 2015] with a class of C*-algebras called approximately finite-dimensional C*-algebras (AF algebras) [Bratteli 1972]. Their work constructs quantum metrics on AF algebras, which is then used to study the topology induced by the propinquity on various natural sets of AF algebras. In

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particular, these quantum metrics can be restricted to the finite-dimensional C*-subalgebras of AF algebras. The present work further studies some of the geometric aspects of this construction on these finite-dimensional algebras when they are full matrix algebras (algebras of complex-valued square matrices). Our focus is to prove that different quantum metrics induced by the first author and Latrémolière's construction on the same full matrix algebras are indeed at positive distance, in the sense of the propinquity. By its nature, it is usually difficult to prove lower bounds on the propinquity, so our work tackles a delicate aspect of this theory. First, we begin by defining many of these objects, from C*-algebras to quantum compact metric spaces, while taking note of some theorems that will prove useful to our efforts. Definitions 1.1–1.10 are contained in [Davidson 1996, Chapter I].

Definition 1.1. An *associative algebra over the complex numbers* $\mathbb{C}$ is a vector space $\mathfrak{A}$ over $\mathbb{C}$ with an associative multiplication, denoted by concatenation, such that

$$a(b+c) = ab+ac \quad \text{and} \quad (b+c)a = ba+ca \quad \text{for all } a, b, c \in \mathfrak{A},$$
$$\lambda(ab) = (\lambda a)b = a(\lambda b) \quad \text{for all } a, b \in \mathfrak{A},\ \lambda \in \mathbb{C}.$$

In other words, the associative multiplication is a bilinear map from $\mathfrak{A} \times \mathfrak{A}$ to $\mathfrak{A}$.

We say that $\mathfrak{A}$ is *unital* if there exists a multiplicative identity, denoted by $1_{\mathfrak{A}}$. That is,

$$1_{\mathfrak{A}} a = a = a 1_{\mathfrak{A}} \quad \text{for all } a \in \mathfrak{A}.$$

Convention 1.2. All algebras are associative algebras over $\mathbb{C}$ unless otherwise specified.

Notation 1.3. When E is a normed vector space, its norm will be denoted by $\|\cdot\|_E$ by default.

Definition 1.4. A *normed algebra* is an algebra $\mathfrak{A}$ with a norm $\|\cdot\|_{\mathfrak{A}}$ such that

$$\|ab\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}} \|b\|_{\mathfrak{A}} \quad \text{for all } a, b \in \mathfrak{A}.$$

$\mathfrak{A}$ is a *Banach algebra* when $\mathfrak{A}$ is complete with respect to the norm $\|\cdot\|_{\mathfrak{A}}$.

Definition 1.5. A *C*-algebra*, $\mathfrak{A}$, is a Banach algebra such that there exists an antimultiplicative conjugate linear involution $^* : \mathfrak{A} \to \mathfrak{A}$, called the *adjoint*. That is, * satisfies

(1) *conjugate linear*: $(\lambda(a+b))^* = \bar{\lambda}(a^* + b^*)$ for all $\lambda \in \mathbb{C},\ a, b \in \mathfrak{A}$;

(2) *involution*: $(a^*)^* = a$ for all $a \in \mathfrak{A}$;

(3) *antimultiplicative*: $(ab)^* = b^* a^*$ for all $a, b \in \mathfrak{A}$.

Furthermore, the norm, multiplication, and adjoint together satisfy the identity

$$\|aa^*\|_{\mathfrak{A}} = \|a\|_{\mathfrak{A}}^2 \quad \text{for all } a \in \mathfrak{A}, \tag{1-1}$$

called the *C*-identity*.

The set of self-adjoint elements of a C*-algebra is the set $\mathfrak{sa}(\mathfrak{A}) = \{a \in \mathfrak{A} : a = a^*\}$.

An element $u \in \mathfrak{A}$ of a unital C*-algebra is *unitary* if $uu^* = u^*u = 1_{\mathfrak{A}}$.

We say that $\mathfrak{B} \subseteq \mathfrak{A}$ is a *C*-subalgebra of* $\mathfrak{A}$ if $\mathfrak{B}$ is a norm-closed subalgebra that is also self-adjoint; i.e., $a \in \mathfrak{B} \iff a^* \in \mathfrak{B}$.

Our main example will be the C*-algebra of $(n \times n)$-matrices over the complex numbers, which we define now.

Example 1.6 [Davidson 1996, Example I.1.1]. Fix $n \in \mathbb{N} \setminus \{0\}$. We let $M_n(\mathbb{C})$ denote the C*-algebra of $(n \times n)$-matrices over the complex numbers called a *full matrix algebra*. The algebra is given by the standard matrix operations and the adjoint is the conjugate transpose. The norm is given by the operator norm

$$\|a\|_{M_n(\mathbb{C})} = \sup\{\|ax\|_2 : x \in \mathbb{C}^n, \|x\|_2 \leqslant 1\} \quad \text{for all } a \in M_n(\mathbb{C}),$$

where ax denotes matrix multiplication of the matrix a and column vector x, and $\|\cdot\|_2$ is the standard Euclidean 2-norm on $\mathbb{C}^n$.

Note that if $a \in \mathfrak{sa}(M_n(\mathbb{C}))$, then $\|a\|_{M_n(\mathbb{C})} = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } a\}$ by [Davidson 1996, Corollary I.3.4]. The unit of $M_n(\mathbb{C})$ is the identity matrix, which we denote by I_n. For $a \in M_n(\mathbb{C})$, we denote the i-row, j-column entry by $a_{i,j}$.

Next, we describe morphisms between C*-algebras.

Definition 1.7. Let $\mathfrak{A}, \mathfrak{B}$ be C*-algebras. A **-homomorphism* $\pi : \mathfrak{A} \to \mathfrak{B}$ is a *-preserving, linear and multiplicative function:

- π is a **-monomorphism* if it is an injective *-homomorphism.
- π is a **-epimorphism* if π is a surjective *-homomorphism.
- π is a **-isomorphism* if π is a bijective *-homomorphism.
- $\mathfrak{A}$ is **-isomorphic* to $\mathfrak{D}$ if there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{D}$, and we then write $\mathfrak{A} \cong \mathfrak{D}$.
- If both $\mathfrak{A}, \mathfrak{D}$ are unital, then we call a *-homomorphism $\pi : \mathfrak{A} \to \mathfrak{D}$ *unital* if $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{D}}$.

The next result shows that there are important analytical properties (such as continuity, contractibility, and isometry) associated to these morphisms without further assumptions. Thus, only algebraic conditions are indeed needed to properly define morphisms between C*-algebras in Definition 1.7.

Proposition 1.8 [Davidson 1996, Theorem I.5.5]. *Let $\mathfrak{A}, \mathfrak{D}$ be C*-algebras. If $\pi : \mathfrak{A} \to \mathfrak{D}$ is a *-homomorphism, then π is continuous and contractive. That is, its operator norm satisfies*

$$\|\pi\|_{\mathfrak{B}(\mathfrak{A},\mathfrak{D})} = \sup\{\|\pi(a)\|_{\mathfrak{D}} : a \in \mathfrak{A}, \|a\|_{\mathfrak{A}} = 1\} \leqslant 1,$$

or equivalently, for all $a \in \mathfrak{A}$, we have $\|\pi(a)\|_{\mathfrak{D}} \leqslant \|a\|_{\mathfrak{A}}$.

*If $\pi : \mathfrak{A} \to \mathfrak{D}$ is a *-homomorphism, then π is an isometry if and only if π is a *-monomorphism. In particular, *-isomorphisms are isometries.*

In the next example, we present some *-isomorphisms related to the C*-algebras $M_n(\mathbb{C})$.

Example 1.9 [Davidson 1996, Lemma III.2.1]. A map $\pi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a *-isomorphism if and only if there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that

$$\pi(a) = UaU^* \quad \text{for all } a \in M_n(\mathbb{C}).$$

In order to define quantum compact metric spaces we need to define another structure associated to C*-algebras.

Definition 1.10. Let $\mathfrak{A}$ be a unital C*-algebra. Let $\mathfrak{A}'$ denote the set of continuous and linear complex-valued functions on $\mathfrak{A}$. The *state space* of $\mathfrak{A}$ is the set

$$\mathscr{S}(\mathfrak{A}) = \{\varphi \in \mathfrak{A}' : 1 = \varphi(1_{\mathfrak{A}}) = \|\varphi\|_{\mathfrak{A}'}\},$$

where $\|\varphi\|_{\mathfrak{A}'} = \sup\{|\varphi(a)| : a \in \mathfrak{A},\ \|a\|_{\mathfrak{A}} = 1\}$ is the operator norm.

A state $\varphi \in \mathscr{S}(\mathfrak{A})$ is called *tracial* if $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathfrak{A}$ [Davidson 1996, p. 114].

As an example, we look to $M_n(\mathbb{C})$.

Example 1.11. For $a \in M_n(\mathbb{C})$, let $\mathrm{Tr}_n(a) = \sum_{j=1}^{n} a_{j,j}$ be the trace of a matrix. Define $\mathrm{tr}_n = \frac{1}{n}\mathrm{Tr}_n$. By [Davidson 1996, Example IV.5.4], the map tr_n is the unique tracial state on $M_n(\mathbb{C})$.

Rieffel [1998] introduced the notion of a quantum compact metric space by providing a particular metric on the state space of a C*-algebra, which serves as a quantum analogue to the Monge–Kantorovich metric on Borel probability measures of a compact Hausdorff space. This metric lies outside the scope of this paper, so we provide an equivalent definition of a quantum compact metric space that utilizes a quantum analogue to the Lipschitz seminorm on continuous functions.

Definition 1.12 [Rieffel 1998; 1999; Ozawa and Rieffel 2005]. Let $\mathfrak{A}$ be a unital C*-algebra. Let L be a seminorm on $\mathfrak{sa}(\mathfrak{A})$ (possibly taking value $+\infty$). The pair $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space if L satisfies the following:

(1) L is lower semicontinuous with respect to $\|\cdot\|_{\mathfrak{A}}$.

(2) The set $\mathrm{dom}(\mathsf{L}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) < \infty\}$ is dense in $\mathfrak{sa}(\mathfrak{A})$.

(3) The kernel of L is $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_A = \{r1_{\mathfrak{A}} \in \mathfrak{sa}(\mathfrak{A}) : r \in \mathbb{R}\}$.

(4) There exists a state $\mu \in \mathscr{S}(\mathfrak{A})$ such that the set

$$\{a \in \mathfrak{sa}(\mathfrak{A}) : \mu(a) = 0 \text{ and } \mathsf{L}(a) \leqslant 1\}$$

is totally bounded with respect to $\|\cdot\|_{\mathfrak{A}}$.

The Lipschitz seminorm on continuous functions satisfies a Leibniz rule and we may generalize this in the following way.

Definition 1.13. Fix $C \geqslant 1$, $D \geqslant 0$. A quantum compact metric space $(\mathfrak{A}, \mathsf{L})$ is a *(C, D)-quasi-Leibniz quantum compact metric space* if L is a *(C, D)-quasi-Leibniz seminorm*; i.e., for all $a, b \in \mathfrak{sa}(\mathfrak{A})$

$$\max\{\mathsf{L}(a \circ b), \mathsf{L}(\{a, b\})\} \leqslant C\big(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \|b\|_{\mathfrak{A}}\mathsf{L}(a)\big) + D\mathsf{L}(a)\mathsf{L}(b),$$

where the Jordan product is $a \circ b = \frac{1}{2}(ab + ba)$ and the Lie product is $\{a, b\} = \frac{1}{2i}(ab - ba)$.

Latrémolière [2016a] introduced a quantum analogue to the Gromov–Hausdorff distance [Burago et al. 2001], the quantum Gromov–Hausdorff propinquity. Indeed, the quantum Gromov–Hausdorff propinquity is a distance between quasi-Leibniz quantum compact metric spaces that preserves the C*-algebraic and metric structures and recovers the topology of the Gromov–Hausdorff distance, and thus provides an appropriate framework for the study of noncommutative metric geometry. The definition is quite involved, so in the following theorem, we summarize the results that pertain to our work in this paper, while also defining the standard notion of isomorphism between two quasi-Leibniz quantum compact metric spaces, a quantum isometry.

Theorem 1.14 [Latrémolière 2016a; 2017]. *The* quantum Gromov–Hausdorff propinquity $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ *between two quasi-Leibniz quantum compact metric spaces* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ *is a metric up to* quantum isometry; *i.e.,* $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ *if and only if there exists a unital *-isomorphism* $\pi : \mathfrak{A} \to \mathfrak{B}$ *with* $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

Moreover, Λ *recovers the Gromov–Hausdorff topology on compact metric spaces.*

2. Gromov–Hausdorff propinquity between isomorphic full matrix algebras

The first author and Latrémolière [Aguilar and Latrémolière 2015] discovered quasi-Leibniz Lip-norms on certain infinite-dimensional C*-algebras called approximately finite-dimensional C*-algebras (AF algebras) with certain tracial states. Now, these AF algebras are built by an inductive sequence of finite-dimensional AF algebras; see [Murphy 1990, Chapter 6]. While it can be the case that two distinct inductive sequences can produce AF algebras that are *-isomorphic, see [Davidson 1996, Example III.2.2], the Lip-norms constructed in [Aguilar and Latrémolière 2015, Theorem 3.5] seem to acknowledge the particular inductive sequence. Therefore, the question arose of whether these Lip-norms can distinguish two *-isomorphic AF algebras with different inductive sequences, where by *distinguish*, we mean by way of a quantum isometry of Theorem 1.14. This would show that these Lip-norms are truly adding further structure to the C*-algebraic structure. Yet, showing that two spaces are not quantum isometric is quite a nontrivial task since the condition in Theorem 1.14 has to be checked for every *-isomorphism to provide a negative result. Indeed, two spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are *not* quantum isometric if and only if for any *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ (if it exists), we have $\mathsf{L}_{\mathfrak{B}} \circ \pi \neq \mathsf{L}_{\mathfrak{A}}$. But, in the case of $M_n(\mathbb{C})$, we have that the *-isomorphisms are well understood, as seen in Example 1.9. Thus, in this paper, we try to tackle this question of quantum isometry in the case of the finite-dimensional C*-algebras $M_n(\mathbb{C})$ with respect to different finite inductive sequences, and we accomplish the task in Theorem 2.11. We begin by defining the particular quantum metric spaces that we will be working with, which requires the following notions.

Definition 2.1. A *conditional expectation* $P : \mathfrak{A} \to \mathfrak{B}$ onto $\mathfrak{B}$, where $\mathfrak{A}$ is a unital C*-algebra and $\mathfrak{B}$ is a unital C*-subalgebra of $\mathfrak{A}$, is a linear map such that

(1) for all $a \in \mathfrak{A}$ there exists $b \in \mathfrak{B}$ such that $P(aa^*) = bb^*$,

(2) for all $a \in \mathfrak{A}$, we have $\|P(a)\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}}$,

(3) for all $b, c \in \mathfrak{B}$ and $a \in \mathfrak{A}$, we have $P(bac) = bP(a)c$, and

(4) $P(b) = b$ for all $b \in \mathfrak{B}$.

Notation 2.2. We write $k \mid n$ to denote that k divides n throughout this paper.

Let $n, k \in \mathbb{N} \setminus \{0\}$, $n > 1$ with $k \mid n$ and $k < n$.

Let $\pi_{k,n} : M_k(\mathbb{C}) \mapsto M_n(\mathbb{C})$ be the unital *-monomorphism of [Davidson 1996, Lemma III.2.1] determined by

$$\pi_{k,n}(a) = \begin{bmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{bmatrix} = \operatorname{diag}(a, \dots, a) \quad \text{for all } a \in M_k(\mathbb{C}),$$

where there are $\frac{n}{k}$ nonoverlapping copies of a filling the block diagonal and 0's elsewhere. Coordinatewise, the map $\pi_{k,n}$ satisfies

$$\pi_{k,n}(a)_{p,q} = \begin{cases} a_{i,j} & \text{if there exists } r \in \left\{0, \dots, \frac{n}{k} - 1\right\} \text{ such that} \\ & \qquad p = i + rk \text{ and } q = j + rk \text{ for some } i, j \in \{1, \dots, k\}, \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in M_k(\mathbb{C})$ and $p, q \in \{1, \dots n\}$.

Note that $\pi_{k,n}(M_k(\mathbb{C}))$ is a unital C*-subalgebra of $M_n(\mathbb{C})$.

In Lemma 2.4, we will present another coordinatewise description of $\pi_{k,n}$, which allows for direct computation rather than finding the terms and is much more suitable and convenient for the many calculations we will work with in this paper. But, for now, we are prepared to define the quantum metrics on $M_n(\mathbb{C})$.

Theorem 2.3. *Let $n \in \mathbb{N} \setminus \{0, 1\}$ such that there exists $k \in \mathbb{N} \setminus \{0, 1\}$ with $k < n$ and $k \mid n$. Recall the definition of the tracial state $\mathrm{tr}_n : M_n(\mathbb{C}) \to \mathbb{C}$ of Example 1.11.*

If $P_{j,n} : M_n(\mathbb{C}) \to \pi_{j,n}(M_j(\mathbb{C}))$ for $j = 1, k$ denotes the unique tr_n-preserving conditional expectation and we define for all $a \in \mathfrak{sa}(M_n(\mathbb{C}))$ the seminorms

$$L_{M_n(\mathbb{C}),1}(a) = \|a - P_{1,n}(a)\|_{M_n(\mathbb{C})},$$
$$L_{M_n(\mathbb{C}),k}(a) = \max\{\|a - P_{1,n}(a)\|_{M_n(\mathbb{C})}, k \cdot \|a - P_{k,n}(a)\|_{M_n(\mathbb{C})}\},$$

then both $(M_n(\mathbb{C}), L_{M_n(\mathbb{C}),1})$ and $(M_n(\mathbb{C}), L_{M_n(\mathbb{C}),k})$ are $(2, 0)$-quasi-Leibniz quantum compact metric spaces.

Proof. This is [Aguilar and Latrémolière 2015, Theorem 3.5] and in particular Step 3 of its proof. □

Our main goal — realized in Theorem 2.11 — is to show that the two quantum metric spaces for a fixed k, n are not quantum isometric. These spaces are motivated by [Aguilar and Latrémolière 2015, Theorem 4.9], where these spaces do in fact form the finite-dimensional quantum metric spaces used in their construction. Therefore, our work is a legitimate step towards understanding the quantum isometries between the infinite-dimensional C*-algebras presented in [Aguilar and Latrémolière 2015]. Next, our proof of Theorem 2.11 requires a detailed and coordinatewise description of the maps $\pi_{k,n}$ and $P_{k,n}$, which could be easily implemented algorithmically. We begin with $\pi_{k,n}$.

Lemma 2.4. *If $k, n \in \mathbb{N} \setminus \{0\}$, $n > 1$ such that $k \mid n$ and $k < n$, then the map $\pi_{k,n} : M_k(\mathbb{C}) \to M_n(\mathbb{C})$ of Notation 2.2 satisfies*

$$\pi_{k,n}(a)_{p,q} = \begin{cases} a_{1+(p-1) \bmod k, 1+(q-1) \bmod k} & \text{if } \left\lfloor \frac{p-1}{k} \right\rfloor = \left\lfloor \frac{q-1}{k} \right\rfloor, \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in M_k(\mathbb{C})$ and $p, q \in \{1, \dots n\}$.

Proof. Let $a \in M_k(\mathbb{C})$ and fix $p, q \in \{1, \dots, n\}$.

Case 1: Assume $\left\lfloor \frac{p-1}{k} \right\rfloor = \left\lfloor \frac{q-1}{k} \right\rfloor$. Now, since $k \mid n$, there exist $r, s \in \left\{0, \dots, \frac{n}{k} - 1\right\}$ and $i, j \in \{1, \dots, k\}$ such that $p = i + rk$ and $q = j + sk$. Therefore, we have

$$\left\lfloor \frac{p-1}{k} \right\rfloor = \left\lfloor \frac{i+rk-1}{k} \right\rfloor = \left\lfloor \frac{i-1}{k} + r \right\rfloor = \left\lfloor \frac{i-1}{k} \right\rfloor + r = r$$

because $0 \leqslant i - 1 < k$.

Also, we have

$$\left\lfloor \frac{q-1}{k} \right\rfloor = \left\lfloor \frac{j+sk-1}{k} \right\rfloor = \left\lfloor \frac{j-1}{k} + s \right\rfloor = \left\lfloor \frac{j-1}{k} \right\rfloor + s = s$$

because $0 \leqslant j - 1 < k$. Thus $r = s$ by the Case 1 assumption and $\pi_{k,n}(a)_{p,q} = a_{i,j}$ by Notation 2.2. However, by modular arithmetic, since $c \mod d = c - d \left\lfloor \frac{c}{d} \right\rfloor$, we gather that $1+p-1-rk=i$ and $1+q-1-rk=j$ imply

$$\pi_{k,n}(a)_{p,q}=a_{1+(p-1) \bmod k, 1+(q-1) \bmod k}.$$

Case 2: Assume $\lfloor \frac{p-1}{k} \rfloor \neq \lfloor \frac{q-1}{k} \rfloor$. By Notation 2.2, assume by way of contradiction that there exists $r \in \{0, \ldots, \frac{n}{k}-1\}$ such that $p=i+rk$ and $q=j+rk$ for some $i, j \in \{1, \ldots, k\}$. Then, the argument of Case 1 implies that $r \neq r$, a contradiction. Hence $\pi_{k,n}(a)_{p,q}=0$ by Notation 2.2. □

Next, we move to understanding the conditional expectations $P_{k,n}$ in a coordinatewise manner. First, we provide some notation for a standard basis for $M_n(\mathbb{C})$.

Notation 2.5. For $n \in \mathbb{N} \setminus \{0\}$, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, n\}$, let $E_{n,j,k} \in M_n(\mathbb{C})$ denote the standard *matrix unit*, see [Davidson 1996, Section III.1], defined coordinatewise by

$$(E_{n,j,k})_{p,q}=\begin{cases} 1, & p=j \text{ and } q=k, \\ 0, & \text{otherwise} \end{cases}$$

for $p, q \in \{1, \ldots, n\}$, and note that the set $\{E_{n,j,k} \in M_n(\mathbb{C}) : j, k \in \{1, \ldots, n\}\}$ forms a basis for $M_n(\mathbb{C})$.

Assume that $l \in \mathbb{N} \setminus \{0\}$ and $l \mid n$. Let

$$B_{l,n}=\{\pi_{l,n}(a) \in M_n(\mathbb{C}) : a \text{ is a matrix unit of } M_l(\mathbb{C})\}.$$

Next, the tracial state tr_n induces an inner product on $M_n(\mathbb{C})$ via $\langle a, b\rangle=\mathrm{tr}_n(b^* a)$. In [Aguilar and Latrémolière 2015], they use this observation and that the matrix units are orthogonal with respect to this inner product to provide a general description of $P_{k,n}$ in terms of the matrix units and $\pi_{k,n}$; see (4.1) of that paper. We will utilize this in Lemma 2.7 to provide an explicit coordinatewise description of $P_{k,n}$. But, first, we prove a lemma about the relationship between the tracial state tr_n and the *-monomorphism $\pi_{k,n}$.

Lemma 2.6. *If* $k, n \in \mathbb{N} \setminus \{0\}$ *and* $k \mid n$, *then* $\mathrm{tr}_n \circ \pi_{k,n}=\mathrm{tr}_k$.

Proof. Let $a \in M_k(\mathbb{C})$. Then, by Lemma 2.4

$$\begin{aligned}
(\mathrm{tr}_n \circ \pi_{k,n})(a) &= \frac{1}{n} \mathrm{Tr}_n(\pi_{k,n}(a))=\frac{1}{n} \sum_{i=1}^{n} \pi_{k,n}(a)_{i,i} \\
&= \frac{1}{n} \sum_{i=1}^{n} a_{1+(i-1) \bmod k, 1+(i-1) \bmod k} \\
&= \frac{1}{n} \sum_{i=1}^{n/k} \sum_{j=1}^{k} a_{j,j}=\frac{1}{n} \sum_{i=1}^{n/k} \mathrm{Tr}_k(a) \\
&= \frac{1}{n} \cdot \frac{n}{k} \mathrm{Tr}_k(a)=\frac{1}{k} \mathrm{Tr}_k(a)=\mathrm{tr}_k(a).
\end{aligned}$$

□

Lemma 2.7. *If $n \in \mathbb{N} \setminus \{0\}$ and there exists $k \in \mathbb{N} \setminus \{0\}$ such that $k \mid n$, then using notation from Theorem 2.3, for all $a \in M_n(\mathbb{C})$, we have*

$$P_{k,n}(a) = \frac{k}{n} \sum_{p=1}^{k} \sum_{q=1}^{k} \left(\sum_{l=0}^{n/k-1} a_{p+kl,q+kl} \right) \pi_{k,n}(E_{k,p,q}),$$

and coordinatewise, this is

$$P_{k,n}(a)_{i,j} = \begin{cases} \frac{k}{n} \sum_{l=0}^{n/k-1} a_{p+kl,q+kl} & \text{if } \left\lfloor \frac{i-1}{k} \right\rfloor = \left\lfloor \frac{j-1}{k} \right\rfloor \text{ and } p-1 = (i-1) \bmod k \\ & \text{and } q-1 = (j-1) \bmod k, \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in \{1, \ldots, n\}$.

Furthermore, if $k = 1$, then

$$P_{1,n}(a) = \operatorname{tr}_n(a) I_n.$$

Proof. By [Aguilar and Latrémolière 2015, (4.1)], we have

$$P_{k,n}(a) = \sum_{b \in B_{k,n}} \frac{\operatorname{tr}_n(b^* a)}{\operatorname{tr}_n(b^* b)} b$$

for all $a \in M_n(\mathbb{C})$, where $B_{k,n}$ was defined in Notation 2.5. Thus, for $b \in B_{k,n}$, we have

$$b = \pi_{k,n}(E_{k,p,q}) \quad \text{for some } p, q \in \{1, \ldots, k\},$$

and thus

$$b^* b = \pi_{k,n}(E_{k,q,p}) \pi_{k,n}(E_{k,p,q}) = \pi_{k,n}(E_{k,q,p} E_{k,p,q}).$$

Now

$$(E_{k,q,p} E_{k,p,q})_{i,j} = \sum_{l=1}^{k} (E_{k,q,p})_{i,l} \cdot (E_{k,p,q})_{l,j}$$

and

$$(E_{k,q,p})_{i,l} \cdot (E_{k,p,q})_{l,j} = \begin{cases} 1 & \text{if } i = q \text{ and } j = q \text{ and } l = p, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$E_{k,q,p} E_{k,p,q} = E_{k,q,q}.$$

Hence, by Lemma 2.6, we gather that

$$\begin{aligned} \operatorname{tr}_n(b^* b) &= \operatorname{tr}_n(\pi_{k,n}(E_{k,q,p} E_{k,p,q})) \\ &= \operatorname{tr}_k(E_{k,q,p} E_{k,p,q}) = \operatorname{tr}_k(E_{k,q,q}) \\ &= \frac{1}{k} \operatorname{Tr}_k(E_{k,q,q}) = \frac{1}{k} \cdot 1 = \frac{1}{k}. \end{aligned}$$

Thus, we now have

$$
\begin{aligned}
P_{k,n}(a) &= \sum_{p=1}^{k}\sum_{q=1}^{k} \frac{\mathrm{tr}_n(\pi_{k,n}(E_{k,q,p})a)}{1/k}\pi_{k,n}(E_{k,p,q}) \\
&= \sum_{p=1}^{k}\sum_{q=1}^{k} k\cdot \mathrm{tr}_n(\pi_{k,n}(E_{k,q,p})a)\pi_{k,n}(E_{k,p,q}) \\
&= \sum_{p=1}^{k}\sum_{q=1}^{k} k\cdot \frac{1}{n}\,\mathrm{Tr}(\pi_{k,n}(E_{k,q,p})a)\pi_{k,n}(E_{k,p,q}) \\
&= \frac{k}{n}\sum_{p=1}^{k}\sum_{q=1}^{k} \mathrm{Tr}(\pi_{k,n}(E_{k,q,p})a)\pi_{k,n}(E_{k,p,q}) \\
&= \frac{k}{n}\sum_{p=1}^{k}\sum_{q=1}^{k} \left(\sum_{i=1}^{n}(\pi_{k,n}(E_{k,q,p})a)_{i,i}\right)\pi_{k,n}(E_{k,p,q}). \qquad (2\text{-}1)
\end{aligned}
$$

Next, fix $i \in \{1, \dots, n\}$. By matrix multiplication

$$\sum_{i=1}^{n}(\pi_{k,n}(E_{k,q,p})a)_{i,i} = \sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{k,n}(E_{k,q,p})_{i,j}\cdot a_{j,i}.$$

But, for $j \in \{1, \dots, n\}$, we have

$$\pi_{k,n}(E_{k,q,p})_{i,j} = \begin{cases} 1 & \text{if there exists } l \in \left\{0, \dots, \frac{n}{k}-1\right\} \text{ such that} \\ & \qquad\qquad i = q+kl \text{ and } j = p+kl, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pi_{k,n}(E_{k,q,p})_{i,j}\cdot a_{j,i} = \begin{cases} a_{j,i} & \text{if there exists } l \in \left\{0, \dots, \frac{n}{k}-1\right\} \text{ such that} \\ & \qquad\qquad j = p+kl \text{ and } i = q+kl, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\pi(E_{k,q,p})_{i,j}\cdot a_{j,i} = \sum_{l=0}^{n/k-1} a_{p+kl,q+kl}.$$

And, by (2-1), we conclude that

$$P_{k,n}(a) = \frac{k}{n}\sum_{p=1}^{k}\sum_{q=1}^{k}\left(\sum_{l=0}^{n/k-1} a_{p+kl,q+kl}\right)\pi_{k,n}(E_{k,p,q}). \qquad (2\text{-}2)$$

The coordinatewise expression follows from Lemma 2.4. Indeed: Let $i, j \in \{1, \dots, n\}$. If $\left\lfloor \frac{i-1}{k} \right\rfloor = \left\lfloor \frac{j-1}{k} \right\rfloor$, then $P_{k,n}(a)_{i,j}$ lies in one of the $(k\times k)$-diagonal blocks of the $(n\times n)$-matrix. So, there exist p, q such that $\pi_{k,n}(E_{k,p,q})_{i,j} = 1$.

Namely $p = 1 + (i-1) \bmod k$ and $q = 1 + (j-1) \bmod k$. However, this pair p, q corresponds to a term in the sum defining $P_{k,n}$ in (2-2); that is, it corresponds to $\frac{k}{n}\big(\sum_{l=0}^{n/k-1} a_{p+kl,q+kl}\big)\pi_{k,n}(E_{k,p,q})$. And, since the (i, j)-th entry of $\pi_{k,n}(E_{k,p,q})$ is 1, this gives us

$$P_{k,n}(a)_{i,j} = \sum_{l=0}^{n/k-1} a_{p+kl,q+kl}.$$

If $\big\lfloor \frac{i-1}{k} \big\rfloor \neq \big\lfloor \frac{j-1}{k} \big\rfloor$, then for all p, q between 1 and k we have $(\pi_{k,n}(E_{k,p,q}))_{i,j} = 0$ by the definition of $\pi_{k,n}$, so $(P_{k,n}(a))_{i,j} = 0$.

The last statement of this lemma follows from this coordinatewise description with $k = 1$. □

For computational purposes, we present an alternative perspective on the projection map before proceeding.

Proposition 2.8. *Let $a \in M_n(\mathbb{C})$, and let k be an integer that divides n. Consider the $(k \times k)$-diagonal blocks of a, denoted from top-left diagonal block to bottom-right diagonal block by $B_1, B_2, \ldots, B_{n/k}$. The projection $P_{k,n}(a)$ is the image of the arithmetic mean of these blocks under the map $\pi_{k,n}$. In other words,*

$$P_{k,n}(a) = \pi_{k,n}\bigg(\frac{k}{n} \cdot \sum_{i=1}^{n/k} B_i\bigg).$$

Proof. This follows from Lemma 2.7. □

Finally, we are in a position to study the Lip-norms of Theorem 2.3.

Lemma 2.9. *Let $n \in \mathbb{N} \setminus \{0, 1\}$. Using notation from Theorem 2.3, if $\pi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a *-isomorphism, then*

$$L_{M_n(\mathbb{C}),1} \circ \pi(a) = L_{M_n(\mathbb{C}),1}(a).$$

Proof. Let $\pi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a *-isomorphism. By [Davidson 1996, Lemma III.2.1], there exists a unitary $U \in M_n(\mathbb{C})$ such that $\pi(a) = UaU^*$ for all $a \in M_n(\mathbb{C})$. Also, by Lemma 2.7, we gather $P_{1,n}(UaU^*) = \mathrm{tr}_n(UaU^*)I_n = \mathrm{tr}_n(U^*Ua)I_n = \mathrm{tr}_n(a)I_n$ for all $a \in M_n(\mathbb{C})$. Now, let $a \in M_n(\mathbb{C})$; thus

$$\begin{aligned}
L_{M_n(\mathbb{C}),1} \circ \pi(a) &= \|UaU^* - P_{1,n}(UaU^*)\|_{M_n(\mathbb{C})} \\
&= \|UaU^* - P_{1,n}(UaU^*)\|_{M_n(\mathbb{C})} \\
&= \|UaU^* - \mathrm{tr}_n(a)I_n\|_{M_n(\mathbb{C})} \\
&= \|UaU^* - U(\mathrm{tr}_n(a)I_n)U^*\|_{M_n(\mathbb{C})} \\
&= \|U(a - \mathrm{tr}_n(a)I_n)U^*\|_{M_n(\mathbb{C})} \\
&= \|U(a - P_{1,n}(a))U^*\|_{M_n(\mathbb{C})} \\
&= \|a - P_{1,n}(a)\|_{M_n(\mathbb{C})} = L_{M_n(\mathbb{C}),1}(a).
\end{aligned}$$ □

Lemma 2.10. *If $k, n \in \mathbb{N}\backslash\{0\}$, $n > 2$ such that $k \mid n$ and $1 < k < n$, then using notation from Theorem 2.3, we have*

$$L_{M_n(\mathbb{C}),1} \neq L_{M_n(\mathbb{C}),k}.$$

Proof. Consider $a \in M_n(\mathbb{C})$:

$$a = \operatorname{diag}(k, 0, \ldots, 0).$$

Now, by Lemma 2.7, we have

$$P_{1,n}(a) = \operatorname{diag}\left(\frac{k}{n}, \ldots, \frac{k}{n}\right)$$

and thus

$$a - P_{1,n}(a) = \operatorname{diag}\left(\frac{k(n-1)}{n}, -\frac{k}{n}, -\frac{k}{n}, \ldots, -\frac{k}{n}\right) \in \mathfrak{sa}(M_n(\mathbb{C})).$$

Therefore by Example 1.6, we have

$$L_{M_n(\mathbb{C}),1}(a) = \|a - P_{1,n}(a)\|_{M_n(\mathbb{C})} = \frac{k(n-1)}{n}$$

since $\frac{k}{n}(n-1) \geqslant \frac{k}{n}$.

Now, we consider $P_{k,n}$. By Proposition 2.8, we need only examine the diagonal $(k \times k)$-blocks of a, which are

$$\operatorname{diag}(k, 0, \ldots, 0), \ \mathbf{0}, \ \ldots, \ \mathbf{0},$$

where $\mathbf{0}$ is the zero matrix. Summing these $\frac{n}{k}$ matrices, we get

$$\operatorname{diag}(k, 0, \ldots, 0),$$

and when we divide by $\frac{n}{k}$ and take the image under $\pi_{k,n}$, we arrive at

$$P_{k,n}(a) = \pi_{k,n} \operatorname{diag}\left(\frac{k^2}{n}, 0, \ldots, 0\right)$$

by Proposition 2.8. Thus

$$a - P_{k,n}(a) = \operatorname{diag}\left(\frac{k(n-k)}{n}, \ldots, 0, -\frac{k^2}{n}, \ldots, 0, \ldots, -\frac{k^2}{n}, \ldots, 0\right) \in \mathfrak{sa}(M_n(\mathbb{C}))$$

by the definition of $\pi_{k,n}$ in Notation 2.2. Since $k \mid n$ and $k < n$, we have $k \leqslant \frac{n}{2}$. This implies $n - k \geqslant k$, and so $k(n-k) \geqslant k^2$. Therefore $\frac{k(n-k)}{n} \geqslant \frac{k^2}{n}$, which implies

$$\|a - P_{k,n}(a)\|_{M_n(\mathbb{C})} = \frac{k(n-k)}{n} \quad \text{and} \quad k \cdot \|a - P_{k,n}(a)\|_{M_n(\mathbb{C})} = \frac{k^2(n-k)}{n}$$

by Example 1.6. Next, since $k \leqslant \frac{n}{2}$, we have $n-k \geqslant \frac{n}{2}$ as $k \geqslant 2$. Hence

$$k(n-k) \geqslant 2 \cdot \frac{n}{2} = n \quad \Longrightarrow \quad k(n-k) \geqslant n > n-1,$$

which means $k^2(n-k) > k(n-1)$, and finally we have

$$\frac{k^2(n-k)}{n} > \frac{k(n-1)}{n},$$

and thus, $k \cdot \|a - P_{k,n}(a)\|_{M_n(\mathbb{C})} > \|a - P_{1,n}(a)\|_{M_n(\mathbb{C})}$. Therefore

$$L_{M_n(\mathbb{C}),k}(a) = \frac{k^2(n-k)}{n}$$

and we conclude that $L_{M_n(\mathbb{C}),1} \neq L_{M_n(\mathbb{C}),k}$. □

Theorem 2.11. *Using notation from Lemma 2.10, if $k, n \in \mathbb{N}\setminus\{0\}$, $n > 2$ such that $k \mid n$ and $1 < k < n$, then the quantum compact metric spaces*

$$(M_n(\mathbb{C}), L_{M_n(\mathbb{C}),1}) \quad \textit{and} \quad (M_n(\mathbb{C}), L_{M_n(\mathbb{C}),k})$$

are not quantum isometric, and therefore

$$\Lambda\big((M_n(\mathbb{C}), L_{M_n(\mathbb{C}),1}), (M_n(\mathbb{C}), L_{M_n(\mathbb{C}),k})\big) > 0,$$

where Λ is the quantum Gromov–Hausdorff propinquity of Theorem 1.14.

Proof. By Theorem 1.14, we show for all unital *-isomorphisms $\pi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ that $L_{M_n(\mathbb{C}),1} \circ \pi \neq L_{M_n(\mathbb{C}),k}$. Let $\pi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, be a unital *-isomorphism. Therefore $L_{M_n(\mathbb{C}),1} \circ \pi = L_{M_n(\mathbb{C}),1}$ by Lemma 2.9. But, Lemma 2.10 implies that $L_{M_n(\mathbb{C}),1} \neq L_{M_n(\mathbb{C}),k}$. Thus, we conclude that $L_{M_n(\mathbb{C}),1} \circ \pi \neq L_{M_n(\mathbb{C}),k}$, and therefore $(M_n(\mathbb{C}), L_{M_n(\mathbb{C}),1})$ and $(M_n(\mathbb{C}), L_{M_n(\mathbb{C}),k})$ are not quantum isometric. □

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