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# Solving Scramble Squares puzzles with repetitions

Jason Callahan and Maria Mota

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A Scramble Squares puzzle consists of nine square pieces with half of an image on each side. To solve the puzzle the pieces are arranged in a $3 \times 3$ grid so that sides of adjacent pieces form complete images. A repetition is a half-image that appears more than once on a piece. Previous research uses a graph-theoretical approach to establish necessary and sufficient conditions for solutions without repetitions to $2 \times 2$ Scramble Squares puzzles. We use a similar approach to establish necessary and sufficient conditions for solutions with repetitions to $2 \times 2$ Scramble Squares puzzles.

## 1. Introduction

Created in the 1990s by b. dazzle, inc. (http://www.b-dazzle.com/profile.asp), a Scramble Squares puzzle consists of nine square pieces with half of an image on each side. To solve the puzzle the pieces are arranged in a $3 \times 3$ grid so that sides of adjacent pieces form complete images. Figure 1 is an example of a solution to a Scramble Squares puzzle.

Puzzle pieces can be arranged many ways. For a $3 \times 3$ puzzle there are $9!$ ways to place the pieces if all nine pieces are distinct. Each piece has four orientations, so there is a total of $4^9 \times 9!$ arrangements of the pieces with orientations.

A *pattern* is a complete image in the puzzle, such as the star, arrow, lightning bolt, and text bubble in Figure 1. Each pattern consists of two *pictures*, each half of the image, such as the two half-images that form the star in Figure 1. The *complement* of a picture is the other half of the pattern; for example, the complement of the top of the arrow is the bottom of the arrow in Figure 1. A *repetition* is a picture that appears more than once on a piece, such as the bottom half of the star that appears twice on a piece in the bottom row of Figure 1.

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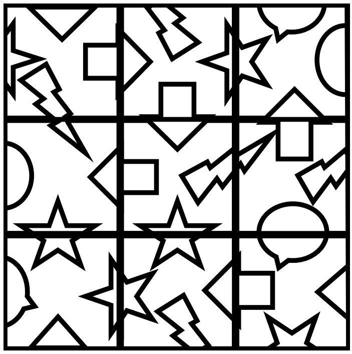

**Figure 1.** A solution to a Scramble Squares puzzle.

## 2. Scramble Squares puzzles without repetitions

For simplicity, [Mason and Zhang 2012] considers $2 \times 2$ Scramble Squares puzzles without repetitions using a graph-theoretical approach inspired by research on the Instant Insanity puzzle [de Carteblanche 1947; Grecos and Gibberd 1971; Van Deventer 1969]. For a $2 \times 2$ puzzle there are $4!$ ways to place the pieces if all four pieces are distinct. Each piece has four orientations, so there is a total of $4^4 \times 4! = 6144$ arrangements of the pieces with orientations.

To solve $2 \times 2$ Scramble Squares puzzles without repetitions, [Mason and Zhang 2012] defines the *recording graph* $G(P)$ of a puzzle $P$ as follows. The vertices of $G(P)$ correspond to the pictures in the puzzle and are arranged in two rows, where the top row represents half of each pattern and the bottom row their complements.

The edges of $G(P)$ are directed from the vertex corresponding to a picture on a piece to the vertex corresponding to the next picture clockwise on that piece, so each piece contributes four edges of a color specific to that piece. Thus, a recording graph $G(P)$ for a $2 \times 2$ puzzle $P$ contains sixteen directed edges.

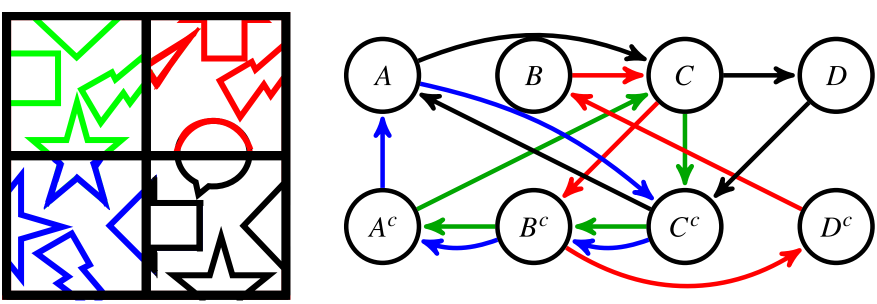


**Figure 2.** A $2 \times 2$ Scramble Squares puzzle and its recording graph.

Figure 2 shows a $2 \times 2$ Scramble Squares puzzle and its recording graph where $A$ represents the star, $B$ the lightning bolt, $C$ the arrow, and $D$ the text bubble, while green edges represent the top-left piece, red edges the top-right piece, blue edges the bottom-left piece, and black edges the bottom-right piece.

A subgraph of a recording graph is *pseudoconnected* if it is connected after identifying each vertex $X$ with its complement $X^c$. Using the recording graph, [Mason and Zhang 2012] proves the following necessary and sufficient conditions for solutions without repetitions to $2 \times 2$ Scramble Squares puzzles.

**Theorem 1.** *A subgraph $G$ of the recording graph $G(P)$ consisting of four edges corresponds to a solution without repetitions to a $2 \times 2$ puzzle $P$ if and only if it is a pseudoconnected subgraph such that*:

(1) *Each edge is a different color.*

(2) *The in-degree of each vertex equals the out-degree of its complement.*

(3) *If $X \to A \to Y$ is a directed path in $G$, then $Y = X^c$.*

## 3. Scramble Squares puzzles with repetitions

We consider $2 \times 2$ Scramble Squares puzzles with repetitions. When the repeated picture appears on adjacent sides of a piece, the repetition is represented by a loop at the vertex corresponding to the repeated picture in the puzzle's recording graph. For a repetition to be part of a solution to a $2 \times 2$ Scramble Squares puzzle, the repeated picture must appear on adjacent sides of a piece, and each of these sides must form complete images with adjacent pieces, i.e., the repetition must be in one of the four central corners in the puzzle's solution; otherwise, Theorem 1 still applies. Using a similar approach, however, we prove the following necessary and sufficient conditions for solutions with repetitions to $2 \times 2$ Scramble Squares puzzles.

**Theorem 2.** *A subgraph $G$ of the recording graph $G(P)$ consisting of four edges, at least one a loop, corresponds to a solution with repetitions to a $2 \times 2$ puzzle $P$ if and only if it is a pseudoconnected subgraph such that*:

(1) *Each edge is a different color.*

(2) *The in-degree of each vertex equals the out-degree of its complement.*

(3) *If $A$ is a repetition and $A^c \to X^c \to X$ is a directed path in $G$, then $X = A$ or $A^c$.*

*Proof.* We first prove that a subgraph $G$ corresponding to a solution with repetitions has the form described above. The subgraph contains exactly four distinctly colored edges, at least one of which is a loop, since a solution uses each of the four pieces, at least one of which involves a repetition.

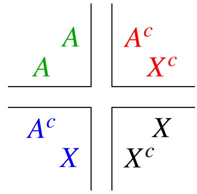


**Figure 3.** If $A$ is a repetition and $X$ is neither $A$ nor $A^c$, then $A^c \to X^c \to X$ cannot be part of a solution.

Next we prove pseudoconnectedness. In a solution to the puzzle, every pair of pieces is either adjacent or diagonally opposite. If two pieces are adjacent, then the pictures on their adjacent sides form a pattern, so the vertices corresponding to pictures on these pieces are pseudoconnected in $G$. If two pieces are diagonally opposite, there is a piece between them whose sides form a pattern with each, so the vertices corresponding to pictures on this piece form a pseudoconnected path between the vertices corresponding to pictures on the diagonally opposite pieces. Hence $G$ is pseudoconnected.

Every vertex in $G$ corresponds to a picture that is matched to its complement. Since one picture is represented by the head of a directed edge and the other by a tail, the in-degree of each vertex equals the out-degree of its complement.

Finally, suppose that $A$ is a repetition and that $G$ contains a directed path $X^c \to X$ such that $X$ is neither $A$ nor $A^c$. Without loss of generality, let the repetition $A \to A$ be the top-left piece of the puzzle solution. Then the piece represented by $X^c \to X$ must be the bottom-right piece since neither $X$ nor $X^c$ is $A^c$, which forces the remaining pieces to be $A^c \to X$ and $X^c \to A^c$ (see Figure 3). Thus, if $A$ is a repetition and $X$ is neither $A$ nor $A^c$, then $A^c \to X^c \to X$ cannot be a directed path in $G$, proving the necessity of the conditions in Theorem 2.

Conversely, we now prove that a subgraph $G$ satisfying the conditions above corresponds to a solution with at least one repetition $A$ represented by a loop in $G$. Each possible subgraph and its corresponding puzzle solution will be shown with a green loop at $A$ for its repetition in the top-left piece, a red edge for the top-right piece, a blue edge the bottom-left piece, and a black edge the bottom-right piece.

Since $G$ has a loop at $A$, three edges remain, so $A^c$ can have at most three loops, but then the in- and out-degrees of $A^c$ are 3 whereas the in- and out-degrees of $A$ are only 1, violating condition (2), so $A^c$ cannot have three or more loops.

If $A^c$ has two loops, then the in- and out-degrees of $A^c$ are both at least 2. By condition (2), $A$ must also have in- and out-degrees at least 2, so $A$ must also have two loops. This subgraph and its puzzle solution are shown in Figure 4.

Now suppose $G$ has one loop at $A^c$ and no other vertices. To satisfy condition (2), the remaining two edges must be in either direction between $A$ and $A^c$. Without

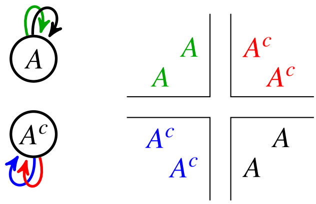


**Figure 4.** A subgraph with two loops at $A^c$ and its puzzle solution.

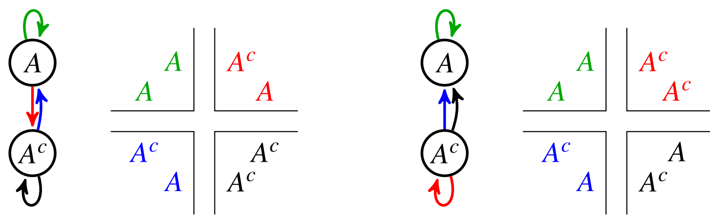


**Figure 5.** Subgraphs with one loop at $A^c$, one pattern, and their puzzle solutions.

loss of generality, by interchanging $A$ with $A^c$, the two possible such subgraphs and their corresponding puzzle solutions are shown in Figure 5.

Next suppose $G$ has one loop at $A^c$ and another vertex $B$. By pseudoconnectedness, an edge must connect $A$ or $A^c$ to $B$ or $B^c$. Without loss of generality, by interchanging $A$ with $A^c$ and/or $B$ with $B^c$, we may assume that the third edge is $A \to B^c$. Since $A$ has in-degree at least 1 and out-degree at least 2, $A^c$ must have in-degree at least 2 and out-degree at least 1 by condition (2). Therefore, $A^c$ must appear at least once more as the head of an edge. Likewise, since $B^c$ has in-degree at least 1, $B$ must have out-degree at least 1, so $B$ must appear at least once as a tail. Thus, the fourth edge must be $B \to A^c$, yielding the subgraph and its corresponding puzzle solution in Figure 6.

Now suppose $G$ has an edge between $A$ and $A^c$ but no loop at $A^c$. Then $A^c$ must connect to another vertex $B$ for its in- and out-degrees to equal the out- and

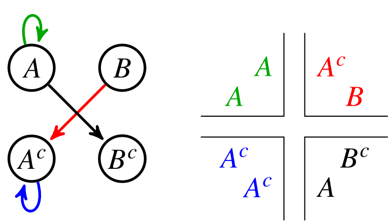


**Figure 6.** A subgraph with one loop at $A^c$, two patterns, and its puzzle solution.

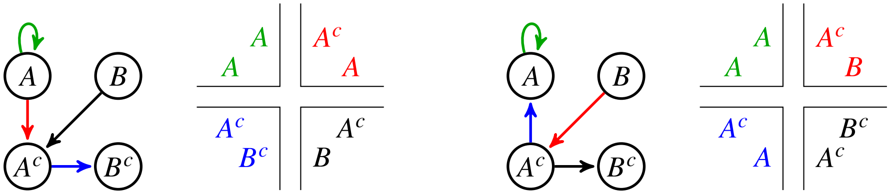


**Figure 7.** Subgraphs with an edge between $A$ and $A^c$, no loop at $A^c$, and their puzzle solutions.

in-degrees of $A$. Without loss of generality by interchanging $B$ with $B^c$, we may assume that $B \to A^c$ and $A^c \to B^c$ are the remaining edges. The direction of the edge between $A$ and $A^c$ yields the two possible subgraphs and their corresponding puzzle solutions in Figure 7.

Finally suppose $G$ has no edge between $A$ and $A^c$ and no loop at $A^c$. Then $A^c$ must connect to another vertex $B$ for its in- and out-degrees to equal the out- and in-degrees of $A$. By pseudoconnectedness, an edge must connect $A$ or $A^c$ to $B$ or $B^c$, but if $A$ connects to $B$ or $B^c$, then $G$ will need more than four edges for the in- and out-degrees of $A^c$ to equal the out- and in-degrees of $A$. Therefore, without loss of generality by interchanging $B$ with $B^c$, we may assume that $B \to A^c$ is an edge. Since $B$ has out-degree at least 1, $B^c$ must have in-degree at least 1 to satisfy condition (2). Similarly, since $A$ has in- and out-degrees at least 1, $A^c$ must also have in- and out-degrees at least 1. Thus, $A^c$ must be the tail of a third edge as follows:

- If the third edge is $A^c \to B$, then $B$ has in- and out-degrees at least 1, so the fourth edge must be a loop at $B^c$ to fulfill condition (2).

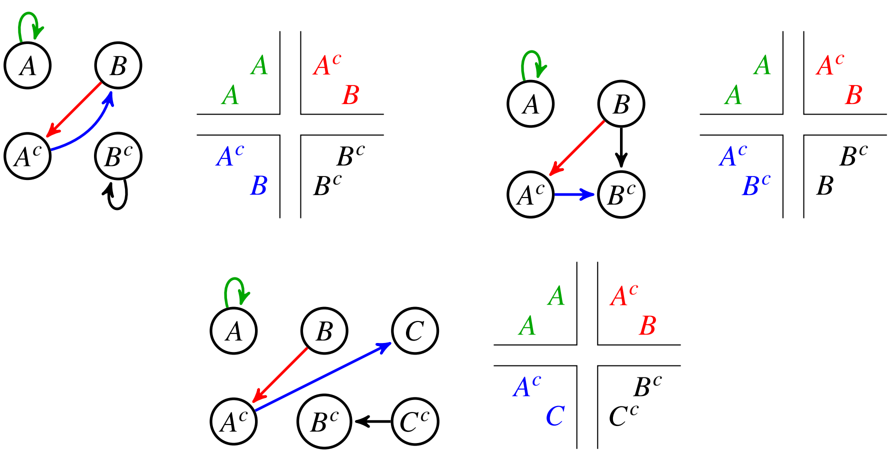


**Figure 8.** Subgraphs with no edge between $A$ and $A^c$, no loop at $A^c$, and their puzzle solutions.

- If the third edge is $A^c \to B^c$, then $B^c \to B$ cannot be an edge because a directed path $A^c \to B^c \to B$ violates condition (3), so the fourth edge must be $B \to B^c$.
- If the third edge is $A^c \to C$, another vertex, then by condition (2), $C^c$ must appear as the tail of an edge and $B^c$ as a head, so the fourth edge must be $C^c \to B^c$.

These three subgraphs and their puzzle solutions are shown in Figure 8, completing our proof that the conditions in Theorem 2 are sufficient for a subgraph to correspond to a $2 \times 2$ Scramble Squares puzzle solution with repetitions. □

## 4. Future directions

Many open questions remain to be explored. For instance, these results can be extended to $3 \times 3$ and larger Scramble Squares puzzles. Another area of exploration is the uniqueness of solutions, i.e., finding conditions under which a puzzle solution is unique. One could also explore the probability that an arbitrary puzzle has a unique solution or solutions at all.

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