

# involve

a journal of mathematics

Toeplitz subshifts with trivial centralizers  
and positive entropy

Kostya Medynets and James P. Talisse





# Toeplitz subshifts with trivial centralizers and positive entropy

Kostya Medynets and James P. Talisse

(Communicated by David Royal Larson)

Given a dynamical system  $(X, G)$ , the centralizer  $C(G)$  denotes the group of all homeomorphisms of  $X$  which commute with the action of  $G$ . This group is sometimes called the automorphism group of the dynamical system  $(X, G)$ . We generalize the construction of Bułatek and Kwiatkowski (1992) to  $\mathbb{Z}^d$ -Toeplitz systems by identifying a class of  $\mathbb{Z}^d$ -Toeplitz systems that have trivial centralizers. We show that this class of  $\mathbb{Z}^d$ -Toeplitz systems with trivial centralizers contains systems with positive topological entropy.

## 1. Introduction

Toeplitz dynamical systems were first introduced by Jacobs and Keane [1969]. They provided a classical definition for a Toeplitz sequence over  $\{0, 1\}$ . Markley [1975] studied these sequences and showed the equivalence of various definitions of them. The orbit closure of a Toeplitz sequence is regarded as a Toeplitz flow. Markley and Paul [1979] showed that these flows were exactly almost one-to-one extensions of odometers, or the group of  $p$ -adic integers. See [Hewitt and Ross 1979] for a general discussion of the group-theoretic properties of the group of  $p$ -adic integers. For a general survey of symbolic dynamics, we refer the reader to [Kitchens 1998]. For a good survey on  $\mathbb{Z}$ -odometers and Toeplitz flows, the reader is referred to [Downarowicz 2005]. Recently the definition of Toeplitz flows was extended to flows over  $\mathbb{Z}^d$  by Cortez [2006], and then to flows over general groups in [Cortez and Petite 2008; Krieger 2010].

The centralizer of a dynamical system is the group of all homeomorphisms of the system which commute with the group action. Sometimes called the *automorphism group* of the dynamical system in the literature, the centralizer of a dynamical system has an intricate relationship with its parent dynamical system. For example,

---

*MSC2010:* 37B05, 37B40, 37B50.

*Keywords:* topological dynamics, symbolic dynamics, automorphism group, centralizer, topological entropy.

The research of Medynets was supported by NSA Young Investigator Grant H98230258656.

in [Boyle et al. 1988], Boyle, Lind and Rudolph studied the centralizers of shifts of finite type and showed that they are countable, residually finite and contain the free group on two generators. Several results have been shown by Cyr and Kra [2015; 2016a; 2016b] which relate varying levels of complexity of symbolic dynamical systems to algebraic properties of their centralizers. We notice that systems with positive entropy tend to have very large centralizers. For example, the centralizer of the full shift contains every finite group and the free group on two generators. On the other hand, Donoso, Durand and Petite [Donoso et al. 2016] showed that some classes of low complexity symbolic dynamical systems have very small centralizers, in the sense that they consist only of powers of  $T$ . Bułatek and Kwiatkowski [1990; 1992] studied the centralizer of a class of high-complexity Toeplitz systems. The centralizer of multidimensional symbolic dynamical systems was studied by Hochman [2010]. For example, he showed that the centralizer of a positive-entropy multidimensional shift of finite type contains a copy of every finite group.

The main question this paper seeks to answer is whether there are multidimensional systems with a trivial centralizer and positive entropy. Following the ideas of [Bułatek and Kwiatkowski 1992], which developed this result in one dimension, we establish this result with a constructive proof. We note that there are several constructions of *one-dimensional* Toeplitz systems with trivial centralizers and positive entropy; see, for example, [Donoso et al. 2017].

In Section 3 we present main facts with proofs regarding general  $G$ -odometers, where  $G$  is a residually finite group. For the reader's convenience, we include the proofs, otherwise scattered across multiple sources. In particular we show that the centralizer group of  $\mathbb{Z}^d$ -Toeplitz systems embeds into the centralizer group of its maximal equicontinuous factor, which is a  $\mathbb{Z}^d$ -odometer, and so is Abelian. This result was originally established in [Auslander 1963, Theorem 9] using the techniques of enveloping semigroups. The proof we present in this note follows the approach developed in [Olli 2013].

In Section 4, we construct a class of  $\mathbb{Z}^d$ -Toeplitz systems that have trivial centralizers. Then in Section 5, we show that this class contains systems of positive entropy, and we provide an explicit construction of a two-dimensional Toeplitz system of positive entropy.

## 2. Definitions and background

By a *dynamical system* we mean a pair  $(X, G)$ , where  $X$  is a compact topological space and  $G$  is a countable discrete group acting on  $X$  by homeomorphisms. The action of a group element  $g \in G$  on  $x \in X$  will be denoted by  $g \cdot x = g(x)$ . The set  $\{g \cdot x \mid g \in G\}$  is called the orbit of the point  $x$ . If every orbit of  $(X, G)$  is dense, we call the system *minimal*. A system  $(X, G)$  is called *equicontinuous* if for all  $\varepsilon > 0$

there exists  $\delta > 0$  such that for all  $x, y \in X$  if  $d(x, y) < \varepsilon$ , then  $d(g \cdot x, g \cdot y) < \delta$  for all  $g \in G$ . Let  $(X, G)$ , and  $(Y, G)$  be two minimal systems. If there exists a continuous surjection  $\pi : X \rightarrow Y$  which preserves the action of  $G$ , we say that  $X$  is an *extension* of  $Y$ , and that  $Y$  is a *factor* of  $X$ . We call  $\pi$  a *factor map*. Given two factor maps  $\pi$  and  $\pi'$ , we say that  $\pi$  is *larger* than  $\pi'$  if there exists a third factor map  $\pi''$  such that  $\pi' = \pi'' \circ \pi$ . As such, we can discuss the *maximal* factor of a system. It is a known fact that every dynamical system has a maximal equicontinuous factor.

In this paper we are interested in symbolic dynamical systems. We start with a finite set  $\Sigma$  called the alphabet. Say  $|\Sigma| = n$ . The set of all bi-infinite sequences over  $\Sigma$  is called the full  $n$ -shift and is denoted by  $\Sigma^{\mathbb{Z}}$ . In general, we denote the full  $d$ -dimensional  $n$ -shift by  $\Sigma^{\mathbb{Z}^d}$ . This set is endowed with the product topology from the discrete topology in each coordinate. Cylinder sets in which we fix a finite number of coordinates form a basis for the topology. For  $x \in \Sigma^{\mathbb{Z}^d}$  we write  $x = \{x(v)\}_{v \in \mathbb{Z}^d}$ . We call  $x$  a  $\mathbb{Z}^d$ -array. The group  $\mathbb{Z}^d$  acts on  $\Sigma^{\mathbb{Z}^d}$ , denoted by  $T^z(x)$  for  $z \in \mathbb{Z}^d$  and  $x \in \Sigma^{\mathbb{Z}^d}$  as follows:  $T^z(x) = \{x(z+v)\}_{v \in \mathbb{Z}^d}$ . The orbit of an array is  $\{T^v(x) \mid v \in \mathbb{Z}^d\}$ . A closed subset  $X \subseteq \Sigma^{\mathbb{Z}^d}$  is called a *subshift* if it is closed under the action of  $\mathbb{Z}^d$ .

For the sake of completeness, we note that symbolic dynamics can be studied over general, discrete groups. In this case, let  $G$  be a discrete group. Then  $\Sigma^G$  is acted on by the group  $G$ . While in this paper we restrict our study of symbolic dynamics to  $\mathbb{Z}^d$ -systems, we note that many of the results can be extended to  $G$ -systems for more general groups  $G$ .

The topological spaces discussed in this note will be topological zero-dimensional compact metric spaces without isolated points, i.e., Cantor sets. Notice that by a theorem of Brouwer [1910] every Cantor set is homeomorphic to the middle-thirds Cantor set, and so all Cantor sets are homeomorphic.

### 3. Odometers

In this section, we will recall some basic facts about odometers and their almost one-to-one extensions. In particular, we show that the centralizer of an odometer is Abelian, and the centralizer of the almost one-to-one extension of an odometer is also Abelian. These results are mostly known, but are scattered. In particular, the proof of Lemma 3.11 appears in [Veech 1970] and the proof of Proposition 3.12 appears in [Olli 2013]. We present slightly modified proofs for clarity and the reader's convenience.

**Definition 3.1.** A group  $G$  is called *residually finite* if the intersection of all its finite-index normal subgroups is trivial.

**Definition 3.2.** Let  $G$  be a residually finite group and  $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$  be nested normal subgroups such that  $\bigcap G_n = \{0\}$ . Let  $\pi_n$  be the natural homomorphism from  $G/G_n$  onto  $G/G_{n-1}$ ; i.e.,  $\pi_n(hG_n) = hG_{n-1}$  for  $h \in G$ . The

*G*-odometer,  $\bar{G}$ , is the inverse limit

$$\bar{G} = \varprojlim(G/G_i; \pi_i) = \{(g_k)_{k=0}^\infty \in \prod_{k=0}^\infty G/G_k \mid \pi_n(g_n) = g_{n-1} \text{ for all } n \geq 1\}.$$

An element  $g \in G$  acts on an element  $y = (y_i)_{i=0}^\infty \in \bar{G}$  as  $g \cdot y = (g \cdot y_i)_{i=0}^\infty$ .

First we prove that  $G$  embeds into  $\bar{G}$ .

**Lemma 3.3.** *Let  $\varphi : G \rightarrow \bar{G}$  be defined as  $g \mapsto (gG_1, gG_2, \dots)$ . Then  $\varphi$  is an embedding.*

*Proof.* Notice that  $\varphi$  is a homomorphism. Let  $g_1, g_2 \in G$ . Suppose

$$\varphi(g_1) = (g_1G_1, g_1G_2, g_1G_3, \dots) = (g_2G_1, g_2G_2, g_2G_3, \dots) = \varphi(g_2).$$

So  $g_1G_i = g_2G_i$  for all  $i$ . Therefore  $g_1^{-1}g_2 \in G_i$  for all  $i$ , and so  $g_1^{-1}g_2 \in \bigcap G_i = \{0\}$ . Thus  $g_1 = g_2$ , which implies that  $\varphi$  is an embedding.  $\square$

So we have shown that  $G$  embeds into  $\bar{G}$  in a natural way. In what follows, we will identify the group  $G$  with its image  $\varphi(G)$ . We now prove that  $(\bar{G}, G)$  is minimal.

**Lemma 3.4.** *The system  $(\bar{G}, G)$  is minimal.*

*Proof.* Consider the identity element,  $e \in \bar{G}$ . In particular,  $e = (G_1, G_2, G_3, \dots)$ . Let  $y = (y_i)_{i=0}^\infty \in \bar{G}$ . So, for each  $n$ , we have  $y_n = \bar{y}_nG_n$ , where  $\bar{y}_n \in G$  is a representative of the coset. Note

$$\begin{aligned} \bar{y}_n \cdot e &= \bar{y}_n(G_1, G_2, G_3, \dots, G_n, \dots) \\ &= (\bar{y}_nG_1, \bar{y}_nG_2, \bar{y}_nG_3, \dots, \bar{y}_nG_n, \dots) \\ &= (\bar{y}_1G_1, \bar{y}_2G_2, \dots, \bar{y}_nG_n, \dots) = (y_1, y_2, \dots, y_n, \dots). \end{aligned}$$

So  $\bar{y}_n \cdot e$  agrees with  $y$  in the first  $n$  coordinates. And so we can get arbitrarily close to  $y$  as we increase  $n$ . It follows that the orbit of  $e$  is dense.

Now let  $a, b \in \bar{G}$ . Note we can find a sequence  $b_n$  of elements of  $G \subset \bar{G}$  such that  $b_n \cdot e \rightarrow ab^{-1}$ , since  $e$  has a dense orbit. Then  $(b_n \cdot e) \cdot b \rightarrow a$  so  $b_n \cdot b \rightarrow a$ . Therefore  $b$  has a dense orbit.  $\square$

**Definition 3.5** (centralizer). Let  $(X, G)$  be a dynamical system. The *centralizer*,  $C(G)$ , is defined as

$$C(G) = \{\varphi \in \text{Homeo}(X) \mid g\varphi = \varphi g \text{ for all } g \in G\}.$$

That is, the centralizer of a system consists of all homeomorphisms of the system which commute with the group action. It can be checked that this is a group under composition.

Next we show that elements of the centralizer of an odometer act as translations of the odometer.

**Lemma 3.6.** *Let  $\varphi \in C(\bar{G}, G)$ . There exists  $g_0 \in \bar{G}$  such that  $\varphi(x) = x \cdot g_0$  for all  $x \in \bar{G}$ .*

*Proof.* Set  $g_0 = \varphi(e)$ . Let  $x \in \bar{G}$ . Since the orbit of  $e$  is dense, by Lemma 3.4, there exists a sequence  $\{g_n\} \subseteq G$  such that  $g_n \cdot e \rightarrow x$ . Since  $\varphi$  is continuous,  $\varphi(g_n \cdot e) \rightarrow \varphi(x)$ . Since  $g_n \cdot e \rightarrow x$ , we have  $g_n \rightarrow x$ . So  $\varphi(g_n \cdot e) = \varphi(g_n) \cdot \varphi(e) \rightarrow x \cdot \varphi(e)$ . Therefore  $\varphi(x) = x \cdot \varphi(e) = x \cdot g_0$ .  $\square$

We are now ready to prove the following proposition. In the following,  $G$  is an Abelian group.

**Proposition 3.7.** *The centralizer  $C(\bar{G}, G) = \{\varphi : \bar{G} \rightarrow \bar{G} \mid \varphi g = g\varphi \text{ for all } g \in G\}$  of an odometer  $\bar{G}$  is isomorphic to  $\bar{G}$ .*

*Proof.* Define  $\psi : C(\bar{G}, G) \rightarrow \bar{G}$  as  $\psi(\varphi) = \varphi(e)$  for all  $\varphi \in C(\bar{G}, G)$ . Let  $\varphi_1, \varphi_2 \in C(\bar{G}, G)$ . Then

$$\psi(\varphi_1 \circ \varphi_2) = \varphi_1 \circ \varphi_2(e) = \varphi_1(\varphi_2(e)) = \varphi_2(e) \cdot \varphi_1(e) = \varphi_1(e) \cdot \varphi_2(e) = \psi(\varphi_1)\psi(\varphi_2).$$

So  $\psi$  is a homomorphism. Let  $y \in \bar{G}$ . Let  $\varphi_y(x) = x \cdot y$  for all  $x \in \bar{G}$ . Note, for  $g \in \bar{G}$ , we have  $\varphi_y(gx) = g\varphi_y(x)$  so  $\varphi_y \in C(\bar{G}, G)$ . Also,  $\psi(\varphi_y) = y$ , so  $\psi$  is onto. Suppose  $\psi(\varphi_1) = \psi(\varphi_2)$ . Then  $\varphi_1(e) = \varphi_2(e)$ . Using Lemma 3.6, we get that for any  $x \in \bar{G}$ ,  $\varphi_1(x) = x \cdot \varphi_1(e) = x \cdot \varphi_2(e) = \varphi_2(x)$ . Therefore  $\psi$  is an isomorphism.  $\square$

We now turn our attention to almost one-to-one extensions of odometers.

**Definition 3.8.** We say  $(X, G)$  is an *almost one-to-one extension* of  $(Y, G)$  if there is a factor map  $\pi : X \rightarrow Y$  such that there is at least one  $y \in Y$  so that  $\pi^{-1}y$  is singleton. Almost one-to-one extensions of odometers are also called *Toeplitz systems*.

We make use of the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{G} & X \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{G} & Y \end{array}$$

Sometimes the context will deem the action of  $G$  on  $X$  or  $Y$  ambiguous, so we will use  $T^g x$  to denote the action of the group element  $g \in G$  on  $x \in X$  and  $S^g y$  to denote the action of  $g$  on  $y \in Y$ . In particular,  $\pi \circ T^g = S^g \circ \pi$ . If the context is clear, the action of  $g$  on a point  $x$  will be denoted by  $g \cdot x$ .

If  $(X, G)$  is a minimal almost one-to-one extension of a minimal equicontinuous system  $(Y, G)$ , then it is known that  $(Y, G)$  is the maximal equicontinuous factor

of  $(X, G)$  [Auslander 1988]. As such, the odometer of which a Toeplitz system  $(X, G)$  is an almost one-to-one extension is its maximal equicontinuous factor.

We will be considering almost one-to-one extensions of  $\mathbb{Z}^d$ -odometers. In this context, we will need the following proposition.

**Proposition 3.9.** *The centralizer  $C(G)$  of the almost one-to-one extension of a  $\mathbb{Z}^d$ -odometer is Abelian.*

To prove Proposition 3.9, we show that the centralizer of the almost one-to-one extension of an odometer embeds into the centralizer of its maximal equicontinuous factor, which we have already shown to be isomorphic to the odometer, which is Abelian in the case of  $G = \mathbb{Z}^d$ .

**Definition 3.10** [Veech 1970]. Given a dynamical system  $(X, G)$  and a metric  $d$  compatible with the topology on  $X$ , two points  $x_1, x_2 \in X$  are called *proximal* if

$$\inf_{g \in G} d(g \cdot x_1, g \cdot x_2) = 0.$$

**Lemma 3.11.** *Let  $(X, G)$  be an almost one-to-one extension of an odometer  $(\bar{G}, G)$  via the factor map  $\pi$ . Then points of  $X$  are proximal if and only if they are in the same  $\pi$ -fiber.*

*Proof.* Let  $x_1, x_2 \in X$  be in the same  $\pi$ -fiber; i.e.,  $\pi(x_1) = \pi(x_2)$ . Let  $y \in \bar{G}$  be such that  $\pi^{-1}y$  is a singleton. Since  $(\bar{G}, G)$  is minimal, there exists a sequence  $\{g_n\}$  of elements in  $G$  such that  $\lim_{n \rightarrow \infty} S^{g_n} \pi x_1 = y$  and so  $\lim_{n \rightarrow \infty} S^{g_n} \pi x_2 = y$ . Since  $X$  is compact, there is a subsequence  $\{g_k\}$  of  $\{g_n\}$  such that  $T^{g_k} x_1$  and  $T^{g_k} x_2$  converge. Suppose  $\lim_{k \rightarrow \infty} T^{g_k} x_1 = z$ . Applying  $\pi$ , we have

$$\pi z = \lim_{k \rightarrow \infty} \pi T^{g_k} x_1 = \lim_{k \rightarrow \infty} S^{g_k} \pi x_1 = y.$$

So we also have

$$\lim_{k \rightarrow \infty} \pi T^{g_k} x_2 = \lim_{k \rightarrow \infty} S^{g_k} \pi x_2 = y.$$

Since  $\pi^{-1}y$  is a singleton, we get that  $\lim_{k \rightarrow \infty} T^{g_k} x_2 = z$ . Now,

$$\begin{aligned} \limsup_{k \rightarrow \infty} d(T^{g_k} x_1, T^{g_k} x_2) &\leq \limsup_{k \rightarrow \infty} (d(T^{g_k} x_1, z) + d(z, T^{g_k} x_2)) \\ &\leq \limsup_{k \rightarrow \infty} d(T^{g_k} x_1, z) + \limsup_{k \rightarrow \infty} d(z, T^{g_k} x_2) = 0. \end{aligned}$$

So the points  $x_1$  and  $x_2$  are proximal.

Now suppose  $x_1, x_2 \in X$  are proximal. Then there is a sequence  $\{g_n\} \subseteq G$  such that  $\lim_{n \rightarrow \infty} T^{g_n} x_1 = \lim_{n \rightarrow \infty} T^{g_n} x_2 = z$ . Applying  $\pi$ , we have  $\lim_{n \rightarrow \infty} \pi T^{g_n} x_1 = \lim_{n \rightarrow \infty} \pi T^{g_n} x_2 = \pi z$ . So  $\lim_{n \rightarrow \infty} S^{g_n} \pi x_1 = \lim_{n \rightarrow \infty} S^{g_n} \pi x_2$ , which implies  $\pi x_1$  and  $\pi x_2$  are proximal in  $\bar{G}$ . But  $\bar{G}$  has no proximal points, so  $\pi x_1 = \pi x_2$ .  $\square$



Finally, we prove that the centralizers of Toeplitz systems embed in the centralizers of the underlying odometers.

**Proposition 3.12.** *Let  $(X, G)$  be an almost one-to-one extension of a  $G$ -odometer  $(Y, G)$ . Every element  $\varphi \in C(X, G)$  determines  $\psi_\varphi \in C(Y, G)$  such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{\psi_\varphi} & Y \end{array}$$

Additionally, this relationship is an embedding; i.e.,  $\psi_{\varphi_1} = \psi_{\varphi_2} \Rightarrow \varphi_1 = \varphi_2$ .

*Proof.* Let  $\varphi \in C(X, G)$ . Let  $x_1, x_2 \in X$  be proximal. So  $\pi x_1 = \pi x_2$ . Since  $x_1$  and  $x_2$  are proximal,  $\inf_{g \in G} d(g \cdot x_1, g \cdot x_2) = 0$ . Thus  $\inf_{g \in G} d(\varphi(g \cdot x_1), \varphi(g \cdot x_2)) = 0$ , which, by Lemma 3.11, implies that  $\varphi(x_1), \varphi(x_2)$  are proximal. So  $\varphi$  preserves the proximal relationship, and so it preserves the  $\pi$ -fibers. Define  $\psi_\varphi : Y \rightarrow Y$  as  $\psi_\varphi = \pi \circ \varphi \circ \pi^{-1}$ . This map is well-defined because  $\varphi$  preserves the  $\pi$ -fibers. Suppose  $\psi_\varphi(y_1) = \psi_\varphi(y_2)$  for  $y_1, y_2 \in Y$ . So  $\pi \circ \varphi \circ \pi^{-1}(y_1) = \pi \circ \varphi \circ \pi^{-1}(y_2)$ , and thus  $\varphi \circ \pi^{-1}(y_1)$  and  $\varphi \circ \pi^{-1}(y_2)$  are in the same  $\pi$ -fibers. Since  $\varphi$  preserves the  $\pi$ -fibers,  $\pi^{-1}(y_1)$  and  $\pi^{-1}(y_2)$  are in the same  $\pi$ -fibers, and so it is clear that  $y_1 = y_2$ . Therefore  $\psi_\varphi$  is one-to-one. Also,  $\psi_\varphi$  is continuous, so it is a homeomorphism; i.e.,  $\psi_\varphi \in C(Y, G)$ .

Now suppose  $\psi_{\varphi_1} = \psi_{\varphi_2}$ . Let  $y \in Y$  be such that  $\pi^{-1}y = \{x\}$  is a singleton. Then  $\varphi_1(x) = \pi^{-1}(\psi_{\varphi_1}(y))$  and  $\varphi_2(x) = \pi^{-1}(\psi_{\varphi_2}(y))$ . Since  $\varphi_i$  preserves  $\pi$ -fibers for  $i \in \{1, 2\}$ , these are singletons. In particular,  $\varphi_1(x) = \varphi_2(x)$ . So it is clear then that  $g \cdot \varphi_1(x) = g \cdot \varphi_2(x)$  for all  $g \in G$ , and so  $\varphi_1(g \cdot x) = \varphi_2(g \cdot x)$  for all  $g \in G$ . But every orbit is dense, so  $\varphi_1$  and  $\varphi_2$  agree on a dense subset of  $X$ , and hence agree everywhere. □

Finally we prove Proposition 3.9.

*Proof.* We have shown in Proposition 3.12 that  $C(X, G)$  embeds into  $C(Y, G)$  and by Proposition 3.7  $C(Y, G)$  is Abelian, so  $C(X, G)$  is Abelian. □

### 4. $\mathbb{Z}^d$ -Toeplitz systems

In this section, we study Toeplitz systems over  $\mathbb{Z}^d$  and generalize the construction of Bułatek and Kwiatkowski. In particular, we present a class of Toeplitz systems over  $\mathbb{Z}^d$  with a trivial centralizer and positive entropy.

Let  $x \in \Sigma^{\mathbb{Z}^d}$ . Note that the topological closure of the orbit of  $x$ ,  $\overline{O(x)}$ , is closed and  $T$ -invariant. So  $(\overline{O(x)}, T)$  is a subshift. This is called the *orbit closure* of  $x$ .

**Definition 4.1.** The centralizer of a symbolic dynamical system is called *trivial* if every element of the centralizer is  $T^s$  for some  $g \in \mathbb{Z}^d$ .

For  $x \in \Sigma^{\mathbb{Z}^d}$ ,  $\sigma \in \Sigma$ , and a subgroup  $Z \subset \mathbb{Z}^d$ , define

$$\begin{aligned} \text{Per}(x, Z, \sigma) &= \{w \in \mathbb{Z}^d \mid x(w+z) = \sigma \text{ for all } z \in Z\}, \\ \text{Per}(x, Z) &= \bigcup_{\sigma \in \Sigma} \text{Per}(x, Z, \sigma). \end{aligned}$$

We say that  $x \in \Sigma^{\mathbb{Z}^d}$  is a *Toeplitz array* if for every  $v \in \mathbb{Z}^d$ , there exists a finite-index subgroup  $Z \subseteq \mathbb{Z}^d$  (note that  $Z$  is necessarily isomorphic to  $\mathbb{Z}^d$ ) such that  $v \in \text{Per}(x, Z)$ .

It can be shown that the orbit closure of a Toeplitz array is an almost one-to-one extension of a  $\mathbb{Z}^d$ -odometer. For details, the reader is referred to Theorem 7 and Proposition 21 in [Cortez 2006]. In fact, almost one-to-one extensions of odometers are exactly those systems which are orbit closures of Toeplitz arrays. In particular, defining a Toeplitz system as the orbit closure of a Toeplitz array is equivalent to Definition 3.8.

**Definition 4.2.** Given a finite alphabet  $\Sigma$ , a *patch* is a pair  $(P, \mathcal{L})$ , where  $P \subseteq \mathbb{Z}^d$  and  $\mathcal{L} : P \rightarrow \Sigma$  is a labeling of  $P$ . For the purposes of this paper, we will only consider rectangular patches (blocks) which can be defined by  $d$  vectors parallel to the coordinate axes.

Given a patch  $(P, \mathcal{L})$ , we denote the coordinate closest to the origin in Cartesian space by  $P[0]$ . Any other location in the patch is denoted by  $P[i]$ , where  $i \in \mathbb{Z}^d$  is a vector pointing to that location, as referenced from  $P[0]$ . A square block within  $P$  is denoted by  $P[i-l, i+k]$ , where  $k, l \in \mathbb{Z}$  and is the (hyper)cube in  $P$  located between  $P[i-l\bar{1}]$  and  $P[i+k\bar{1}]$ , where  $\bar{1} = (1, 1, \dots, 1)$ .

For a finite block  $D$  in  $d$  dimensions, we denote the size of  $D$  along the  $i$ -th dimension as  $|D|_i$ . Note that the left-most and bottom-most entry of  $D$  is identified with  $D(0, 0, \dots, 0)$ .

We now show how Toeplitz arrays can be constructed over an alphabet  $\Sigma$  borrowing ideas from [Downarowicz 2005].

Let  $\{p_{t,i}\}_{t=0}^{\infty}$ ,  $1 \leq i \leq d$ , be  $d$  sequences of positive integers such that  $p_{0,i} \geq 2$  and  $p_{t,i}$  divides  $p_{t+1,i}$  for all  $1 \leq i \leq d$ . Define  $\lambda_{t+1,i} = p_{t+1,i}/p_{t,i}$  and  $\lambda_{0,i} = p_{0,i}$  for all  $1 \leq i \leq d$  and  $t \geq 0$ .

Specify blocks  $A_t$  as follows:

- (1)  $|A_t|_i = p_{t,i}$ .
- (2) Some spaces in  $A_t$  are filled with elements from  $\Sigma$  and others are left unfilled. The unfilled spaces are called *holes*.
- (3) The block  $A_{t+1}$  is the concatenation of  $\lambda_{t+1,i}$  copies of  $A_t$  along the  $i$ -th dimension for all  $1 \leq i \leq d$ , where some holes are filled by symbols from  $\Sigma$ .

- (4) For every  $(i_1, i_2, \dots, i_d) \in \mathbb{N}^d$  there exists a  $t \geq 0$  such that  $A_t(i_1, i_2, \dots, i_d) \in \Sigma$  and  $A_t(p_{t,1}-i_1, p_{t,2}-i_2, \dots, p_{t,d}-i_d) \in \Sigma$ .

Denote by  $\omega_t$  the periodic tiling of  $\mathbb{Z}^d$  by the block  $A_t$  with the bottom-left corner of  $A_t$  appearing at the origin. Set  $\omega = \lim \omega_t$ . The fourth condition assures that  $\omega \in \Sigma^{\mathbb{Z}^d}$ . We will additionally assume that  $p_t$  is the smallest period of  $\omega_t$ , which ensures that  $\omega$  is a nonperiodic Toeplitz array.

Essentially, in this construction we build finite blocks, each of which contains multiple copies of the block built in the previous step. As we copy these blocks, we fill in some of the holes, and leave some of them as holes. As we continue this process forever, we will have a Toeplitz array covering  $\mathbb{Z}^d$ .

**Example 4.3** (one-dimensional Toeplitz array [Downarowicz 2005]). We will construct a Toeplitz array over  $\mathbb{Z}$  from the alphabet  $\Sigma = \{0, 1\}$ . Let  $\{p_t\} = \{2, 4, 8, 16, \dots\}$  and so  $\lambda_t = 2$  for all  $t \geq 0$ . Let  $A_0 = 0\_$ , where the  $\_$  symbol indicates a hole. To get  $A_1$ , we copy  $A_0$  twice and fill in some of the holes. Say  $A_1 = 0\underline{1}0\_$ . The underline indicates a hole that was filled in at that step. In each step we will have two holes. For this construction, at each step we will alternately fill in the first hole with 0 and 1. Let the limiting sequence of this process be  $\omega$ . Continuing, we have

$$A_2 = 010\underline{0}010\_,$$

$$A_3 = 0100010\underline{1}0100010\_,$$

$$A_4 = 0100010101000\underline{1}00010001010100010\_,$$

⋮

$$\omega = 01000101010001000100010101000101010001010100010001000100010001000100\dots,$$

and so we have a Toeplitz array  $\omega$ . The orbit closure of this point is a Toeplitz system.

**Example 4.4** (two-dimensional Toeplitz array). Again we will use the alphabet  $\Sigma = \{0, 1\}$  and we will construct a Toeplitz array over  $\mathbb{Z}^2$ . Let  $\{p_{t,1}\} = \{p_{t,2}\} = \{2, 4, 8, 16, \dots\}$ . Then  $\lambda_{t,1} = \lambda_{t,2} = 2$  for all  $t \geq 0$ .

Let

$$A_0 = \begin{matrix} \boxed{1} & \boxed{1} \\ \boxed{0} & \blacksquare \end{matrix}, \quad A_1 = \begin{matrix} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{0} & \blacksquare & \boxed{0} & \boxed{0} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{0} & \boxed{1} & \boxed{0} & \blacksquare \end{matrix}, \quad A_2 = \begin{matrix} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{0} & \blacksquare & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{0} & \boxed{1} & \boxed{0} & \blacksquare & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \blacksquare & \boxed{0} & \boxed{0} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{0} & \blacksquare \end{matrix}.$$

The black squares indicate where the holes are. Continuing this process, we will have a coloring of the whole plane, which will be a Toeplitz array, say  $\omega$ .

We call subblocks of  $A_{t+1}$  which coincide with indices of the location of concatenated  $A_t$  blocks *t-blocks*. We note that  $\omega$  consists of the concatenation of *t-blocks* in all directions for any  $t$ , where all *t-blocks* agree in all locations except for where the holes were. In Example 4.4, the thick lines in  $A_1$  indicate the 0-blocks, and the thick lines in  $A_2$  indicate the 1-blocks.

Now we introduce a condition on constructing Toeplitz arrays which will give rise to Toeplitz systems with a trivial centralizer.

**Condition (\*)**. We say a Toeplitz array satisfies the condition (\*) if:

- Every *t*-block in  $A_{t+1}$  is composed of either  $A_t$  where no hole remaining from  $A_t$  is filled in or  $A_t$  with all holes filled.
- The perimeter of  $A_{t+1}$  is composed of *t*-blocks which are all filled in.
- For every  $i \in \mathbb{Z}^d$  such that  $A_t[i]$  is a hole, there are two *t*-blocks  $B_1$  and  $B_2$  with  $B_1[i] \neq B_2[i]$ .

Let  $e_1, e_2, \dots, e_d$  be the generators of  $\mathbb{Z}^d$ . For  $1 \leq i \leq d$ , let  $T_i$  denote a shift by the vector  $e_i$ . In this context, the shift action on the system can be considered  $d$  independent shift actions; i.e.,  $T^g = T^{(g_1, g_2, \dots, g_d)} = T_1^{g_1} \times T_2^{g_2} \times \dots \times T_d^{g_d}$ .

**Theorem 4.5.** *Let  $\omega$  be a Toeplitz array satisfying the condition (\*). Then the centralizer  $C(T)$  of  $(\overline{O(\omega)}, T)$  is trivial.*

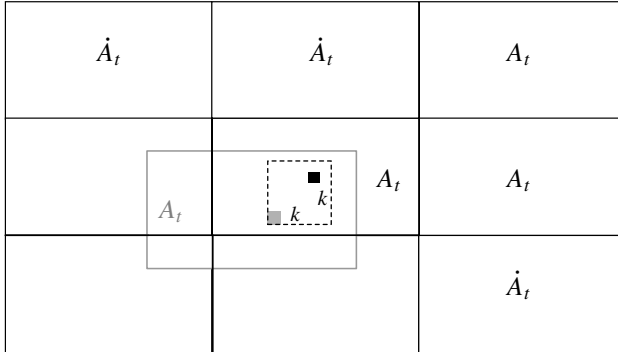
*Proof.* Let  $(\overline{G}, T_1 \times T_2 \times \dots \times T_d)$  be the maximal equicontinuous factor of  $(\overline{O(\omega)}, T)$ . Denote by  $\pi : (\overline{O(\omega)}, T) \rightarrow (\overline{G}, T_1 \times T_2 \times \dots \times T_d)$  the almost one-to-one factor map. Let  $S \in C(T)$ . By Proposition 3.12, this determines an element  $S' \in C(\overline{G}, T_1 \times T_2 \times \dots \times T_d)$  which acts as a translation by some element  $h \in \overline{G}$ , by Lemma 3.6. By a result of [Hedlund 1969], we note  $S$  is determined by a block code  $f$  of window size  $k \in \mathbb{N}$ . In particular, if  $u \in \overline{O(\omega)}$  and  $z = S(u)$ , then

$$z[i] = f(u[i - k, i + k]) \quad \text{for all } i \in \mathbb{Z}^d. \tag{1}$$

In particular, the automorphism determines what to put in a specific location by looking at a block around that location in the preimage. Increasing  $k$  if necessary, we can assume that  $S^{-1}$  is also determined by a block code of the same window size  $k$ .

Note  $\overline{G}$  is a product odometer, so  $h = (h_1, h_2, \dots, h_d)$ , where  $h_i = \sum_{t=0}^{\infty} h_{t,i} p_{t-1,i}$  for  $1 \leq i \leq d$  with  $0 \leq h_{t,i} \leq \lambda_{t,i} - 1$ ,  $p_{-1,i} = 1$ . Each  $h_i$  is an element of the one-dimensional odometer occurring in the  $i$ -th coordinate of  $h$ . Let  $m_{t,i} = \sum_{j=0}^t h_{j,i} p_{j-1,i}$  and  $m_t = (m_{t,1}, m_{t,2}, \dots, m_{t,d}) \in \mathbb{Z}^d$ . For each  $g \in G$ , denote by  $X_g$  the preimage  $\pi^{-1}(\{g\})$  under the factor map  $\pi$ . Then  $S(X_g) = X_{g+h}$ .

We claim that for all  $1 \leq i \leq d$  either  $m_{t,i} \leq k$  or  $m_{t,i} \geq p_{t,i} - k - 1$ .



**Figure 1.** The two-dimensional case of the argument in the proof of Theorem 4.5.  $\dot{A}_t$  indicates  $A_t$  blocks with all holes filled and the solid black and gray squares indicate holes in  $x_{t+1}$  and  $y_{t+1}$ , respectively.

Let  $x \in \overline{O(\omega)}$  and  $y = S(x)$ . Suppose that  $x$  has a  $t$ -block appearing at a location  $x[i]$ . Then by the construction of Toeplitz subshifts and almost one-to-one extensions,  $y$  necessarily has a  $t$ -block at the location  $y[i - m_t]$ . Note that for every  $t \geq 1$  the array  $x$  can be written as a concatenation of  $(t+1)$ -blocks, which are made of  $t$ -blocks. Recall that all  $t$ -blocks are the same, except they may disagree where the holes are located. Denote by  $x_{t+1}$  the  $d$ -dimensional array over the alphabet  $\Sigma \cup \{\text{hole}\}$  consisting of copies of the block  $A_{t+1}$  at the locations where they appear in  $x$ . Note that  $\lim x_t = x$ . Similarly, we can define  $y_{t+1}$ , a  $(t+1)$ -block skeleton of the array  $y$ . Note that  $x_{t+1}$  and  $y_{t+1}$  are  $p_{t+1}$ -periodic and shifted by the vector  $m_{t+1}$  relative to each other. Let  $A$  denote any  $(t+1)$ -block of  $y$ . This block looks like  $A_{t+1}$ , which in turn is the concatenation of  $A_t$  blocks both filled and not. Thus, we can view both  $y_{t+1}$  and  $x_{t+1}$  as the concatenations of copies of  $A_t$  and copies of  $\dot{A}_t$ , filled versions of  $A_t$ . The  $t$ -blocks in  $x_{t+1}$  and  $y_{t+1}$  appear shifted by the vector  $m_t$ .

Fix  $t > 0$  such that  $p_{t-1,i} > 2k + 1$  for every  $i = 1, \dots, d$ . Fix  $j \in \mathbb{Z}^d$  such that  $y_{t+1}[j]$  is a hole. Note that this hole would correspond to a hole in  $A_t$ . Then the hypercube  $x_{t+1}[j - k, j + k]$  must also contain a hole. For otherwise,  $x_{t+1}[j - k, j + k]$  would be the same for every  $j$  with  $j \equiv i \pmod{p_t}$  and, thus,  $y[j] = S(x)[j] = f(x[j - k, j + k])$  would be the same for every such  $j$ . This would contradict the last property of the condition (\*). Applying the same argument to  $S^{-1}$ , we see that if  $x_{t+1}[j]$  is a hole for some  $j \in \mathbb{Z}^d$ , then  $y[j - k, j + k]$  must also contain a hole.

This argument is demonstrated for the two-dimensional case and for the forward-looking centralizers in Figure 1.

Now, the  $t$ -blocks in the arrays  $x_{t+1}$  and  $y_{t+1}$  are shifted by the vector  $m_t$  relative to each other. At the same time, by the argument above, the filled  $t$ -blocks  $\dot{A}_t$  in  $x_{t+1}$  and  $y_{t+1}$  must appear under each other and can be shifted by a vector of length at

most  $k$ . Since  $p_i$  is the smallest period of  $\omega_i$ , we conclude that for all  $1 \leq i \leq d$  either  $m_{t,i} \leq k$  or  $m_{t,i} \geq p_{t,i} - k - 1$ . It follows that  $h_i$  is an integer for every  $i = 1, \dots, d$ . So,  $h = S'(0) = T^h(0)$ ; i.e.,  $S'$  and  $T^h$  agree on one point. Furthermore,  $S'$  agrees with the action of  $T^h$  on the entire orbit of 0, which is dense. Therefore,  $S' = T^h$ .

Let  $\alpha$  be in the orbit of  $\omega$  in  $(\overline{O(\omega)}, T)$ ; i.e.,  $\alpha = T^g \omega$  for some  $g \in \mathbb{Z}^d$ . Note

$$\pi S(\alpha) = \pi S(T^g \omega) = S' \pi(T^g \omega) = S' T^g(0) = T^h T^g(0) = \pi T^h T^g \omega = \pi T^h(\alpha).$$

So  $S(\alpha)$  and  $T^h(\alpha)$  are in the same  $\pi$ -fiber. Since  $\alpha$  is in the orbit of  $\omega$ , it has a unique preimage under  $\pi$ . Therefore  $S(\alpha) = T^h(\alpha)$ . And so  $S$  and  $T^h$  agree on the entire orbit of  $\omega$ , which is dense. So  $S = T^h$ . □

### 5. Positive-entropy Toeplitz subshift

We now construct an explicit example of a two-dimensional Toeplitz subshift which has positive entropy. This example is constructed so that it obeys the condition (\*), thus ensuring that it has a trivial centralizer.

Let  $h > 0$  and choose an integer  $l_0$  such that  $\log(l_0 - 1) \leq h \leq \log(l_0)$ . For  $i \geq 0$ , let  $\varepsilon_i > 0$  and  $\{\varepsilon_i\}$  be such that  $\sum_{i=0}^\infty \varepsilon_i < h/2$ .

We note that for any  $l$  and any  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  sufficiently large such that

$$\frac{\log(l^{n^2})}{(n+2)^2} \geq \log(l) - \varepsilon \tag{2}$$

since  $(n/(n+2))^2 \rightarrow 1$ .

Let  $q_0$  be chosen so that

$$\frac{\log(l_0^{q_0^2})}{(q_0+2)^2} \geq \log(l_0) - \frac{\varepsilon_0}{2}.$$

Also require  $q_0^2 \geq l_0$ . Define  $l_1 = l_0^{q_0^2}$ . We notice that there are  $l_0^{q_0^2}$  square blocks of side length  $q_0$  over the alphabet  $\{0, 1, \dots, l_0 - 1\}$ . We enumerate these blocks as  $B_i^{(0)}$  for  $0 \leq i \leq l_1 - 1$ . Furthermore, we require that  $B_0^{(0)}$  and  $B_1^{(0)}$  contain every letter from the alphabet. Let  $C_i^{(0)}$  be the square block of side length  $q_0 + 2$  with the block  $B_i^{(0)}$  surrounded by a 0 in the top left corner, a 1 in the bottom right corner, and 0's below the main diagonal and 1's above it, as in the diagram below. We will denote this as  $C_i^{(0)} = 0B_i^{(0)}1$  for  $0 \leq i \leq l_1 - 1$ :

$$C_i^{(0)} = \begin{array}{|c|c|c|c|} \hline 0 & 1 & \cdots & 1 \\ \hline 0 & B_i^{(0)} & & \vdots \\ \hline \vdots & & & 1 \\ \hline 0 & \cdots & 0 & 1 \\ \hline \end{array}.$$

For  $k \geq 1$ , define  $l_k = l_{k-1}^{q_{k-1}^2}$  and let  $q_k$  be such that

$$\frac{\log(l_k^{q_k^2})}{(q_k + 2)^2} \geq \log(l_k) - \frac{\varepsilon_k}{2}. \tag{3}$$

Additionally, require that  $q_k^2 \geq l_k$ . Let  $B_i^{(k)}$  be all the square blocks of side length  $q_k$  over the alphabet  $\{0, 1, \dots, l_k - 1\}$  for  $0 \leq i \leq l_{k+1} - 1$ . Require that  $B_0^{(k)}$  and  $B_1^{(k)}$  contain every letter from the alphabet. Let  $C_i^{(k)} = 0B_i^{(k)}1$  for  $0 \leq i \leq l_{k+1} - 1$ .

Consider the following operation on finite blocks. Let  $\{A_1, A_2, \dots, A_n\}$  be square blocks of the same side length  $|A|$  over some alphabet. Let  $B$  be a square block over an alphabet containing  $\{1, 2, \dots, n\}$ . We define the block

$$C = \{A_1, A_2, \dots, A_n\} * B$$

as  $C[i, j] = A_{B[i, j]}$ . In particular,  $C$  will be a square block of side length  $|B| \cdot |A|$ .

Our goal is to construct a sequence of blocks  $\{A_i\}$  that defines a system of  $k$ -blocks and that satisfies the condition  $(*)$ . We proceed as follows: Let  $A_i^{(0)} = C_i^{(0)}$  and

$$A_i^{(k)} = \{A_0^{(k-1)}, A_1^{(k-1)}, \dots, A_{l_{k-1}}^{(k-1)}\} * C_i^{(k)}.$$

We note that since  $C_0^{(1)}$  and  $C_1^{(1)}$  have every letter of the alphabet  $\{0, 1, \dots, l_1 - 1\}$ , the blocks  $A_0^{(1)}$  and  $A_1^{(1)}$  will have every 0-block as a subblock. Similarly,  $C_0^{(2)}$  and  $C_1^{(2)}$  contain every letter in  $\{0, 1, \dots, l_2 - 1\}$  and so the blocks  $A_0^{(2)}$  and  $A_1^{(2)}$  will contain every 1-block as a subblock. In general, we note that each block  $A_i^{(k)}$  for  $i = 0, 1$  has every  $(k-1)$ -block as a subblock.

We let

$$A_0 = \begin{array}{|c|c|c|c|} \hline 0 & 1 & \cdots & 1 \\ \hline 0 & & & \vdots \\ \hline \vdots & & - & 1 \\ \hline 0 & \cdots & 0 & 1 \\ \hline \end{array},$$

where the side length of the square box  $A_0$  is  $q_0 + 2$ , and the dash in the center square indicates a square of side length  $q_0$  consisting of all holes. We note that  $A_i^{(0)}$ ,  $i = 0, \dots, l_1 - 1$ , are 0-blocks corresponding to  $A_0$ .

Inductively, define

$$A_{k+1} = \begin{array}{|c|c|c|c|c|} \hline A_0^{(k)} & A_1^{(k)} & \cdots & A_1^{(k)} & A_1^{(k)} \\ \hline A_0^{(k)} & A_k & \cdots & A_k & A_1^{(k)} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline A_0^{(k)} & A_k & \cdots & A_k & A_1^{(k)} \\ \hline A_0^{(k)} & A_0^{(k)} & \cdots & A_0^{(k)} & A_1^{(k)} \\ \hline \end{array},$$

where there is a square block consisting of  $q_k^2$  copies of  $A_k$  surrounded by  $4q_k + 4$  copies of  $A_i^{(k)}$  for  $i = 0$  or  $1$  on each side positioned similarly to 0's and 1's in  $A_0$ . Notice that  $A_0^{(k)}$  and  $A_1^{(k)}$  have no holes, so all the holes are contained in the middle block of  $A_k$  blocks. Note that  $A_i^{(k)}$ ,  $i = 0, \dots, l_k - 1$ , are the  $k$ -blocks corresponding to the pattern  $A_k$ .

Let  $\omega$  be the limiting array from the above process. Note that  $\omega \in \{0, \dots, l_0 - 1\}^{\mathbb{Z}^2}$  and  $\omega$  satisfies the condition (\*).

**Proposition 5.1.** *The Toeplitz system  $(\overline{O(\omega)}, T)$  has positive entropy.*

*Proof.* Define  $\lambda_k = q_k + 2$  and  $p_k = \lambda_1 \lambda_2 \cdots \lambda_k$ . Let  $h_\omega$  be the entropy of  $(\overline{O(\omega)}, T)$  and let  $\Theta(n)$  be the number of square blocks of side length  $n$  appearing in  $\omega$ . We note that

$$h_\omega = \lim_{n \rightarrow \infty} \frac{\log(\Theta(n))}{n^2} = \lim_{k \rightarrow \infty} \frac{\log(\Theta(p_k))}{p_k^2},$$

by switching to a subsequence.

There are  $l_{k+1}$  many  $k$ -blocks. We note that every block  $A_k$  contains every  $(k-1)$ -block as a subblock. This is because the blocks  $C_i^{(k)}$  for  $i = 0$  or  $i = 1$  contain every letter of the alphabet in them. This means that as we do the shuffling process described above, the blocks  $A_i^{(k)}$  for  $i = 0$  or  $i = 1$  contain every single block  $A_i^{(k-1)}$  for  $0 \leq i \leq l_k - 1$ . Furthermore, since  $k$ -blocks are squares of side length  $p_k$ , there are at least as many blocks of side length  $p_k$  occurring in  $\omega$  as there are  $k$ -blocks. Specifically, square blocks of length  $p_k$  can occur at any position within  $\omega$ , while  $k$ -blocks only occur at specific positions. Hence we have

$$\Theta(p_k) \geq l_{k+1}. \tag{4}$$

So we have

$$h_\omega \geq \limsup_{k \rightarrow \infty} \frac{\log(l_{k+1})}{p_k^2}. \tag{5}$$

By (3) we have

$$\frac{\log(l_{k+1})}{\lambda_k^2} \geq \log(l_k) - \frac{\varepsilon_k}{2}.$$

It then follows by (2) that

$$\frac{\log(l_{k+1})}{p_k^2} \geq \frac{\lambda_k^2(\log(l_k) - \varepsilon_k/2)}{p_k^2} = \frac{\log(l_k) - \varepsilon_k/2}{p_{k-1}^2} \geq \frac{\log(l_k)}{p_{k-1}^2} - \varepsilon_k.$$

Continuing, we have

$$\frac{\log(l_{k+1})}{p_k^2} \geq h - \sum_{i=0}^k \varepsilon_i.$$

Taking the limit as  $k \rightarrow \infty$ , from (5), we have  $h_\omega \geq h/2 > 0$ .



It is a basic fact that every Toeplitz system is minimal, so this system is minimal. It is either finite or uncountable, and since it has positive entropy, it cannot be finite. So this is an infinite minimal Toeplitz system.  $\square$

## References

- [Auslander 1963] J. Auslander, “Endomorphisms of minimal sets”, *Duke Math. J.* **30**:4 (1963), 605–614. MR Zbl
- [Auslander 1988] J. Auslander, *Minimal flows and their extensions*, North-Holland Mathematics Studies **153**, North-Holland, Amsterdam, 1988. MR Zbl
- [Boyle et al. 1988] M. Boyle, D. Lind, and D. Rudolph, “The automorphism group of a shift of finite type”, *Trans. Amer. Math. Soc.* **306**:1 (1988), 71–114. MR Zbl
- [Brouwer 1910] L. E. J. Brouwer, “On the structure of perfect sets of points”, *KNAW, Proc.* **12** (1910), 785–794.
- [Bułatek and Kwiatkowski 1990] W. Bułatek and J. Kwiatkowski, “The topological centralizers of Toeplitz flows and their  $Z_2$ -extensions”, *Publ. Mat.* **34**:1 (1990), 45–65. MR Zbl
- [Bułatek and Kwiatkowski 1992] W. Bułatek and J. Kwiatkowski, “Strictly ergodic Toeplitz flows with positive entropies and trivial centralizers”, *Studia Math.* **103**:2 (1992), 133–142. MR Zbl
- [Cortez 2006] M. I. Cortez, “ $Z^d$  Toeplitz arrays”, *Discrete Contin. Dyn. Syst.* **15**:3 (2006), 859–881. MR Zbl
- [Cortez and Petite 2008] M. I. Cortez and S. Petite, “ $G$ -odometers and their almost one-to-one extensions”, *J. Lond. Math. Soc.* (2) **78**:1 (2008), 1–20. MR Zbl
- [Cyr and Kra 2015] V. Cyr and B. Kra, “The automorphism group of a shift of linear growth: beyond transitivity”, *Forum Math. Sigma* **3** (2015), art. id. e5. MR Zbl
- [Cyr and Kra 2016a] V. Cyr and B. Kra, “The automorphism group of a minimal shift of stretched exponential growth”, *J. Mod. Dyn.* **10** (2016), 483–495. MR Zbl
- [Cyr and Kra 2016b] V. Cyr and B. Kra, “The automorphism group of a shift of subquadratic growth”, *Proc. Amer. Math. Soc.* **144**:2 (2016), 613–621. MR Zbl
- [Donoso et al. 2016] S. Donoso, F. Durand, A. Maass, and S. Petite, “On automorphism groups of low complexity subshifts”, *Ergodic Theory Dynam. Systems* **36**:1 (2016), 64–95. MR Zbl
- [Donoso et al. 2017] S. Donoso, F. Durand, A. Maass, and S. Petite, “On automorphism groups of Toeplitz subshifts”, *Discrete Anal.* (2017), art. id. 11. MR
- [Downarowicz 2005] T. Downarowicz, “Survey of odometers and Toeplitz flows”, pp. 7–37 in *Algebraic and topological dynamics* (Bonn, 2004), edited by S. Kolyada et al., Contemp. Math. **385**, Amer. Math. Soc., Providence, RI, 2005. MR Zbl
- [Hedlund 1969] G. A. Hedlund, “Endomorphisms and automorphisms of the shift dynamical system”, *Math. Systems Theory* **3** (1969), 320–375. MR Zbl
- [Hewitt and Ross 1979] E. Hewitt and K. A. Ross, *Abstract harmonic analysis, I: Structure of topological groups, integration theory, group representations*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **115**, Springer, 1979. MR Zbl
- [Hochman 2010] M. Hochman, “On the automorphism groups of multidimensional shifts of finite type”, *Ergodic Theory Dynam. Systems* **30**:3 (2010), 809–840. MR Zbl
- [Jacobs and Keane 1969] K. Jacobs and M. Keane, “0-1-sequences of Toeplitz type”, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **13** (1969), 123–131. MR Zbl

- [Kitchens 1998] B. P. Kitchens, *Symbolic dynamics: one-sided, two-sided and countable state Markov shifts*, Springer, 1998. MR Zbl
- [Krieger 2010] F. Krieger, “Toeplitz subshifts and odometers for residually finite groups”, pp. 147–161 in *École de Théorie Ergodique* (Marseilles, 2006), edited by Y. Lacroix et al., Sémin. Congr. **20**, Soc. Math. France, Paris, 2010. MR Zbl
- [Markley 1975] N. G. Markley, “Substitution-like minimal sets”, *Israel J. Math.* **22**:3-4 (1975), 332–353. MR Zbl
- [Markley and Paul 1979] N. G. Markley and M. E. Paul, “Almost automorphic symbolic minimal sets without unique ergodicity”, *Israel J. Math.* **34**:3 (1979), 259–272. MR Zbl
- [Olli 2013] J. Olli, “Endomorphisms of Sturmian systems and the discrete chair substitution tiling system”, *Discrete Contin. Dyn. Syst.* **33**:9 (2013), 4173–4186. MR Zbl
- [Veech 1970] W. A. Veech, “Point-distal flows”, *Amer. J. Math.* **92** (1970), 205–242. MR Zbl

Received: 2017-05-03

Revised: 2017-06-13

Accepted: 2018-06-25

medynets@usna.edu

*Mathematics Department, United States Naval Academy,  
Annapolis, MD, United States*

jptalisse@gmail.com

*Mathematics Department, United States Naval Academy,  
Annapolis, MD, United States*

# involve

msp.org/involve

## INVOLVE YOUR STUDENTS IN RESEARCH

*Involve* showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

### MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

### BOARD OF EDITORS

Colin Adams	Williams College, USA	Chi-Kwong Li	College of William and Mary, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Robert B. Lund	Clemson University, USA
Martin Bohner	Missouri U of Science and Technology, USA	Gaven J. Martin	Massey University, New Zealand
Nigel Boston	University of Wisconsin, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of N Carolina, Chapel Hill, USA	Frank Morgan	Williams College, USA
Pietro Cerone	La Trobe University, Australia	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Scott Chapman	Sam Houston State University, USA	Zuhair Nashed	University of Central Florida, USA
Joshua N. Cooper	University of South Carolina, USA	Ken Ono	Emory University, USA
Jem N. Corcoran	University of Colorado, USA	Yuval Peres	Microsoft Research, USA
Toka Diagana	Howard University, USA	Y.-F. S. Pétermann	Université de Genève, Switzerland
Michael Dorff	Brigham Young University, USA	Jonathon Peterson	Purdue University, USA
Sever S. Dragomir	Victoria University, Australia	Robert J. Plemmons	Wake Forest University, USA
Joel Foisy	SUNY Potsdam, USA	Carl B. Pomerance	Dartmouth College, USA
Errin W. Fulp	Wake Forest University, USA	Vadim Ponomarenko	San Diego State University, USA
Joseph Gallian	University of Minnesota Duluth, USA	Bjorn Poonen	UC Berkeley, USA
Stephan R. Garcia	Pomona College, USA	József H. Przytycki	George Washington University, USA
Anant Godbole	East Tennessee State University, USA	Richard Rebarber	University of Nebraska, USA
Ron Gould	Emory University, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Javier Rojo	Oregon State University, USA
Jim Haglund	University of Pennsylvania, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Johnny Henderson	Baylor University, USA	Hari Mohan Srivastava	University of Victoria, Canada
Glenn H. Hurlbert	Arizona State University, USA	Andrew J. Sterge	Honorary Editor
Charles R. Johnson	College of William and Mary, USA	Ann Trenk	Wellesley College, USA
K. B. Kulasekera	Clemson University, USA	Ravi Vakil	Stanford University, USA
Gerry Ladas	University of Rhode Island, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
David Larson	Texas A&M University, USA	John C. Wierman	Johns Hopkins University, USA
Suzanne Lenhart	University of Tennessee, USA	Michael E. Zieve	University of Michigan, USA

### PRODUCTION

Silvio Levy, Scientific Editor


Cover: Alex Scorpan

See inside back cover or [msp.org/involve](http://msp.org/involve) for submission instructions. The subscription price for 2019 is US \$195/year for the electronic version, and \$260/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

# involve

2019 vol. 12 no. 3

Darboux calculus	361
MARCO ALDI AND ALEXANDER MCCLEARY	
A countable space with an uncountable fundamental group	381
JEREMY BRAZAS AND LUIS MATOS	
Toeplitz subshifts with trivial centralizers and positive entropy	395
KOSTYA MEDYNETS AND JAMES P. TALISSE	
Associated primes of $h$ -wheels	411
COREY BROOKE, MOLLY HOCH, SABRINA LATO, JANET STRIULI AND BRYAN WANG	
An elliptic curve analogue to the Fermat numbers	427
SKYE BINEGAR, RANDY DOMINICK, MEAGAN KENNEY, JEREMY ROUSE AND ALEX WALSH	
Nilpotent orbits for Borel subgroups of $SO_5(k)$	451
MADELEINE BURKHART AND DAVID VELLA	
Homophonic quotients of linguistic free groups: German, Korean, and Turkish	463
HERBERT GANGL, GIZEM KARAALI AND WOOHYUNG LEE	
Effective moments of Dirichlet $L$ -functions in Galois orbits	475
RIZWANUR KHAN, RUOYUN LEI AND DJORDJE MILIĆEVIĆ	
On the preservation of properties by piecewise affine maps of locally compact groups	491
SERINA CAMUNGOL, MATTHEW MORISON, SKYLAR NICOL AND ROSS STOKKE	
Bin decompositions	503
DANIEL GOTSHALL, PAMELA E. HARRIS, DAWN NELSON, MARIA D. VEGA AND CAMERON VOIGT	
Rigidity of Ulam sets and sequences	521
JOSHUA HINMAN, BORYS KUCA, ALEXANDER SCHLESINGER AND ARSENIY SHEYDVASSER	



1944-4176(2019)12:3;1-2