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On the preservation of properties by piecewise affine maps of locally compact groups

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As shown by Cohen (1960) and Ilie and Spronk (2005), for locally compact groups G and H, there is a one-to-one correspondence between the completely bounded homomorphisms of their respective Fourier and Fourier–Stieltjes algebras $\varphi : A(G) \to B(H)$ and piecewise affine continuous maps $\alpha : Y \subseteq H \to G$. Using elementary arguments, we show that several (locally compact) group-theoretic properties, including amenability, are preserved by certain continuous piecewise affine maps. We discuss these results in relation to Fourier algebra homomorphisms.

Piecewise affine maps are, loosely speaking, finite unions of translations of subgroup homomorphisms. They seem to have been exclusively studied in connection with their applications to abstract harmonic analysis; see for example [Cohen 1960; Rudin 1962; Ilie 2004; Ilie and Spronk 2005; Ilie and Stokke 2008]. Our motivation in writing this paper has been to view piecewise affine maps as weak types of "generalized homomorphisms" and to study of them, in their own right, accordingly. Observe that most of our topologically imposed conditions are automatically satisfied by (discrete) groups and our results are also new in this situation.

Throughout this note, G and H are locally compact groups, and $\mathcal{P}$ will denote a property of locally compact groups. If E is a coset of a closed subgroup H_0 of H, we will say that E has $\mathcal{P}$ when H_0 has $\mathcal{P}$, and we define the *index* of E in H to be the index of H_0 in H. As noted in [Ilie 2004], a subset E of H is a coset of some subgroup of H exactly when $EE^{-1}E = E$, and a map $\alpha : E \to G$ is called *affine* if for any $x, y, z \in E$, $\alpha(xy^{-1}z) = \alpha(x)\alpha(y)^{-1}\alpha(z)$. Thus, the affine maps are the natural morphisms of cosets and the affine image of a coset is also a coset.

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Note that for any $y_0 \in E$, $H_0 = y_0^{-1}E = E^{-1}E$ is a subgroup of H, and the map defined by $\beta(h) = \alpha(y_0)^{-1}\alpha(y_0 h)$ ($h \in H_0$) is a homomorphism of H_0 into G when α is an affine map; conversely, if $\beta : H_0 \to G$ is a homomorphism and $x_0 \in G$, then $\alpha(x) = x_0\beta(y_0^{-1}x)$ defines an affine map on E; see [Ilie 2004, Remark 2.2]. Thus, affine maps are exactly the translates of subgroup homomorphisms. A map $\alpha : E \to G$ is *antiaffine* if for any $x, y, z \in E$, $\alpha(xy^{-1}z) = \alpha(z)\alpha(y)^{-1}\alpha(x)$. Hence, the antiaffine image of a coset is also a coset and, as with the affine case, one can readily check that the antiaffine maps on E are precisely the translates of subgroup antihomomorphisms on $H_0 = E^{-1}E$.

We let $\Omega(H)$ denote the ring of sets generated by the open cosets of H. Then every set in $\Omega(H)$ can be expressed as a finite union of disjoint sets in

$$\Omega_0(H) = \left\{ E_0 \setminus \left(\bigcup_1^m E_k \right) : \begin{array}{l} E_0 \subseteq H \text{ an open coset,} \\ E_1, \ldots, E_m \text{ open subcosets of infinite index in } E_0 \end{array} \right\}$$

[Cohen 1960; Ilie 2004]. If $Y = E_0 \setminus \left(\bigcup_1^m E_k\right) \in \Omega_0(H)$, Ilie showed that Aff$(Y)$, the coset generated by Y, is exactly E_0 and that there is a finite subset F of $E_0^{-1}E_0$ such that $E_0 = YF$ [Ilie 2004]. A map $\alpha : Y \to G$ is *piecewise affine* if

(†) there exist pairwise disjoint sets $Y_1, \ldots, Y_n \in \Omega_0(H)$ such that $Y = \bigcup_{i=1}^n Y_i$ and for each i, $\alpha|_{Y_i}$ has an affine extension α_i mapping $E_i = \text{Aff}(Y_i)$ into G;

when each α_i is antiaffine, α is *piecewise antiaffine*, and when each α_i is affine *or* antiaffine, α is *mixed piecewise affine*.

Notation. Whenever we say that $\alpha : Y \subseteq H \to G$ is a (mixed) piecewise affine map, we shall use precisely the notation found in (†).

The continuous (mixed) piecewise affine maps can be viewed as the natural morphisms to consider on sets in the open coset ring $\Omega(H)$ of H. Equivalent definitions of piecewise affine maps on nonabelian groups are found in [Ilie 2004]. We note that if α is proper, open, closed or injective then so is α_i for each $i = 1, \ldots, n$ [Ilie 2004, Proposition 4.6], [Ilie and Stokke 2008, Lemma 3.3] (the same argument works for closed maps) and [Pham 2010, proof of Theorem 6.4]. As well, if α is continuous, then so is each α_i (and the converse also holds). This is almost certainly known but the authors were unable to locate the statement in the literature; note that continuity of the affine extensions seems to be implicitly assumed in the definition of a piecewise affine map in [Ilie 2004]. Nevertheless, this is easy to see: Let F_i be a finite subset of $E_i^{-1}E_i$ such that $E_i = Y_i F_i$ [Ilie 2004, Lemma 4.5]. Let $x \in F_i$, say $x = u^{-1}v$ where $u, v \in E_i$. Then, for each $y \in Y_i x$, $\alpha_i(y) = \alpha(yx^{-1})\alpha_i(u)^{-1}\alpha_i(v)$, so α_i is continuous on $Y_i x$. By the pasting lemma, α_i is continuous on $Y_i F_i = E_i$.

Since the terminology of group extensions varies in the literature, we note that when N is a closed normal subgroup of G, we will call G an extension of G/N by N, (whereas in [Palmer 2001], for example, G is called an extension of N by G/N).

1. Properties preserved by mixed piecewise affine maps

One typically begins studying a property $\mathcal{P}$ of locally compact groups by asking if $\mathcal{P}$ is preserved by closed subgroups, quotients and extensions. Phrased in terms of homomorphic images, $\mathcal{P}$ is preserved by closed subgroups if whenever there exists a continuous, injective homomorphism $\phi : H \to G$ such that ϕ is a homeomorphism onto $\phi(H)$ with its relative topology and G has $\mathcal{P}$, then H has $\mathcal{P}$. Additionally, $\mathcal{P}$ is preserved by closed quotients if whenever there exists a continuous, surjective homomorphism $\phi : H \to G$ such that ϕ is an open map and H has $\mathcal{P}$, then G has $\mathcal{P}$. More generally, given a continuous mixed piecewise affine map $\alpha : Y \subseteq H \to G$, the main purpose of this section is to address the following two questions:

(a) If α has dense image in G and H has $\mathcal{P}$, when does G have $\mathcal{P}$?

(b) If $Y = H$ and α is injective (or proper) and G has $\mathcal{P}$, when does H have $\mathcal{P}$?

Since any homomorphism is a piecewise affine map, properties for which there is a positive answer to (a) must be preserved by quotients and properties for which there is a positive answer to (b) must be preserved by closed subgroups. As the following example shows, other restrictions on $\mathcal{P}$ must also be imposed.

Example 1. Suppose that G contains a finite-index closed — and therefore open — normal subgroup N. Let $\beta : N \to G/N \times N$ be the continuous open homomorphism defined by $\beta(z) = (e_{G/N}, z)$. Let $F \subseteq G$ be a complete set of representatives of distinct cosets of N and for each $x \in F$ define

$$\alpha_x : xN \to G/N \times N \quad \text{by} \quad \alpha_x(y) = (xN, e)\beta(x^{-1}y) = (xN, x^{-1}y).$$

Then α_x is an affine homeomorphism of xN onto $\{xN\} \times N$, so $\alpha : G \to G/N \times N$, defined by putting $\alpha|_{xN} = \alpha_x$ $(x \in F)$, is a homeomorphic piecewise affine bijection. Observe that since the inverse of an affine bijection between cosets is also affine, $\alpha^{-1} : G/N \times N \to G$ is also a piecewise affine homeomorphism. Thus, if $\mathcal{P}$ is a property for which there is a positive answer to either question (a) or (b) above, $G/N \times N$ has $\mathcal{P}$ exactly when G has $\mathcal{P}$ in this situation.

In particular, if $N \rtimes H$ is a semidirect product of a locally compact group N and a finite group H, the identity map is a piecewise affine homeomorphism of $N \times H$ onto $N \rtimes H$. However, $N \rtimes H$ may fail to be a homomorphic image of $N \times H$, such as when N and H are chosen to be abelian groups with $N \rtimes H$ nonabelian. As a specific example, consider $G = \mathbb{R} \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\pm 1\}$ acts on $\mathbb{R}$ via $(\pm 1)t = \pm t$. Then G not nilpotent or $[\mathrm{FC}]^-$ (and fails to have any property implying either of these properties) [Palmer 2001, Chapter 12], but since $\mathbb{R} \times \mathbb{Z}_2$ is abelian, it is both nilpotent and $[\mathrm{FC}]^-$. Hence, these are examples of properties $\mathcal{P}$ that are preserved by both quotients and closed subgroups, yet fail to provide a positive answer to either question (a) or (b), even when the piecewise affine maps involved are homeomorphisms with piecewise affine inverses.

Definition 2. We say that a property $\mathcal{P}$ of locally compact groups is *preserved by*

(a) *direct products with finite groups* if $H \times F$ has $\mathcal{P}$ whenever H has $\mathcal{P}$ and F is a finite group;

(b) *locally compact extensions of finite groups* if G has $\mathcal{P}$ whenever it contains a closed (and open) normal subgroup N such that G/N is finite and N has $\mathcal{P}$;

(c) *finite coset unions* if G has $\mathcal{P}$ whenever it can be written as a finite union of closed cosets, each of which has $\mathcal{P}$;

(d) *$\mathcal{P}$-by-compact extensions* if G has $\mathcal{P}$ whenever it contains a compact normal subgroup K such that G/K has $\mathcal{P}$.

The meaning of the statements "$\mathcal{P}$ is preserved by open (closed) subgroups", "$\mathcal{P}$ is preserved by dense-range continuous homomorphisms" and "$\mathcal{P}$ is preserved by dense-range continuous mixed piecewise affine maps" will be clear.

We remark that if $\mathcal{P}$ is preserved by direct products with finite groups and the trivial group has $\mathcal{P}$, then every finite group must have $\mathcal{P}$. Also, $\mathcal{P}$ satisfies condition (a) in Definition 2 whenever it satisfies condition (b), but not conversely: since $[\mathrm{FC}]^-$ is trivially closed under the formation of direct products and contains all finite groups, $\mathcal{P} = [\mathrm{FC}]^-$ satisfies (a), but $\mathbb{R} \rtimes \mathbb{Z}_2$ is not in $[\mathrm{FC}]^-$, so $[\mathrm{FC}]^-$ does not satisfy (b). Observe as well that $\mathcal{P}$ satisfies condition (b) in Definition 2 whenever it satisfies condition (c). In the proof of the following lemma, which establishes a partial converse to this last implication, we will use a theorem due to Neumann [1954] that says that a group cannot be expressed as a finite union of cosets of infinite index. An elegant, analytic proof of this theorem can be found in [Ilie and Spronk 2005]. Recall that open subgroups are always closed, and finite-index closed subgroups are always open.

Lemma 3. *If $\mathcal{P}$ is preserved by finite-index closed* (*equivalently open*) *normal subgroups and locally compact extensions of finite groups, then $\mathcal{P}$ is preserved by finite coset unions.*

Proof. Suppose that G can be expressed as a finite union of closed cosets, each with property $\mathcal{P}$. By Neumann's theorem, G contains a finite-index closed subgroup M such that M has $\mathcal{P}$. Then $N = \bigcap_{g \in G} gMg^{-1}$, the core of M in G, is a finite-index closed normal subgroup of G that is contained in M; see, e.g., [Isaacs 1994, Corollary 4.6]. Hence, N has $\mathcal{P}$ and G is an extension of the finite group G/N. Therefore, G also has $\mathcal{P}$. □

If $\phi : H \to G$ is an antihomomorphism, then $\check{\phi}(x) := \phi(x^{-1})(= \phi(x)^{-1})$ is a homomorphism with the same range as ϕ. Therefore if $\mathcal{P}$ is preserved by dense-range homomorphisms, it is also preserved by dense-range antihomomorphisms.

Proposition 4. *The following statements are equivalent*:

(i) *$\mathcal{P}$ is preserved by continuous dense-range homomorphisms* (*i.e.*, *quotients in the discrete case*), *locally compact extensions of finite groups and finite-index closed normal subgroups.*

(ii) *$\mathcal{P}$ is preserved by continuous dense-range mixed piecewise affine maps and products with finite groups.*

Proof. Assume that statement (i) holds, H has $\mathcal{P}$, and $\alpha : Y \subseteq H \to G$ is a continuous mixed piecewise affine map with dense range in G. Employing the notation (†), each of the cosets $\overline{\alpha_i(E_i)}$ has $\mathcal{P}$ and $G = \bigcup_{i=1}^{n} \overline{\alpha_i(E_i)}$. By Lemma 3, G has $\mathcal{P}$. Hence (ii) holds. Suppose, conversely, that statement (ii) holds, and let N be a finite-index closed normal subgroup of G. Then, as shown in Example 1, there is a piecewise affine homeomorphic mapping of $G/N \times N$ onto G with piecewise affine inverse. Hence, if N has $\mathcal{P}$, then $G/N \times N$ has $\mathcal{P}$, and therefore G has $\mathcal{P}$. If G has $\mathcal{P}$, then $G/N \times N$ has $\mathcal{P}$, whence N, as a quotient of $G/N \times N$, has $\mathcal{P}$. □

Remarks 5. If we replace the assumption that $\mathcal{P}$ is preserved by continuous dense-range homomorphisms in statement (i) of Proposition 4 with the statement that $\mathcal{P}$ is preserved by continuous open (closed) epimorphisms — i.e., quotients in the case of open epimorphisms — then we can conclude that $\mathcal{P}$ is preserved by continuous open (closed) surjective mixed piecewise affine maps: when α is open (closed) in the above proof, so is each α_i and therefore $\overline{\alpha_i(E_i)} = \alpha(E_i)$. For discrete groups, each of these conditions is equivalent to the statement that $\mathcal{P}$ is preserved by quotients.

We say that G is *virtually* $\mathcal{P}$ if G contains a finite-index closed (equivalently, open) subgroup with property $\mathcal{P}$.

Proposition 6. *The following statements hold*:

(i) *Virtually $\mathcal{P}$ is preserved by locally compact extensions of finite groups.*

(ii) *If $\mathcal{P}$ is preserved by finite-index closed normal subgroups*, *then so is virtually $\mathcal{P}$.*

(iii) *If $\mathcal{P}$ is preserved by continuous dense-range homomorphisms*, *then so is virtually $\mathcal{P}$.*

(iv) *If $\mathcal{P}$ is preserved by finite-index closed normal subgroups and locally compact extensions of finite groups*, *then every virtually-$\mathcal{P}$ group has property $\mathcal{P}$.*

Proof. (i) This is obvious.

(ii) Let M be a finite-index closed subgroup of G with property $\mathcal{P}$, and let N be a finite-index closed normal subgroup of G. Then $N \cap M$ is a finite-index closed subgroup of N, since — by a standard (readily verified) fact — $|N : N \cap M| \leq |G : M| < \infty$. Moreover, $|M : N \cap M| \leq |G : N| < \infty$, so $N \cap M$ is a finite-index closed normal subgroup of M. Since M has $\mathcal{P}$, so does $N \cap M$. Hence, N is virtually $\mathcal{P}$.

(iii) Let $\phi : H \to G$ be a continuous dense-range homomorphism and suppose that M is a closed subgroup of H with finite index — say $H = \bigcup_{i=1}^{n} h_i M$ — with property $\mathcal{P}$. Then $\overline{\phi(M)}$ has $\mathcal{P}$ and

$$\bigcup_{i=1}^{n} \phi(h_i)\overline{\phi(M)} = \overline{\bigcup_{i=1}^{n} \phi(h_i M)} = \overline{\phi(H)} = G.$$

Hence, G is virtually $\mathcal{P}$.

(iv) A virtually-$\mathcal{P}$ group is a finite union of cosets with $\mathcal{P}$, so this is an immediate consequence of Lemma 3. □

We note as well that if $\mathcal{P}$ is preserved by open (respectively closed) subgroups, then so is virtually $\mathcal{P}$. The following, which is an immediate consequence of Propositions 4 and 6, shows that virtually $\mathcal{P}$ is often preserved by mixed piecewise affine maps.

Corollary 7. *If $\mathcal{P}$ is preserved by continuous dense-range homomorphisms and finite-index closed normal subgroups, then virtually $\mathcal{P}$ is preserved by continuous dense-range mixed piecewise affine maps.*

We were unable to find a reference for the following lemma.

Lemma 8. *Let $\phi : H \to G$ be a continuous (anti-)homomorphism, $K = \ker \phi$, $\phi_K : H/K \to G : xK \mapsto \phi(x)$. Then ϕ is proper if and only if K is compact and ϕ_K is proper.*

Proof. Suppose that K is compact and A is a compact subset of H/K. Choose a compact subset L of H such that $\pi(L) = A$, where $\pi : H \to H/K$ is the quotient map [Fell and Doran 1988, Proposition III.2.5]. Since $\pi^{-1}(A) = LK$, which is compact, π is proper. Hence, if K is compact and ϕ_K is proper, then $\phi = \phi_K \circ \pi$ is proper. Conversely, if ϕ is proper, then $K = \phi^{-1}(\{e_G\})$ is compact and given any compact subset C of G, $\phi^{-1}(C) = \pi^{-1}(\phi_K^{-1}(C))$ is compact, whence $\phi_K^{-1}(C) = \pi(\phi^{-1}(C))$ is compact. Hence, ϕ_K is proper. □

Lemma 9. *A property $\mathcal{P}$ of locally compact groups is preserved by closed subgroups and $\mathcal{P}$-by-compact extensions if and only if*

(∗) *H has $\mathcal{P}$ whenever there exists a proper continuous (anti-)homomorphism mapping H into a locally compact group G that has $\mathcal{P}$.*

Proof. Suppose that $\alpha : H \to G$ is a proper continuous (anti-)homomorphism where G has $\mathcal{P}$ and $\mathcal{P}$ is preserved by closed subgroups and $\mathcal{P}$-by-compact extensions. Letting $K = \ker \alpha$, K is compact and α_K is a continuous proper (anti-)isomorphism of H/K onto its image by Lemma 8. Since proper maps are closed, α_K is, in fact, a topological (anti-)isomorphism of H/K onto $\alpha_K(H/K)$. We can conclude that H/K has $\mathcal{P}$, and therefore H has $\mathcal{P}$. Conversely, if $\mathcal{P}$ satisfies $(*)$, then $\mathcal{P}$ is obviously preserved by closed subgroups and, since the quotient map of G onto G/K is proper when K is compact, it is preserved by $\mathcal{P}$-by-compact extensions. □

Suppose that H has $\mathcal{P}$ whenever there exists a continuous mixed piecewise affine proper mapping of H into a locally compact group G that has $\mathcal{P}$ and, further, that $\mathcal{P}$ is preserved by the formation of direct products with finite groups. If N, a closed finite-index normal subgroup of G, has $\mathcal{P}$, then $G/N \times N$ has $\mathcal{P}$ and, by Example 1, one can define a homeomorphic piecewise affine mapping of $G/N \times N$ onto G; hence G has $\mathcal{P}$. This, together with Lemma 9, establishes "(ii) implies (i)" of the following proposition.

Proposition 10. *The following statements are equivalent*:

(i) *$\mathcal{P}$ is preserved by closed subgroups, $\mathcal{P}$-by-compact extensions, and locally compact extensions of finite groups.*

(ii) *$\mathcal{P}$ is preserved by the formation of direct products with finite groups, and H has $\mathcal{P}$ whenever there exists a continuous mixed piecewise affine proper mapping of H into a locally compact group G that has $\mathcal{P}$.*

Proof. We only need to show that statement (i) implies the second condition found in statement (ii). To this end, let $\alpha : H \to G$ be a continuous mixed piecewise affine proper map. Using the notation (†), each α_i is a proper continuous affine, or antiaffine, mapping of E_i into G [Ilie 2004, Proposition 4.6]. Since each α_i can be obtained through translation of a continuous proper homomorphism, or antihomomorphism, on the subgroup $E_i^{-1}E_i$ of H, each coset E_i has $\mathcal{P}$ by Lemma 9. Since H is the union of the closed cosets E_i $(i = 1, \ldots, n)$, Lemma 3 allows us to conclude that H has $\mathcal{P}$. □

Example 11. Some examples of properties of locally compact groups that are preserved by open subgroups (and therefore by finite-index closed normal subgroups), continuous dense-range homomorphisms, and locally compact extensions of finite groups are amenability and compactness; within the class of discrete groups, torsion, local finiteness, polynomial growth and exponential growth have these hereditary properties. Thus, for each of these properties, virtually $\mathcal{P}$ and $\mathcal{P}$ are equivalent, and each is preserved by continuous dense-range mixed piecewise affine maps.

Examples of some properties that are preserved by open subgroups, quotients, and locally compact extensions of finite groups are those listed in the last paragraph and the properties of being an [IN]-group or a [SIN]-group. These properties are all preserved by continuous, open mixed piecewise affine surjections.

Examples of some properties $\mathcal{P}$ that are preserved by closed subgroups, $\mathcal{P}$-by-compact extensions, and locally compact extensions of finite groups are amenability, compactness and the property of being an [IN]-group. For each of these properties, H has $\mathcal{P}$ whenever there exists a continuous mixed piecewise affine proper mapping of H into G for some G with $\mathcal{P}$. A reference for these assertions is [Palmer 2001, Chapter 12].

2. Remarks concerning Fourier algebra homomorphisms

With pointwise-defined operations and a particular norm that dominates the uniform norm, the Fourier–Stieltjes algebra $B(G)$ is a Banach algebra of continuous complex-valued functions on G containing the Fourier algebra $A(G)$ as a closed ideal [Eymard 1964]. A long-standing open problem in abstract harmonic analysis asks for a description of every homomorphism mapping $A(G)$ into $B(H)$ and, as we have already noted, piecewise affine maps have primarily been studied in relation to this problem. A solution was obtained by Cohen [1960] in the abelian case, a solution that was generalized by Ilie and Spronk [2005] when G is amenable and the homomorphism is completely bounded, and Pham [2010] when the homomorphism is norm decreasing.

Using the fact that $A(G)$ separates points and closed sets, i.e., $A(G)$ is a regular algebra of continuous functions on G, and the fact that the Gelfand spectrum of $A(G)$ — the set of nonzero multiplicative linear functionals on $A(G)$ — is exactly the set of point-evaluation maps $\delta_g(u) := u(g)$ ($g \in G$, $u \in A(G)$), one can see that for any homomorphism $\varphi : A(G) \to B(H)$ there is an open subset Y of H and a continuous map $\alpha : Y \to G$ such that $\varphi = j_\alpha$, where for $u \in A(G)$

$$j_\alpha(u) = \begin{cases} u \circ \alpha & \text{on } Y, \\ 0 & \text{on } H \setminus Y. \end{cases}$$

(For each $h \in H$, either $\delta_h \circ \varphi = 0$ or $\delta_h \circ \varphi$ belongs to the Gelfand spectrum of $A(G)$, whence $\delta_h \circ \varphi = \delta_{\alpha(h)}$ for some $\alpha(h) \in G$. Letting $Y = \{h \in H : \delta_h \circ \varphi \neq 0\}$, one obtains $\alpha : Y \to G$ such that $\varphi = j_\alpha$.) By [Ilie 2004, Proposition 3.9], which does not require that G be amenable or that α be piecewise affine, $j_\alpha : A(G) \to B(H)$ maps $A(G)$ into $A(H)$ exactly when α is a proper map. An easy application of the regularity of $A(G)$ is that a map $\varphi = j_\alpha : A(G) \to B(H)$ is injective exactly when $\alpha : Y \subseteq H \to G$ has dense range. Observe as well that $Y = H$ exactly when $\delta_h \circ \varphi \neq 0$ for each $h \in H$. These facts are used below without comment.

As preduals of von Neumann algebras, $A(G)$ and $B(H)$ have operator space structures with respect to which they are completely contractive Banach algebras

[Effros and Ruan 2000], so it makes sense to speak of completely bounded homomorphisms $\varphi : A(G) \to B(H)$. If $\alpha : Y \subseteq H \to G$ is continuous and piecewise affine, Ilie and Spronk showed that j_α is a completely bounded homomorphism of $A(G)$ into $B(H)$ and, moreover, when G is amenable, every completely bounded homomorphism $\varphi : A(G) \to B(H)$ equals j_α for some continuous piecewise affine map $\alpha : Y \subseteq H \to G$ [Ilie and Spronk 2005, Theorem 3.7]. Thus, the following statement is an immediate consequence of Proposition 4, Corollary 7 and Proposition 10.

Proposition 12. *Suppose that G is amenable and there exists a completely bounded homomorphism φ mapping $A(G)$ into $B(H)$:*

(i) *Suppose that $\mathcal{P}$ is preserved by continuous dense-range homomorphisms, locally compact extensions of finite groups, and finite-index closed normal subgroups. If φ is injective and H has $\mathcal{P}$, then so does G.*

(ii) *Suppose that $\mathcal{P}$ is preserved by continuous dense-range homomorphisms and finite-index closed normal subgroups. If φ is injective and H is virtually $\mathcal{P}$, then so is G.*

(iii) *Suppose that $\mathcal{P}$ is preserved by closed subgroups, $\mathcal{P}$-by-compact extensions and locally compact extensions of finite groups. Suppose further that φ maps $A(G)$ into $A(H)$ and for each $h \in H$, $\delta_h \circ \varphi \neq 0$. If G has $\mathcal{P}$, then so does H.*

Amenability of Banach algebras is not, in general, preserved by closed subalgebras, much less injective homomorphisms; for example, the semigroup algebra $\ell^1(\mathbb{N})$ is a nonamenable subalgebra of the (Connes) amenable Banach algebra $\ell^1(\mathbb{Z})$. However, since $A(H)$ is an amenable Banach algebra ($B(H)$ is a Connes amenable Banach algebra) exactly when H is virtually abelian [Forrest and Runde 2005; Runde and Uygul 2015] and the property of being abelian is preserved by subgroups and continuous dense-range homomorphisms, the following is an immediate corollary of Proposition 12(ii).

Corollary 13. *Suppose that G is amenable and $A(H)$ is amenable (equivalently, $B(H)$ is Connes amenable). If there exists an injective completely bounded homomorphism φ mapping $A(G)$ into $A(H)$ or $B(H)$, then $A(G)$ is amenable.*

We remark that by applying the main result in [Pham 2010], we obtain the same conclusions in Proposition 12 and Corollary 13 if we drop the condition that G is amenable and replace the assumption of the existence of a completely bounded homomorphism with that of a norm-decreasing homomorphism.

N. Spronk [2010, Conjecture 4.8] has conjectured that when G is amenable, every homomorphism $\varphi : A(G) \to B(H)$ takes the form $\varphi = j_\alpha$ for some mixed piecewise affine map $\alpha : Y \subseteq H \to G$. If correct, then Propositions 4, 7 and 10 would imply that the statements of Proposition 12 and Corollary 13 hold without the assumption that

φ is completely bounded. Thus, the results of this note suggest a possible method of testing the conjecture: for instance, an example of an amenable, but not virtually abelian, group G and a virtually abelian group H for which there exists an injective homomorphism φ mapping $A(G)$ into $B(H)$ would disprove the conjecture. On the other hand, since the conjecture may well be correct, Proposition 12 suggests that when G is amenable, every Fourier algebra homomorphism $A(G) \to B(H)$ preserves certain properties $\mathcal{P}$, as described in the proposition. Explicitly, we have the following question:

Question 14. Given a specific property $\mathcal{P}$ satisfying the conditions described in one of the statements in Proposition 12, does the corresponding statement of Proposition 12 hold if the homomorphism φ is not assumed to be completely bounded? That is, can such a statement be established without necessarily verifying the Spronk conjecture?

Any positive answer would lend evidence in support of the conjecture (and a negative answer would disprove it). For example, since it is known that the property of being an amenable locally compact group satisfies all of the conditions considered in this note, Proposition 12(iii) suggests the following, which, as we now observe, is a consequence of [Kaniuth and Ülger 2010, Theorem 5.1]: this theorem states that a locally compact group G is amenable if and only if $A(G)$ contains a bounded net $(e_i)_i$ converging pointwise on G to 1 (i.e., $A(G)$ contains a "Δ-weak bounded approximate identity").

Proposition 15. *Suppose there exists a homomorphism $\varphi : A(G) \to A(H)$ such that for each $h \in H$, $\delta_h \circ \varphi \neq 0$. If G is amenable, then so is H.*

Proof. Since G is amenable, $A(G)$ has a Δ-weak bounded approximate identity $(e_i)_i$. As noted above, $\varphi = j_\alpha$ for some (continuous, proper) map $\alpha : H \to G$. As noted by Pham [2010], since $A(H)$ is semisimple, φ is automatically bounded, so $\varphi(e_i)$ is a bounded net in $A(H)$ such that for each $h \in H$, $\varphi(e_i)(h) = e_i(\alpha(h)) \to 1$. Thus, $\varphi(e_i)$ is a Δ-weak bounded approximate identity in $A(H)$, whence H is amenable by [Kaniuth and Ülger 2010, Theorem 5.1]. □

We remark that when G is amenable, $A(G)$ actually has a bounded approximate identity $(e_i)_i$ (and the converse holds) by Leptin's theorem, but it is not clear that $\varphi(e_i)$ in the proof of Proposition 15 is then a bounded approximate identity for $A(H)$. That is, more than Leptin's theorem was required to prove the above proposition. Observe that in establishing Proposition 15, we did not assume that amenability actually satisfies any of the hereditary properties described in Propositions 10 and 12(iii), because these hereditary properties are not employed in the proof of [Kaniuth and Ülger 2010, Theorem 5.1] (and the theory on which it depends). Since whenever $\alpha : H \to G$ is a proper continuous mixed piecewise affine map, $\varphi = j_\alpha$ is a homomorphism of $A(G)$ into $A(H)$ such that, for each $h \in H$, $\delta_h \circ \varphi \neq 0$, we obtain — *independent of the hereditary properties of amenability* (and therefore independent of Proposition 10) — the following immediate corollary of Proposition 15.

Corollary 16. *If G is amenable and there exists a proper continuous mixed piecewise affine map α of H into G, then H is amenable. In particular, closed subgroups of amenable locally compact groups are amenable.*

Thus, [Kaniuth and Ülger 2010, Theorem 5.1] and the basic fact that proper continuous group homomorphisms determine Fourier algebra homomorphisms yield a new proof that closed subgroups of locally compact groups are amenable. This seems interesting because although [Kaniuth and Ülger 2010, Theorem 5.1] is certainly not at all obvious, the standard proof of this fundamental hereditary property, which in the nondiscrete case involves the construction of a Bruhat function for H on G (e.g., see [Pier 1984, Section 13] or [Runde 2002, Section 1.2]), is also not at all obvious.

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