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A countable space with an uncountable fundamental group

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Traditional examples of spaces that have an uncountable fundamental group (such as the Hawaiian earring space) are path-connected compact metric spaces with uncountably many points. We construct a T_0 compact, path-connected, locally path-connected topological space H with countably many points but with an uncountable fundamental group. The construction of H, which we call the "coarse Hawaiian earring" is based on the construction of the usual Hawaiian earring space $\mathbb{H} = \bigcup_{n\geq 1} C_n$ where each circle C_n is replaced with a copy of the four-point "finite circle".

1. Introduction

Since fundamental groups are defined in terms of maps from the unit interval [0, 1], students are often surprised to learn that spaces with finitely many points can be path connected and have nontrivial fundamental groups. In fact, it has been known since the 1960s that the homotopy theory of finite spaces is quite rich [McCord 1966; Stong 1966]. The algebraic topology of finite topological spaces has gained significant interest since Peter May's Research Experience for Undergraduates (REU) Summer Program at the University of Chicago in 2003; see [May 2003a; 2003b; 2003c]. For more recent theory and applications of the algebraic topology of finite spaces, we refer to [Barmak 2011; Barmak and Minian 2008; Cianci and Ottina 2016].

While it is reasonable to expect that all finite connected spaces have finitely generated fundamental groups, it is rather remarkable that for every finitely generated group G one can construct a finite space X so that $\pi_1(X, x_0) \cong G$. In fact, every finite simplicial complex is weakly homotopy equivalent to a finite space [McCord 1966]. In the same spirit, we consider fundamental groups of spaces with coarse topologies.

MSC2010: primary 54D10, 55Q52; secondary 57M05, 57M10.

Keywords: fundamental group, countable topological space, finite topological space, Hawaiian earring, coarse Hawaiian earring.

It is well known that there are connected, locally path-connected compact metric space whose fundamental groups are uncountable [Cannon and Conner 2000]. Since finite spaces can only have finitely generated fundamental groups, we must extend our view to spaces with countably many points. We prove the following theorem.

Theorem 1. *There exists a connected, locally path-connected, compact, T_0 topological space H with countably many points such that $\pi_1(H, w_0)$ is uncountable.*

Since countable simplicial complexes have countable fundamental groups, Theorem 1 shows that countable spaces need not be weakly homotopy equivalent to countable simplicial complexes. Thus the relationship between weak homotopy types of finite spaces and finite simplicial complexes cannot be fully generalized to countable spaces.

To construct the space H in Theorem 1, we must consider spaces which are not locally finite, that is, spaces which have a point such that every neighborhood of that point contains infinitely many other points. Additionally, since our example must be path connected, the following lemma demands that such a space cannot have the T_1 separation axiom.

Lemma 2. *Every countable T_1 space is totally path disconnected.*

Proof. If X is countable and T_1 and $\alpha : [0, 1] \to X$ is a nonconstant path, then $\{\alpha^{-1}(x) \mid x \in X\}$ is a nontrivial, countable partition of $[0, 1]$ into closed sets. However, it is a classical result in general topology that such a partition of $[0, 1]$ is impossible [Sierpinski 1918]. □

Ultimately, we construct the space H by modeling the construction of the traditional Hawaiian earring space $\mathbb{H}$, which is the prototypical space that fails to be semilocally simply connected and which does not admit a traditional universal covering space. The fundamental group of the Hawaiian earring is an uncountable group which plays a key role in the homotopy classification of one-dimensional Peano continua given in [Eda 2010]. Due to the similarities between $\mathbb{H}$ and H, we call H the *coarse Hawaiian earring*.

2. Fundamental groups

Let X be a topological space with basepoint $x_0 \in X$. A *path* in X is a continuous function $\alpha : [0, 1] \to X$. We say X is *path connected* if every pair of points $x, y \in X$ can be connected by a path $p : [0, 1] \to X$ with $p(0) = x$ and $p(1) = y$. All spaces in this paper will be path connected.

We say a path p is a *loop* based at x_0 if $\alpha(0) = \alpha(1)$. Let $\Omega(X, x_0)$ be the set of continuous functions $p : [0, 1] \to X$ such that $p(0) = p(1) = x_0$. Let $\alpha^- : [0, 1] \to X$ be the reverse path of α defined as $\alpha^-(t) = \alpha(1 - t)$. If α and β are paths in X

satisfying $\alpha(1)=\beta(0)$, let $\alpha\cdot\beta$ be the concatenation defined piecewise as

$$\alpha\cdot\beta(t)=\begin{cases}\alpha(2t), & 0\le t\le\frac{1}{2},\\ \beta(2t-1), & \frac{1}{2}\le t\le 1.\end{cases}$$

More generally, if $\alpha_1,\dots,\alpha_n$ is a sequence of paths such that $\alpha_i(1)=\alpha_{i+1}(0)$ for $i=1,\dots,n-1$, let $\prod_{i=1}^{n}\alpha_i$ be the path defined as α_i on the interval $[(i-1)/n, i/n]$.

Two loops α and β based at x_0 are said to be *homotopic* if there is a map $H:[0,1]\times[0,1]\to X$ such that $H(s,0)=\alpha(s)$, $H(s,1)=\beta(s)$ and $H(0,t)=H(1,t)=x_0$ for all $s,t\in[0,1]$. We write $\alpha\simeq\beta$ if α and β are homotopic. Homotopy $\simeq$ is an equivalence relation on the set of loops $\Omega(X,x_0)$. The equivalence class $[\alpha]$ of a loop α is called the *homotopy class* of α. The set of homotopy classes $\pi_1(X,x_0)=\Omega(X,x_0)/\simeq$ is called the *fundamental group* of X at x_0. It is a group when it has multiplication $[\alpha]*[\beta]=[\alpha\cdot\beta]$ and $[\alpha]^{-1}=[\alpha^-]$ is the inverse of $[\alpha]$ [Munkres 2000]. A space X is *simply connected* if X is path connected and $\pi_1(X,x_0)$ is isomorphic to the trivial group. Finally, a map $f:X\to Y$ such that $f(x_0)=y_0$ induces a well-defined homomorphism $f_*:\pi_1(X,x_0)\to\pi_1(Y,y_0)$ given by $f_*([\alpha])=[f\circ\alpha]$.

Fundamental groups are often studied using maps called covering maps. For this theory and other aspects of algebraic topology, we refer to [Munkres 2000; Spanier 1966], taking our conventions primarily from the former.

Definition 3. Let $p:\widetilde{X}\to X$ be a map. An open set $U\subseteq X$ is *evenly covered* by p if $p^{-1}(U)\subseteq\widetilde{X}$ is the disjoint union $\bigsqcup_{\lambda\in\Lambda}V_\lambda$, where V_λ is open in $\widetilde{X}$ and $p|_{V_\lambda}:V_\lambda\to U$ is a homeomorphism for every $\lambda\in\Lambda$. A *covering map* is a map $p:\widetilde{X}\to X$ such that every point $x\in X$ has an open neighborhood which is evenly covered by p. The space $\widetilde{X}$ is called a *covering space* of X. We call p a *universal covering map* if $\widetilde{X}$ is simply connected.

Remark 4. An alternative definition of universal covering map appears in [Spanier 1966] where a covering map $p:\widetilde{X}\to X$ is defined to be universal if it is an initial object in the category of coverings over X, that is, if $\widetilde{X}$ is a covering space of every covering space of X. For general spaces (even locally path-connected compact metric spaces) the two definitions differ. Example 18 in Chapter 2 of [Spanier 1966] describes the twin cone $C\mathbb{H}\vee C\mathbb{H}$ over the Hawaiian earring $\mathbb{H}$ (sometimes called the Griffiths twin cone), which is a non-simply connected space whose only covering space is itself. Thus the identity map of the twin cone is a universal covering in the sense of [Spanier 1966] but not in the sense of [Munkres 2000]. On the other hand, one can use the covering space theory developed in [Munkres 2000] to confirm that the two definitions of "universal covering map" agree when X is locally path connected and semilocally simply connected. Since we only consider covering maps over such spaces in this paper, we will not need to worry about the difference in the definitions.

An important property of covering maps is that for every path $\alpha : [0, 1] \to X$ such that $\alpha(0) = x_0$ and point $y \in p^{-1}(x_0)$ there is a unique path $\tilde{\alpha}_y : [0, 1] \to \widetilde{X}$ (called a *lift* of α) such that $p \circ \tilde{\alpha}_y = \alpha$ and $\tilde{\alpha}_y(0) = y$.

Lemma 5 [Munkres 2000, Theorem 54.6]. *A covering map $p : \widetilde{X} \to X$ such that $p(y) = x_0$ induces an injective homomorphism $p_* : \pi_1(\widetilde{X}, y) \to \pi_1(X, x_0)$. If $\alpha : [0, 1] \to X$ is a loop based at x_0, then $[\alpha] \in p_*(\pi_1(\widetilde{X}, y))$ if and only if $\tilde{\alpha}_y(1) = y$.*

A covering map $p : \widetilde{X} \to X$ induces a *lifting correspondence map* $\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ from the fundamental group of X to the fiber over x_0 defined by the formula $\phi([\alpha]) = \tilde{\alpha}_y(1)$.

Lemma 6 [Munkres 2000, Theorem 54.4]. *If $p : \widetilde{X} \to X$ is a covering map, then the lifting correspondence $\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ is surjective. If p is a universal covering map, then p is bijective.*

Example 7. Let $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ be the unit circle and $b_0 = (1, 0)$. The exponential map $\epsilon : \mathbb{R} \to S^1$, $\epsilon(t) = (\cos(2\pi t), \sin(2\pi t))$, defined on the real line is a covering map such that $\epsilon^{-1}(b_0) = \mathbb{Z}$ is the set of integers. The lifting correspondence for this covering map $\phi : \pi_1(S^1, b_0) \to \epsilon^{-1}(b_0) = \mathbb{Z}$ is an isomorphism when $\mathbb{Z}$ is the additive group of integers. See [Munkres 2000, Theorem 54.5] for a proof.

3. Some basic finite spaces

A *finite space* is a topological space $X = \{x_1, x_2, \ldots, x_n\}$ with finitely many points.

Example 8. The *coarse interval* is the three-point space $I = \{0, \frac{1}{2}, 1\}$ with topology generated by the basic sets the sets $\{0\}$, $\{1\}$, and I (See Figure 1). In other words, the topology of I is $T_I = \{I, \{0\}, \{1\}, \{0, 1\}, \varnothing\}$.

The coarse interval clearly satisfies the T_0 separation axiom. It is also path connected since we can define a continuous surjection $p : [0, 1] \to I$ by

$$p(t) = \begin{cases} 0, & t \in [0, \frac{1}{2}), \\ \frac{1}{2} & t = \frac{1}{2}, \\ 1, & t \in (\frac{1}{2}, 1], \end{cases}$$

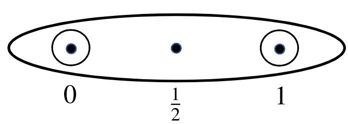

Figure 1. The coarse interval I. A basic open set is illustrated here as a bounded region whose interior contains the points of the set.

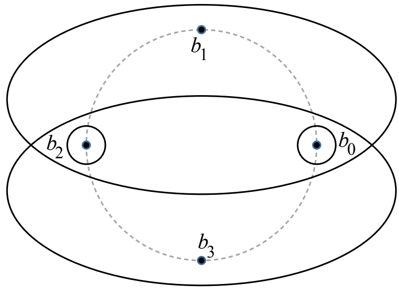

Figure 2. The coarse circle S and it's basic open sets.

and the continuous image of a path-connected space is path connected. A space X is contractible if the identity map $\mathrm{id} : X \to X$ is homotopic to a constant map $X \to X$. Every contractible space is simply connected.

Lemma 9. *The coarse interval I is contractible.*

Proof. To show I is contractible we define a continuous map $G : I \times [0, 1] \to I$ such that $G(x, 0) = x$ for $x \in I$ and $G(x, 1) = \frac{1}{2}$. The set $C = \left(\{0, 1\} \times \left[\frac{1}{2}, 1\right]\right) \cup \left(\left\{\frac{1}{2}\right\} \times [0, 1]\right)$ is closed in $I \times [0, 1]$. Define G by

$$G(s, t) = \begin{cases} 0, & (s, t) \in \{0\} \times \left[0, \frac{1}{2}\right), \\ \frac{1}{2}, & (s, t) \in C, \\ 1, & (s, t) \in \{1\} \times \left[0, \frac{1}{2}\right). \end{cases}$$

This function is well-defined and continuous since $\{0\}$ and $\{1\}$ are open in I. □

Corollary 10. *I is simply connected.*

For $n = 0, 1, 2, 3$, let $b_n = \left(\cos\left(\frac{1}{2}n\pi\right), \sin\left(\frac{1}{2}n\pi\right)\right) \in S^1$ be the points of the unit circle on the coordinate axes; i.e., $b_0 = (1, 0)$, $b_1 = (0, 1)$, $b_2 = (-1, 0)$, and $b_3 = (0, -1)$.

Example 11. The *coarse circle* is the four-point set $S = \{b_i \mid i = 0, 1, 2, 3\}$ with the topology generated by the basic sets $\{b_0, b_1, b_2\}$, $\{b_2, b_3, b_0\}$, $\{b_0\}$, and $\{b_2\}$ (see Figure 2). The entire topology of S may be written as

$$T_S = \{S, \{b_0, b_1, b_2\}, \{b_2, b_3, b_0\}, \{b_0, b_2\}, \{b_0\}, \{b_2\}, \varnothing\}.$$

Observe that the open sets $U_1 = \{b_0, b_1, b_2\}$ and $U_2 = \{b_2, b_3, b_0\}$ are homeomorphic to I when they are given the subspace topology. Since S is the union of two path-connected subsets with nonempty intersection, S is also path connected.

Remark 12. The spaces I and S have appeared many times in the literature. The space S is sometimes called the "finite circle". We use the term "coarse circle" since we are considering it within the broader context of infinite spaces with non-T_1

topologies. The space S is the smallest finite space having the same weak homotopy type as the usual circle S^1. In fact, this is a special case of a more general result on minimal $(2n+2)$-point models of n-spheres (see [Barmak 2011, Chapter 3]): there exists a space with $2n+2$ points weakly homotopy equivalent to S^n and moreover any finite space that is weakly homotopy equivalent to the n-sphere S^n must have at least $2n+2$ points.

4. The coarse line as a covering space

As indicated in Remark 12, it follows from the much more sophisticated theory in [Stong 1966] that S has the weak homotopy type of S^1. To keep the current paper self-contained, we devote this section to a direct proof of the fact that $\pi_1(S, b_0)$ is isomorphic to the infinite cyclic group $\mathbb{Z}$, i.e., the additive group of integers, by constructing a map $g : S^1 \to S$ that induces an isomorphism on fundamental groups.

Example 13. The *coarse line* is the set $L = \left\{\frac{1}{4}n \in \mathbb{R} \,\middle|\, n \in \mathbb{Z}\right\}$ with the topology generated by the basis consisting of the sets $A_n = \left\{\frac{1}{2}n\right\}$ and $B_n = \left\{\frac{1}{2}n, \frac{1}{4}(2n+1), \frac{1}{2}(n+1)\right\}$ for each $n \in \mathbb{Z}$ (see Figure 3). Even though L is not a finite space, it is a countable space with a T_0 but non-T_1 topology.

Lemma 14. *L is simply connected.*

Proof. The set $L_n = L \cap \left[-\frac{1}{2}n, \frac{1}{2}n\right]$ is open in L since it is the union of the basic sets $B_k = \left\{\frac{1}{2}k, \frac{1}{4}(2k+1), \frac{1}{2}(k+1)\right\}$, $k = -n, \dots, n-1$, with the subspace topology of L.

It follows from the classical van Kampen theorem [Munkres 2000, Theorem 70.2] that if $X = U \cup V$, where U, V are open in X and U, V, $U \cap V$ are simply connected, then X is simply connected. We will apply this fact inductively to prove that L_n is simply connected for all $n \geq 1$.

Since $B_k \cong I$ for each k, we know B_k is simply connected for each k. Observe that $L_1 = B_{-1} \cup B_0$, where $B_{-1} \cap B_0 = \{0\}$ is simply connected since it only has one point. Thus L_1 is simply connected by the van Kampen theorem. Now suppose L_n is simply connected. Since L_n, B_n, and $L_n \cap B_n = \left\{\frac{1}{2}n\right\}$ are all simply connected, $L_n \cup B_n$ is simply connected by the van Kampen theorem. Similarly, since $L_n \cup B_n$, B_{-n-1}, and $(L_n \cup B_n) \cap B_{-k-1} = \left\{-\frac{1}{2}n\right\}$ are all simply connected, $L_{n+1} = B_{-n-1} \cup L_n \cup B_n$ is simply connected by the van Kampen theorem. Thus L_n is simply connected for all $n \geq 1$.

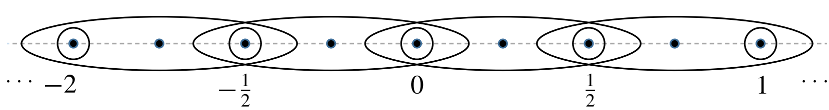

Figure 3. The basic open sets generating the topology of the coarse line L.

Since L is the union of the path-connected sets L_n, all of which contain 0, it follows that L is path connected. Now suppose $\alpha : [0, 1] \to L$ is a path such that $\alpha(0) = \alpha(1)$. Since $[0, 1]$ is compact, the image $\alpha([0, 1])$ is compact. But $\{L_n \mid n \geq 1\}$ is an open cover of L such that $L_n \subseteq L_{n+1}$. Since α must have image in a finite subcover of $\{L_n \mid n \geq 1\}$, we must have $\alpha([0, 1]) \subseteq L_n$ for some n. But L_n is simply connected, showing that α is homotopic to the constant loop at 0. This proves $\pi_1(L, 0)$ is the trivial group; i.e., L is simply connected. □

Just like the usual covering map $\epsilon : \mathbb{R} \to S^1$ used to compute $\pi_1(S^1, b_0)$, we define a similar covering map in the coarse situation.

Example 15. Consider the function $p : L \to S$ from the coarse line to the coarse circle which is the restriction of the covering map $\epsilon : \mathbb{R} \to S^1$. More directly, define $p\left(\frac{1}{4}n\right) = b_{n \bmod 4}$. We check that the preimage of each basic open set in S can be written as a union of basic open sets in L. Since

- $p^{-1}(\{b_0\}) = \mathbb{Z} = \bigcup_{k \in \mathbb{Z}} A_{2k}$,
- $p^{-1}(\{b_2\}) = \frac{1}{2} + \mathbb{Z} = \bigcup_{k \in \mathbb{Z}} A_{2k+1}$,
- $p^{-1}(U_1) = \bigcup_{k \in \mathbb{Z}} B_{2k}$,
- $p^{-1}(U_2) = \bigcup_{k \in \mathbb{Z}} B_{2k+1}$,

we can conclude that p is continuous.

Lemma 16. *The function $p : L \to S$ is a covering map.*

Proof. We claim that the sets U_1, U_2 are evenly covered by p. Notice that $p^{-1}(U_1) = \bigcup_{k \in \mathbb{Z}} B_{2k}$ is a disjoint union where each B_{2k} is open. Recall that both B_{2k} and U_1 are homeomorphic to I; specifically $p|_{B_{2k}} : B_{2k} \to U_1$ is a homeomorphism. Thus U_1 is evenly covered. Similarly, $p^{-1}(U_2)$ is the disjoint union $\bigcup_{k \in \mathbb{Z}} B_{2k+1}$ where each B_{2k+1} is open and is mapped homeomorphically on to U_2 by p. □

Since $p : L \to S$ is a covering map and L is simply connected, p is a universal covering map. The proof of the following theorem is similar to the proof that the lifting correspondence for ϵ is a group isomorphism. We remark that even though L is not a topological group, the shift map $\sigma_n : L \to L$, $\sigma(t) = t + n$, is a homeomorphism satisfying $p \circ \sigma_n = p$ for each $n \in \mathbb{Z}$.

Theorem 17. *The lifting correspondence $\phi : \pi_1(S, b_0) \to p^{-1}(b_0) = \mathbb{Z}$ is a group isomorphism where $\mathbb{Z}$ has the usual additive group structure.*

Proof. Since $p : L \to S$ is a covering map and L is simply connected, ϕ is bijective by Lemma 6. Suppose $\alpha, \beta : [0, 1] \to S$ are loops based at b_0. Respectively, let $\tilde{\alpha}_0 : [0, 1] \to L$ and $\tilde{\beta}_0 : [0, 1] \to L$ be the unique lifts of α and β starting at 0. Since $\tilde{\alpha}_0(1) \in p^{-1}(b_0) = \mathbb{Z}$, we have $\phi([\alpha]) = \tilde{\alpha}_0(1) = n$ for some integer n. Similarly, $\phi([\beta]) = \tilde{\beta}_0(1) = m$ for some integer m.

Consider the concatenated path $\gamma = \tilde{\alpha}_0 \cdot (\sigma_n \circ \tilde{\beta}_0) : [0, 1] \to L$ from 0 to $m+n$. Since $p \circ \sigma_n = p$, we have

$$\begin{aligned} p \circ \gamma &= p \circ (\tilde{\alpha}_0 \cdot (\sigma_n \circ \tilde{\beta}_0)) \\ &= (p \circ \tilde{\alpha}_0) \cdot (p \circ \sigma_n \circ \tilde{\beta}_0) \\ &= (p \circ \tilde{\alpha}_0) \cdot (p \circ \tilde{\beta}_0) = \alpha \cdot \beta, \end{aligned}$$

which means that γ is a lift of $\alpha \cdot \beta$ starting at 0. Since lifts are unique, this means $\gamma = \widetilde{\alpha \cdot \beta}_0$. It follows that $\phi([\alpha][\beta]) = \phi([\alpha \cdot \beta]) = \widetilde{\alpha \cdot \beta}_0(1) = \gamma(1) = m+n$. This proves ϕ is a group homomorphism. □

Both $\pi_1(S^1, b_0)$ and $\pi_1(S, b_0)$ are isomorphic to the infinite cyclic group $\mathbb{Z}$. In fact, we can define maps which induce the isomorphism between the two fundamental groups.

Let $f : \mathbb{R} \to L$ be the map defined so that $f\big(\big(\frac{1}{2}n - \frac{1}{4}, \frac{1}{2}n + \frac{1}{4}\big)\big) = \frac{1}{2}n$ and $f\big(\frac{1}{2}n + \frac{1}{4}\big) = \frac{1}{2}n + \frac{1}{4}$ for each $n \in \mathbb{Z}$. Notice that $p \circ f$ is constant on each fiber $\epsilon^{-1}(x)$, $x \in S^1$. Therefore, there is an induced map $g : S^1 \to S$ such that $g \circ \epsilon = p \circ f$.

As mentioned at the end of the previous section, the following proposition is a special case of more general results on weak homotopy types of finite spaces in [Stong 1966].

Proposition 18. *The induced homomorphism $g_* : \pi_1(S^1, b_0) \to \pi_1(S, b_0)$ is a group isomorphism.*

Proof. Recall that $\epsilon^{-1}(b_0) = \mathbb{Z}$ and $p^{-1}(b_0) = \mathbb{Z}$ and notice that the restriction to the fibers $f|_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ is the identity map. Let $i : [0, 1] \to \mathbb{R}$ be the inclusion and note $f \circ i : [0, 1] \to L$ is a path from 0 to 1. The group $\pi_1(S^1, b_0)$ is freely generated by the homotopy class of $\alpha = \epsilon \circ i$ and $\pi_1(S, b_0)$ is freely generated by the homotopy class of $p \circ f \circ i$. Since $g_*([\epsilon \circ i]) = [g \circ \epsilon \circ i] = [p \circ f \circ i]$, the homomorphism g_* maps one free generator to the other and it follows that g_* is an isomorphism. □

5. The coarse Hawaiian earring

Let $C_n = \{(x, y) \in \mathbb{R}^2 \mid (x - 1/n)^2 + y^2 = 1/n^2\}$ be the circle of radius $1/n$ centered at $(1/n, 0)$. The *Hawaiian earring* is the countably infinite union $\mathbb{H} = \bigcup_{n \geq 1} C_n$ of these circles over the positive integers (see Figure 4, left). We construct our countable version of $\mathbb{H}$ by replacing the usual circle with the coarse circle studied in the previous sections.

Let $w_0 = (0, 0)$, and for integers $n \geq 1$ define $x_n = (1/n, -1/n)$, $y_n = (2/n, 0)$, and $z_n = (1/n, 1/n)$. Let $D_n = \{w_0, x_n, y_n, z_n\}$ and $H = \bigcup_{n \geq 1} D_n$. Note that H is a countable subset of $\mathbb{H}$ (see Figure 4, right).

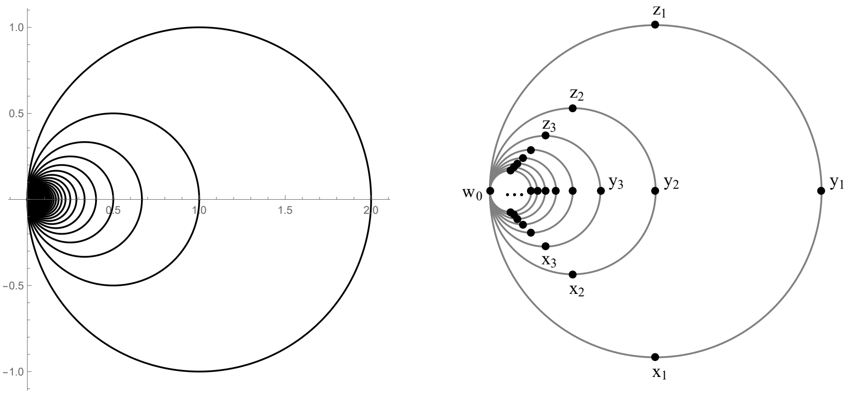

Figure 4. Left: the Hawaiian earring $\mathbb{H}$. Right: the underlying set of H as a subset of $\mathbb{H}$. The intersection of the n-th circle C_n and H is the four-point set $D_n = \{w_0, x_n, y_n, z_n\}$.

Proposition 19. *Let $\mathscr{B}$ be the collection of subsets of H of the form $\{x_n\}$, $\{z_n\}$, $\{x_n, y_n, z_n\}$, and $V_n = \bigcup_{j \ge n} D_j \cup \{x_n \mid n \ge 1\} \cup \{z_n \mid n \ge 1\}$ for $n \ge 1$. Then $\mathscr{B}$ is a basis for a topology on H.*

Proof. Since $H = V_1$, it is clear that every element of H is contained in at least one element of $\mathscr{B}$. Suppose $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathscr{B}$. We must show there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. We complete the proof by defining B_3 for all possible cases of intersection:

(1) If one of B_1 or B_2 is of the form $\{x_n\}$ or $\{z_n\}$ then we may take B_3 to be this singleton.

(2) If $B_1 = \{x_m, y_m, z_m\}$ and $B_2 = \{x_n, y_n, z_n\}$, then we must have $n = m$ since these sets are disjoint if $n \ne m$. Set $B_3 = \{x_m, y_m, z_m\}$.

(3) Note that $V_n \subseteq V_m$ if $n \ge m$. Thus if $B_1 = V_m$ and $B_2 = V_n$, we may set $B_3 = V_m \cap V_n = V_{\max\{m,n\}} \in \mathscr{B}$.

(4) If $B_1 = \{x_m, y_m, z_m\}$ and $B_2 = V_n$, then $B_1 \cap B_2 = \{x_m, z_m\}$ and we may take B_3 to be the singleton (either $\{x_m\}$ or $\{z_m\}$) containing x. □

Definition 20. The *coarse Hawaiian earring* is the set H with the topology generated by the basis consisting of subsets of the form $\{x_n\}$, $\{z_n\}$, $\{x_n, y_n, z_n\}$, and V_n for $n \ge 1$.

A topological space whose topology is closed under arbitrary intersection is called an *Alexandroff space* [Arenas 1999]. Such spaces were introduced by P. Alexandroff [1937] and may also appear in modern literature under the name “A-space” or “Alexandroff-discrete space”. Regarding the coarse Hawaiian earring, notice that

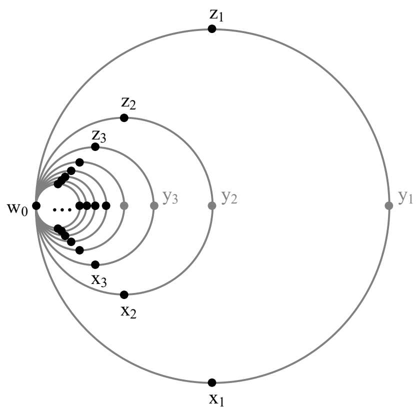

Figure 5. The basic open neighborhood V_5 of w_0 contains all points of H except y_1, y_2, y_3, y_4, which are shaded lighter. In particular, V_5 contains the four-point set D_n for all $n \geq 5$.

for all $n \geq 2$, the basic open neighborhood $V_n = H \backslash \{y_1, \ldots, y_{n-1}\}$ contains all but finitely many of the coarse circles D_j (see Figure 5). These sets form a neighborhood base at w_0 so that H is not an Alexandroff space.

Remark 21. Notice that the four-point subset $D_n \subset H$ is homeomorphic to the coarse circle S when equipped with the subspace topology inherited from H. An explicit homeomorphism $S \to D_n$ is given by taking $b_0 \mapsto x_n$, $b_1 \mapsto w_0$, $b_2 \mapsto z_n$, and $b_3 \mapsto y_n$.

Proposition 22. *H is a path-connected, locally path-connected, compact, T_0 space which is not T_1.*

Proof. Since D_n is homeomorphic to S, we know D_n is path connected for all $n \geq 1$. Moreover, since $w_0 \in \bigcap_{n \geq 1} D_n$ and $H = \bigcup_{n \geq 1} D_n$, it follows that H is path connected. To see that H is locally path connected, we check that every basic open set is path connected. Certainly, $\{x_n\}$ and $\{z_n\}$ are path connected. Since $\{x_n, y_n, z_n\}$ is homeomorphic to I when it is given the subspace topology of H, this basic open set is path connected. Additionally, the subspace $\{w_0, x_n, y_n\} \subseteq H$ is homeomorphic to I and is path connected. Therefore, since V_n is the union $\bigcup_{j \geq n} D_n \cup \bigcup_{n \geq 1} \{w_0, x_n, y_n\}$ of sets all of which are path connected and contain w_0, we can conclude that V_n is path connected. This proves H is locally path connected.

To see that H is compact let $\mathscr{U}$ be an open cover of H. Since the only basic open sets containing w_0 are the sets V_n, there must be a $U_0 \in \mathscr{U}$ such that $w_0 \in V_n \subseteq U_0$ for some n. For $i = 1, \ldots, n-1$, find a set $U_i \in \mathscr{U}$ such that $y_i \in U_i$. Now $\{U_0, U_1, \ldots, U_{n-1}\}$ is a finite subcover of $\mathscr{U}$. This proves H is compact.

To see that H is T_0, we pick two points $a, b \in H$. If $a = w_0$ and $b = y_n$, then $a \in V_{n+1}$ but $b \notin V_{n+1}$. If $a = w_0$ and $b \in \{x_n, z_n\}$, then b lies in the open set

$\{x_n, y_n, z_n\}$ but a does not. If $a \in \{x_n, z_n\}$ and $a \neq b$, then $\{a\}$ is open and does not contain b. This concludes all the possible cases of distinct pairs of points in H, proving that H is T_0.

Lastly, H is not T_1 since the every open neighborhood V_n of w_0 contains the infinite set $\bigcup_{n\geq 1}\{w_0, x_n, z_n\}$. □

Since $D_n \cong S$, we have by Theorem 17 that $\pi_1(D_n, w_0) \cong \mathbb{Z}$ for all $n \geq 1$. Recall that if A is a subspace of X, then a retraction is a map $r : X \to A$ such that the restriction $r|_A : A \to A$ is the identity map.

Proposition 23. *For each $n \geq 1$, the function $r_n : H \to D_n$ which is the identity on D_n and collapses $\bigcup_{j\neq n} D_j$ to w_0 is a retraction.*

Proof. Since D_n is a subspace of H, it suffices to show r_n is continuous. We have

$$r_n^{-1}(\{x_n\}) = \{x_n\}, \quad r_n^{-1}(\{x_n, y_n, z_n\}) = \{x_n, y_n, z_n\},$$
$$r_n^{-1}(\{z_n\}) = \{z_n\}, \quad r_n^{-1}(\{w_0, x_n, y_n\}) = \{x_n\} \cup \{y_n\} \cup V_{n+1} \cup \bigcup_{j<n}\{x_j, y_j, z_j\}.$$

Since the pullback of each basic open set in D_n is the union of basic open sets in H, r_n is continuous. □

Corollary 24. *H is not semilocally simply connected at w_0.*

Proof. Fix $n \geq 1$. We show that V_n contains a loop α which is not null-homotopic in H. Let $\alpha : [0, 1] \to D_n$ be any loop based at w_0 such that $[\alpha]$ is not the identity element of $\pi_1(D_n, w_0)$. Let $i : D_n \to H$ be the inclusion map so that $r_n \circ i = \mathrm{id}_{D_n}$ is the identity map. Since π_1 is a functor, $(r_n)_* \circ i_* = (r_n \circ i)_* = \mathrm{id}_{\pi_1(D_n, w_0)}$ is the identity homomorphism of $\pi_1(D_n, w_0)$. In particular, $i \circ \alpha$ is a loop in H with image in $D_n \subseteq V_n$ such that $(r_n)_*([i \circ \alpha]) = [\alpha]$ is not the identity of $\pi_1(D_n, w_0)$. Since homomorphisms preserve identity elements, $[i \circ \alpha]$ cannot be the identity element of $\pi_1(H, w_0)$. □

Definition 25. The *infinite product* of a sequence of groups $G_1, G_2, \ldots$ is denoted by $\prod_{n\geq 1} G_n$ and consists of all infinite sequences $(g_1, g_2, \ldots)$ with $g_n \in G_n$ for each $n \geq 1$. Group multiplication and inversion are evaluated componentwise. If $G_n = \mathbb{Z}$ for each $n \geq 1$, then the group $\prod_{n\geq 1} \mathbb{Z}$ consisting of sequences $(n_1, n_2, \ldots)$ of integers is called the *Baer–Specker group*.

Infinite products of groups have the useful property that if G is a fixed group and $f_n : G \to G_n$ is a sequence of homomorphisms, then there is a well-defined homomorphism $f : G \to \prod_{n\geq 1} G_n$ given by $f(g) = (f_1(g), f_2(g), \ldots)$.

Lemma 26. *The infinite product $\prod_{n\geq 1} \pi_1(D_n, w_0)$ is uncountable.*

Proof. If each G_n is nontrivial, then G_n contains at least two elements. Therefore the product $\prod_{n\geq 1} G_n$ is uncountable since the Cantor set $\{0, 1\}^{\mathbb{N}} = \prod_{n\geq 1}\{0, 1\}$ can be injected as a subset. In particular, the Baer–Specker group is uncountable. Since $\pi_1(D_n, w_0) \cong \mathbb{Z}$ for each $n \geq 1$, the infinite product $\prod_{n\geq 1} \pi_1(D_n, w_0)$ is isomorphic to the Baer–Specker group and is therefore uncountable. □

Let $\lambda_n : [0, 1] \to D_n$ be the loop defined as

$$\lambda_n(t) = \begin{cases} w_0, & t \in \{0, 1\}, \\ x_n, & t \in \left(0, \frac{1}{2}\right), \\ y_n, & t = \frac{1}{2}, \\ z_n, & t \in \left(\frac{1}{2}, 1\right). \end{cases}$$

This function is continuous and therefore a loop in D_n. In particular, our description of the universal covering of S in the previous section shows that the homotopy class $[\lambda_n]$ is a generator of the cyclic group $\pi_1(D_n, b_0)$.

Definition 27. Suppose for each $n \geq 1$ we have a continuous loop $\alpha_n : [0, 1] \to H$ based at w_0 with image in D_n. The *infinite concatenation* of this sequence of loops is the loop $\alpha_\infty : [0, 1] \to H$ defined as follows: for each $n \geq 1$, the restriction of α_∞ to $[(n-1)/n, n/(n+1)]$ is the path α_n and $\alpha_\infty(1) = w_0$.

Lemma 28. *The loop α_∞ is continuous and $[r_n \circ \alpha_\infty] = [\alpha_n]$ for all $n \geq 1$.*

Proof. Since each loop α_n is continuous and each concatenation $\alpha_n \cdot \alpha_{n+1}$ is continuous, it is enough to show that α_∞ is continuous at 1. Consider a basic open neighborhood V_n of $\alpha_\infty(1) = w_0$. Since α_i has image in V_n for each $i \geq n$, we have $\alpha_\infty([(n-1)/n, 1]) \subseteq V_n$. In particular, $1 \in ((n-1)/n, 1] \subseteq f^{-1}(V_n)$. This proves that α_∞ is continuous.

Notice that $r_1 \circ \alpha_\infty$ is defined to be α_1 on $\left[0, \frac{1}{2}\right]$ and is constant at w_0 on $\left[\frac{1}{2}, 1\right]$. If $n \geq 2$, then $r_n \circ \alpha_\infty$ is defined as α_n on $[(n-1)/n, n/(n+1)]$ and is constant at w_0 on $[0, (n-1)/n] \cup [n/(n+1), 1]$. Thus for all $n \geq 1$, we have $r_n \circ \alpha_\infty$ is homotopic to α_n. □

Theorem 29. *The fundamental group $\pi_1(H, w_0)$ is uncountable.*

Proof. We have a sequence of homomorphisms $(r_n)_* : \pi_1(H, w_0) \to \pi_1(D_n, w_0)$ induced by the retractions r_n. Together, these induce a homomorphism $r : \pi_1(H, w_0) \to \prod_{n\geq 1} \pi_1(D_n, w_0)$ given by

$$r([\alpha]) = ((r_1)_*([\alpha]), (r_2)_*([\alpha]), \dots) = ([r_1 \circ \alpha], [r_2 \circ \alpha], \dots).$$

By Lemma 26, the infinite product $\prod_{n\geq 1} \pi_1(D_n, w_0)$ is uncountable. We claim that r is onto.

Suppose $(g_1, g_2, \dots) \in \prod_{n\geq 1} \pi_1(D_n, w_0)$, where $g_n \in \pi_1(D_n, w_0)$. Since g_n is an element of the infinite cyclic group $\pi_1(D_n, w_0)$ generated by $[\lambda_n]$, we may write $g_n = [\lambda_n]^{m_n}$ for some integer $m_n \in \mathbb{Z}$.

For each $n \geq 1$, define the loop α_n by

$$\alpha_n = \begin{cases} \prod_{i=1}^{m_n} \lambda_n & \text{if } m_n > 0, \\ \text{constant at } w_0 & \text{if } m_n = 0, \\ \prod_{i=1}^{|m_n|} \lambda_n^- & \text{if } m_n < 0. \end{cases}$$

Notice that α_n is defined so that $g_n = [\lambda_n]^{m_n} = [\alpha_n]$. Let $\alpha_\infty : [0, 1] \to H$ be the loop based at w_0 which is the infinite concatenation as in Definition 27. By Lemma 28, we have $[r_n \circ \alpha_\infty] = [\alpha_n] = g_n$ for each $n \geq 1$. Therefore, $r([\alpha_\infty]) = (g_1, g_2, \dots)$. This proves that r is onto.

Thus, since $\pi_1(H, b_0)$ surjects onto an uncountable group, it must also be uncountable. □

We conclude that there is a T_0 space with countably many points but which has an uncountable fundamental group.

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