involve
a journal of mathematics

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msp

2019 vol. 12, no. 4

INVOLVE 12:4 (2019)
dx.doi.org/10.2140/involve.2019.12.541

Euler's formula for the zeta function at the positive even integers

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(Communicated by Filip Saidak)

We give a new proof of Euler's formula for the values of the Riemann zeta function at the positive even integers. The proof involves estimating a certain integral of elementary functions two different ways and using a recurrence relation for the Bernoulli polynomials evaluated at $\frac{1}{2}$.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta function and let $\eta(s) = (1-2^{1-s})\zeta(s)$. Then the series representations

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{and} \quad \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

converge absolutely in the half-plane $\mathrm{Re}(s) > 1$. For $n \in \mathbb{N}$, we define the Bernoulli polynomials $B_n(x)$ via the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!},$$

and (as usual) we call $B_n := B_n(0)$ the n-th Bernoulli number. It follows that

$$B_2 = \tfrac{1}{6}, \quad B_4 = -\tfrac{1}{30}, \quad B_6 = \tfrac{1}{42}, \quad \ldots, \quad B_{12} = -\tfrac{691}{2730}, \tag{1-1}$$

and that

$$B_{2n+1} = 0 \quad \text{for } n \in \mathbb{N}. \tag{1-2}$$

These and other standard properties of the Bernoulli numbers and Bernoulli polynomials can be found in [Montgomery and Vaughan 2007, Appendix B]. In this note we give an apparently new proof of Euler's well-known result which states that

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!} \quad \text{for } k \in \mathbb{N}. \tag{1-3}$$

MSC2010: primary 11M06; secondary 11B68, 11B37.
Keywords: Riemann zeta function, Euler, Basel problem, Bernoulli numbers, Bernoulli polynomials.

From (1-1) and (1-3), we see (as Euler did) that

$$\zeta(2) = \tfrac{1}{6}\pi^2, \quad \zeta(4) = \tfrac{1}{90}\pi^4, \quad \zeta(6) = \tfrac{1}{945}\pi^6, \quad \ldots, \quad \zeta(12) = \tfrac{691}{638512875}\pi^{12}.$$

In 1734, before realizing the connection to the Bernoulli numbers, Euler derived the values of $\zeta(2k)$ for $k = 1, 2, \ldots, 6$. A few years later, in 1740, Euler discovered the formula in (1-3) relating $\zeta(2k)$ to B_{2k} for $k \in \mathbb{N}$. Some historical remarks about Euler's work on the Riemann zeta function and on other infinite series can be found in [Weil 1984, Chapter 3], see also [Ayoub 1974; Kline 1983; Varadarajan 2007], while references to numerous proofs of Euler's formula in (1-3) can be found in [de Amo et al. 2011].

Instead of evaluating $\zeta(2k)$ directly, our proof naturally evaluates the function $\eta(s)$ at the positive even integers. Since

$$B_n\big(\tfrac{1}{2}\big) = -(1-2^{1-n})B_n \quad \text{for } n \geq 0, \tag{1-4}$$

we note that Euler's result in (1-3) is equivalent to the formula

$$\eta(2k) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{2k}} = (-1)^k \frac{(2\pi)^{2k} B_{2k}\big(\tfrac{1}{2}\big)}{2(2k)!} \quad \text{for } k \in \mathbb{N}. \tag{1-5}$$

We derive (1-5) in Section 3. Note that (1-1), (1-4), and (1-5) imply

$$\eta(2) = \tfrac{1}{12}\pi^2, \quad \eta(4) = \tfrac{7}{720}\pi^4, \quad \eta(6) = \tfrac{31}{30240}\pi^6, \quad \ldots, \quad \eta(12) = \tfrac{1414477}{1307674368000}\pi^{12}.$$

Since our proof of (1-5) is more straightforward in the special case $k = 1$, we discuss this situation separately at the end of this article.

There is a striking resemblance between Euler's formula (1-3), relating the values of $\zeta(2k)$ to B_{2k}, and the formula (1-5), relating the values of $\eta(2k)$ to $B_{2k}\big(\tfrac{1}{2}\big)$. We have chosen to write the expression in (1-5) in this manner for more than simply aesthetic reasons; indeed our proof of (1-5) relies naturally on a recursive formula for the sequence $\big\{B_{2k}\big(\tfrac{1}{2}\big)\big\}_{k=0}^{\infty}$.

2. A recursive formula for $B_{2k}\big(\tfrac{1}{2}\big)$

The Bernoulli polynomials satisfy the inversion formula

$$x^n = \frac{1}{n+1} \sum_{\ell=0}^{n} \binom{n+1}{\ell} B_\ell(x)$$

for every integer $n \geq 0$. Setting $x = \tfrac{1}{2}$ and then observing that (1-2) and (1-4) imply $B_n\big(\tfrac{1}{2}\big) = 0$ if n is odd, we derive the recursive formula

$$\frac{1}{2^{2k}} = \frac{1}{2k+1} \sum_{j=0}^{k} \binom{2k+1}{2j} B_{2j}\big(\tfrac{1}{2}\big) \quad \text{for } k \in \mathbb{N}. \tag{2-1}$$

3. Proof of (1-5)

We prove (1-5) by evaluating the integral

$$I_{2k} = \int_0^1 \frac{x(\log x)^{2k}}{(x^2+1)^2}\,\mathrm{d}x \quad \text{for } k \in \mathbb{N}$$

in two different ways. On one hand, we show that

$$I_{2k} = \frac{(2k)!}{2^{2k+1}}\eta(2k) \tag{3-1}$$

by expressing the integrand as a series and then integrating term-by-term. The formula (3-1) actually holds for $k = 0$ as well, since $I_0 = \frac{1}{4}$ and it can be shown that $\eta(0) = \frac{1}{2}$. On the other hand, using the residue theorem in a relatively standard way, we derive the recursive formula

$$\frac{1}{2^{2k}} = \frac{1}{2k+1}\sum_{j=0}^{k}\binom{2k+1}{2j}(-1)^j\,\frac{4I_{2j}}{\pi^{2j}}. \tag{3-2}$$

Comparing this expression to the recurrence relation for $B_{2k}\left(\frac{1}{2}\right)$ from the previous section, we can derive our desired expression for $\eta(2k)$ from (3-1) and (3-2).

Proof of (1-5). Evidently, from (2-1) and (3-2), the sequences

$$\left\{B_{2j}\left(\tfrac{1}{2}\right)\right\}_{j=0}^{\infty} \quad \text{and} \quad \left\{(-1)^j\,\frac{4I_{2j}}{\pi^{2j}}\right\}_{j=0}^{\infty}$$

satisfy the same recursion relation. Moreover, since

$$4I_0 = 4\int_0^1 \frac{x}{(x^2+1)^2}\,\mathrm{d}x = 1 = B_0\left(\tfrac{1}{2}\right),$$

the initial terms in these sequences agree and therefore these sequences are equal. Hence, from (3-1), we see that

$$B_{2k}\left(\tfrac{1}{2}\right) = (-1)^k\frac{4I_{2k}}{\pi^{2k}} = (-1)^k\frac{2(2k)!}{(2\pi)^{2k}}\eta(2k) \quad \text{for every } k \in \mathbb{N}. \qquad \square$$

It remains to establish (3-1) and (3-2).

3.1. ***Relating I_{2k} to $\eta(2k)$.*** Integrating by parts $2k$ times, we derive that

$$\int_0^1 x^{2n-1}(\log x)^{2k}\,\mathrm{d}x = \frac{(2k)!}{(2n)^{2k+1}} \tag{3-3}$$

for positive integers k and n. Alternatively, we can prove this estimate by using that the gamma function,

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,\mathrm{d}x \quad \text{for } \operatorname{Re}(z) > 0,$$

satisfies the relation $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. To see this, note that the variable change $x \mapsto e^{-t/(2n)}$ implies

$$\int_0^1 x^{2n}(\log x)^{2k}\,\frac{\mathrm{d}x}{x} = \frac{1}{(2n)^{2k+1}}\int_0^\infty e^{-t}t^{2k}\,\mathrm{d}t = \frac{\Gamma(2k+1)}{(2n)^{2k+1}} = \frac{(2k)!}{(2n)^{2k+1}}.$$

We now express the integrand of I_{2k} as a series, interchange the sum and the integral, and then use (3-3) to integrate term-by-term. Since

$$\begin{aligned}\frac{x}{(x^2+1)^2} &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1+x^2}\right\}\\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=0}^\infty(-1)^n x^{2n}\right\} = \sum_{n=1}^\infty n(-1)^{n-1}x^{2n-1}\end{aligned}\tag{3-4}$$

for $|x| < 1$, we have

$$\begin{aligned}I_{2k} = \int_0^1 \frac{x(\log x)^{2k}}{(x^2+1)^2}\,\mathrm{d}x &= \int_0^1 \sum_{n=1}^\infty n(-1)^{n-1}x^{2n-1}(\log x)^{2k}\,\mathrm{d}x\\ &= \sum_{n=1}^\infty n(-1)^{n-1}\int_0^1 x^{2n-1}(\log x)^{2k}\,\mathrm{d}x\\ &= \sum_{n=1}^\infty n(-1)^{n-1}\frac{(2k)!}{(2n)^{2k+1}} = \frac{(2k)!}{2^{2k+1}}\eta(2k)\end{aligned}$$

for every $k \in \mathbb{N}$. This proves (3-1). Note that the interchange of summation and integration is justified using Fubini's theorem since, for every $k \in \mathbb{N}$, (3-3) implies

$$\begin{aligned}\sum_{n=1}^\infty \int_0^1 |n(-1)^{n-1}x^{2n-1}\log^{2k} x|\,\mathrm{d}x &= \sum_{n=1}^\infty n\int_0^1 x^{2n-1}(\log x)^{2k}\,\mathrm{d}x\\ &= \frac{(2k)!}{2^{2k+1}}\sum_{n=1}^\infty \frac{1}{n^{2k}} < \infty.\end{aligned}$$

3.2. ***A recursive formula for I_{2k}.*** Making the variable change $x \mapsto 1/x$, it follows that

$$\int_0^1 \frac{x(\log x)^{2k}}{(x^2+1)^2}\,\mathrm{d}x = \int_1^\infty \frac{x(\log x)^{2k}}{(x^2+1)^2}\,\mathrm{d}x,$$

$$\int_0^1 \frac{x(\log x)^{2k+1}}{(x^2+1)^2}\,\mathrm{d}x = -\int_1^\infty \frac{x(\log x)^{2k+1}}{(x^2+1)^2}\,\mathrm{d}x$$

for integers $k \geq 0$. Therefore

$$I_{2k} = \frac{1}{2}\int_0^\infty \frac{x(\log x)^{2k}}{(x^2+1)^2}\,\mathrm{d}x \quad\text{and}\quad \int_0^\infty \frac{x(\log x)^{2k+1}}{(x^2+1)^2}\,\mathrm{d}x = 0.\tag{3-5}$$

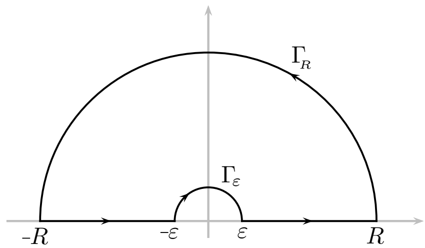

Figure 1

Now we introduce the complex-valued function

$$f(z) = \frac{z(\log z)^{2k+1}}{(1+z^2)^2},$$

where $\log z$ denotes the branch of the logarithm in $\mathbb{C}$ with $|z| > 0$ and $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$. Note that the power of $\log z$ in the numerator of $f(z)$ is one power higher than the power of $\log x$ appearing in the integrand of I_{2k}. We integrate $f(z)$ around the positively oriented simple closed contour (shown in Figure 1) composed of the line segment $[\varepsilon, R]$ along the real-axis, the semicircle Γ_R centered at 0 of radius R starting at $z = R$ passing through $z = iR$ and ending at $z = -R$, the line segment $[-R, -\varepsilon]$ along the real-axis, and finally the semicircle Γ_ε centered at 0 of radius ε starting at $z = -\varepsilon$ passing through $z = i\varepsilon$ and ending at $z = \varepsilon$. Here ε and R denote real numbers satisfying $0 < \varepsilon < 1 < R < \infty$. The only singularity of $f(z)$ inside this contour is a double pole at $z = i$. Therefore the residue theorem implies

$$\begin{aligned}2\pi i \operatorname*{Res}_{z=i} f(z) = \int_\varepsilon^R \frac{x(\log x)^{2k+1}}{(1+x^2)^2}\,\mathrm{d}x + \int_{\Gamma_R} f(z)\,\mathrm{d}z \\ + \int_{-R}^{-\varepsilon} \frac{x(\log(-x)+i\pi)^{2k+1}}{(1+x^2)^2}\,\mathrm{d}x + \int_{\Gamma_\varepsilon} f(z)\,\mathrm{d}z,\end{aligned} \tag{3-6}$$

where the logarithms in the first and third integrals on the right-hand side denote the natural logarithm. Estimating trivially, we have

$$\left|\int_{\Gamma_\varepsilon} f(z)\,\mathrm{d}z\right| \le \mathrm{length}(\Gamma_\varepsilon) \cdot \max_{z\in\Gamma_\varepsilon} |f(z)| \le (\pi\varepsilon)\left(\frac{\varepsilon(\log(-\varepsilon)+\pi)^{2k+1}}{(1-\varepsilon^2)^2}\right) \to 0$$

as $\varepsilon \to 0^+$ and

$$\left|\int_{\Gamma_R} f(z)\,\mathrm{d}z\right| \le \mathrm{length}(\Gamma_R) \cdot \max_{z\in\Gamma_R} |f(z)| \le (\pi R)\left(\frac{R(\log R+\pi)^{2k+1}}{(1-R^2)^2}\right) \to 0$$

as $R \to +\infty$. It follows that

$$2\pi i \operatorname*{Res}_{z=i} f(z) = \int_0^\infty \frac{x(\log x)^{2k+1}}{(1+x^2)^2}\,\mathrm{d}x + \int_{-\infty}^0 \frac{x(\log(-x)+i\pi)^{2k+1}}{(1+x^2)^2}\,\mathrm{d}x.$$

By the second expression in (3-5), the first integral on the right-hand side equals 0. Sending $x \mapsto -x$, the second integral on the right-hand side equals

$$-\int_0^\infty \frac{x(\log x + i\pi)^{2k+1}}{(1+x^2)^2}\,\mathrm{d}x = -\sum_{\ell=0}^{2k+1}\binom{2k+1}{\ell}(i\pi)^{2k-\ell+1}\int_0^\infty \frac{x(\log x)^\ell}{(1+x^2)^2}\,\mathrm{d}x.$$

Again by (3-5), the terms in the sum with ℓ odd vanish. Hence, for even ℓ, letting $\ell = 2j$ and using the first expression in (3-5), we have

$$2\pi i \operatorname*{Res}_{z=i} f(z) = -(i\pi)^{2k+1}\sum_{j=0}^{k}\binom{2k+1}{2j}(-1)^j\,\frac{2I_{2j}}{\pi^{2j}}. \tag{3-7}$$

On the other hand, a straightforward calculation shows that

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{z(\log z)^{2k+1}}{(z+i)^2}\right\} = -\frac{(2k+1)(i\pi)^{2k}}{2^{2k+2}}.$$

Inserting this into (3-7) and dividing by $-(2k+1)(i\pi)^{2k+1}/2$, we conclude that

$$\frac{1}{2^{2k}} = \frac{1}{2k+1}\sum_{j=0}^{k}\binom{2k+1}{2j}(-1)^j\,\frac{4I_{2j}}{\pi^{2j}},$$

as claimed.

3.3. ***Remarks on the case k = 1.*** Historically, the *Basel problem* asked for a closed-form evaluation of the sum

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

As mentioned in the Introduction, this problem was solved by Euler in 1734. Therefore, there is perhaps special interest in a direct proof of the equivalent problem of showing that

$$\eta(2) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

In this special case, our proof above can be simplified since there is no need to appeal to properties of the Bernoulli polynomials, the gamma function, or recursion relations. We sketch the details of this calculation for the interested reader.

In this case, we evaluate the integral

$$I_2 = \int_0^1 \frac{x(\log x)^2}{(x^2+1)^2}\,\mathrm{d}x$$

in two different ways. Integrating by parts twice, it can be shown that

$$\int_0^1 x^{2n-1}(\log x)^2\,\mathrm{d}x = \frac{1}{4n^3} \quad \text{for } n \in \mathbb{N}. \tag{3-8}$$

Therefore, using the series expansion in (3-4), it follows that

$$\begin{aligned} I_2 = \int_0^1 \frac{x(\log x)^2}{(x^2+1)^2}\,\mathrm{d}x &= \int_0^1 \sum_{n=1}^{\infty} n(-1)^{n-1} x^{2n-1}(\log x)^2\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} n(-1)^{n-1} \int_0^1 x^{2n-1}(\log x)^2\,\mathrm{d}x \\ &= \frac{1}{4}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\eta(2)}{4}. \end{aligned}$$

As in Section 3.1, the interchange of summation and integration can be justified using Fubini's theorem. On the other hand, making the variable change $x \mapsto 1/x$, it follows that

$$I_2 = \int_0^1 \frac{x(\log x)^2}{(x^2+1)^2}\,\mathrm{d}x = \int_1^{\infty} \frac{x(\log x)^2}{(x^2+1)^2}\,\mathrm{d}x.$$

Therefore

$$I_2 = \frac{1}{2}\int_0^{\infty} \frac{x(\log x)^2}{(x^2+1)^2}\,\mathrm{d}x. \tag{3-9}$$

In order to evaluate this integral, we apply the residue theorem in a manner similar to that in the previous section. We integrate the complex-valued function

$$f(z) = \frac{z(\log z)^3}{(1+z^2)^2}$$

around the positively oriented simple closed contour shown in Figure 1. As before, $\log z$ denotes the branch of the logarithm in $\mathbb{C}$ with $|z| > 0$ and $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$, while ε and R denote real numbers satisfying $0 < \varepsilon < 1 < R < \infty$. Then the residue theorem implies

$$\begin{aligned} 2\pi i \operatorname*{Res}_{z=i} f(z) = \int_{\varepsilon}^{R} \frac{x(\log x)^3}{(1+x^2)^2}\,\mathrm{d}x + \int_{\Gamma_R} f(z)\,\mathrm{d}z \\ + \int_{-R}^{-\varepsilon} \frac{x(\log(-x)+i\pi)^3}{(1+x^2)^2}\,\mathrm{d}x + \int_{\Gamma_\varepsilon} f(z)\,\mathrm{d}z, \end{aligned}$$

where the logarithms in the first and third integrals on the right-hand side denote the natural logarithm. As was shown in the previous section, the second and fourth integrals on the right-hand side tend to 0 as $R \to +\infty$ and $\varepsilon \to 0^+$, respectively. Since the only singularity of $f(z)$ inside this contour is a double pole at $z = i$ with

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} \frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{z(\log z)^3}{(z+i)^2}\right\} = \frac{3\pi^2}{16},$$

it follows that

$$\begin{aligned}\frac{3\pi^3 i}{8} &= \int_0^\infty \frac{x(\log x)^3}{(1+x^2)^2}\,\mathrm{d}x + \int_{-\infty}^0 \frac{x(\log(-x)+i\pi)^3}{(1+x^2)^2}\,\mathrm{d}x \\ &= \int_0^\infty \frac{x(\log x)^3}{(1+x^2)^2}\,\mathrm{d}x - \int_0^\infty \frac{x(\log x+i\pi)^3}{(1+x^2)^2}\,\mathrm{d}x.\end{aligned}$$

Here we have made the variable change $x \mapsto -x$ in the second integral. Expanding the factor $(\log x + i\pi)^3$, taking imaginary parts of both sides of the equation, and then using (3-9), we deduce that

$$\frac{3\pi^3}{8} = \pi^3 \int_0^\infty \frac{x}{(1+x^2)^2}\,\mathrm{d}x - 3\pi \int_0^\infty \frac{x(\log x)^2}{(1+x^2)^2}\,\mathrm{d}x = \frac{\pi^3}{2} - 6\pi I_2.$$

This implies $I_2 = \pi^2/48$. Combining this with our previous observation that $I_2 = \eta(2)/4$, we conclude that $\eta(2) = \pi^2/12$.

Acknowledgements

Milinovich is supported in part by the NSA Young Investigator Grant H98230-16-1-0311.

References

[de Amo et al. 2011] E. de Amo, M. Díaz Carrillo, and J. Fernández-Sánchez, "Another proof of Euler's formula for $\zeta(2k)$", *Proc. Amer. Math. Soc.* **139**:4 (2011), 1441–1444. MR Zbl

[Ayoub 1974] R. Ayoub, "Euler and the zeta function", *Amer. Math. Monthly* **81** (1974), 1067–1086. MR Zbl

[Kline 1983] M. Kline, "Euler and infinite series", *Math. Mag.* **56**:5 (1983), 307–314. MR Zbl

[Montgomery and Vaughan 2007] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory, I: Classical theory*, Cambridge Studies in Advanced Mathematics **97**, Cambridge University Press, 2007. MR Zbl

[Varadarajan 2007] V. S. Varadarajan, "Euler and his work on infinite series", *Bull. Amer. Math. Soc.* (*N.S.*) **44**:4 (2007), 515–539. MR Zbl

[Weil 1984] A. Weil, *Number theory: an approach through history, from Hammurapi to Legendre*, Birkhäuser, Boston, MA, 1984. MR Zbl

Received: 2017-06-12 Revised: 2018-07-30 Accepted: 2018-10-28

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2019 vol. 12 no. 4