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Descents and des-Wilf equivalence of permutations avoiding certain nonclassical patterns

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A frequent topic in the study of pattern avoidance is identifying when two sets of patterns Π, Π' are Wilf equivalent, that is, when $|\mathrm{Av}_n(\Pi)| = |\mathrm{Av}_n(\Pi')|$ for all n. In recent work of Dokos et al. the notion of Wilf equivalence was refined to reflect when avoidance of classical patterns preserves certain statistics. We continue their work by examining des-Wilf equivalence when avoiding certain nonclassical patterns.

1. Introduction

Let $\mathfrak{S}_n$ denote the set of permutations of $[n] := \{1, \dots, n\}$, and let $\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \cdots$ be the set of all permutations of finite length. We write $\sigma \in \mathfrak{S}_n$ as $\sigma = a_1 a_2 \cdots a_n$ to indicate that $\sigma(i) = a_i$. A function $\mathrm{st} : \mathfrak{S}_n \to \mathbb{N}$ is called a *statistic*, and the systematic study of permutation statistics is generally accepted to have begun with MacMahon [1960, Volume I, Section III, Chapter V]. Four of the most well-known statistics are the *descent*, *inversion*, *major*, and *excedance* statistics, defined respectively by

$$\begin{aligned}
\mathrm{des}(\sigma) &= |\mathrm{Des}(\sigma)|, \\
\mathrm{inv}(\sigma) &= |\{(i, j) \in [n]^2 \mid i < j \text{ and } a_i > a_j\}|, \\
\mathrm{maj}(\sigma) &= \sum_{i \in \mathrm{Des}(\sigma)} i, \\
\mathrm{exc}(\sigma) &= |\{i \in [n] \mid a_i > i\}|,
\end{aligned}$$

where $\mathrm{Des}(\sigma) = \{i \in [n-1] \mid a_i > a_{i+1}\}$. Given any statistic st, one may form the generating function

$$F_n^{\mathrm{st}}(q) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{st}\,\sigma}.$$

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A famous result due to [MacMahon 1960] states that $F_n^{\mathrm{des}}(q) = F_n^{\mathrm{exc}}(q)$, and that both are equal to the *Eulerian polynomial* $A_n(q)$. Similarly, it is known that $F_n^{\mathrm{inv}}(q) = F_n^{\mathrm{maj}}(q) = [n]_q!$, where

$$[n]_q = 1 + q + \cdots + q^{n-1} \quad \text{and} \quad [n]_q! = [n]_q\,[n-1]_q \cdots [1]_q.$$

Let $A \subseteq [n]$, and denote by $\mathfrak{S}_A$ the set of permutations of the elements of A. The *standardization* of $\sigma = a_1 \cdots a_{|A|} \in \mathfrak{S}_A$ is the element of $\mathfrak{S}_{|A|}$ whose letters are in the same relative order as those of σ; we denote this permutation by $\mathrm{std}(\sigma)$. Now, we say that a permutation $\sigma \in \mathfrak{S}_n$ *contains the pattern* $\pi \in \mathfrak{S}_k$ if there exists a subsequence $\sigma' = a_{i_1} \cdots a_{i_k}$ of σ such that $\mathrm{std}(\sigma') = \pi$. If no such subsequence exists, then we say that σ *avoids the pattern* π. Since we will introduce additional notions of patterns, we may call such a pattern a *classical pattern* to avoid confusion. If $\Pi \subseteq \mathfrak{S}$, then we say σ *avoids* Π if σ avoids every element of Π. The set of all permutations of $\mathfrak{S}_n$ avoiding Π is denoted by $\mathrm{Av}_n(\Pi)$. In a mild abuse of notation, if $\Pi = \{\pi\}$, we will write $\mathrm{Av}_n(\pi)$. If Π, Π' are two sets of patterns and $|\mathrm{Av}_n(\Pi)| = |\mathrm{Av}_n(\Pi')|$ for all n, then we say Π and Π' are *Wilf equivalent* and write $\Pi \equiv \Pi'$.

These ideas may be combined by setting

$$F_n^{\mathrm{st}}(\Pi; q) = \sum_{\sigma \in \mathrm{Av}_n(\Pi)} q^{\mathrm{st}\,\sigma}.$$

This allows one to say that Π, Π' are st-*Wilf equivalent* if $F_n^{\mathrm{st}}(\Pi; q) = F_n^{\mathrm{st}}(\Pi'; q)$ for all n, and write this as $\Pi \overset{\mathrm{st}}{\equiv} \Pi'$. Thus, Π and Π' may be Wilf equivalent without being st-Wilf equivalent. As a concrete example, 123 and 321 are clearly not des-Wilf equivalent, even though they are Wilf equivalent. It is straightforward to check that st-Wilf equivalence is indeed an equivalence relation on $\mathfrak{S}$.

Since it is generally a difficult question to determine whether two sets are nontrivially Wilf equivalent, one should not expect it to be any easier to determine st-Wilf equivalence. However, it is certainly possible to obtain some results; see [Dokos et al. 2012] for results regarding F_n^{inv} and F_n^{maj}, and [Baxter 2014; Cameron and Killpatrick 2015] for further results, including a study of enumeration strategies for questions of this nature. In this article, we will study $F_n^{\mathrm{des}}(\Pi; q)$ for certain nonclassical patterns, called mesh patterns and barred patterns. Special cases will allow us to identify des-Wilf equivalences. We will also present several conjectural des-Wilf equivalences and provide computational evidence for these.

2. Pattern avoidance background

Classical patterns. In order to work most efficiently, it is important to recognize that certain Wilf equivalences are almost immediate to establish. For example, it is obvious that $|\mathrm{Av}_n(123)| = |\mathrm{Av}_n(321)|$, since $a_1 \cdots a_n \in \mathrm{Av}_n(123)$ if and only if $a_n a_{n-1} \cdots a_1 \in \mathrm{Av}_n(321)$. This idea can be generalized significantly.

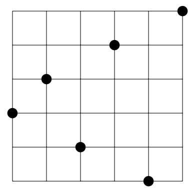


Figure 1. The plot of 342516.

The *plot* of $\sigma \in \mathfrak{S}_n$ is the set of pairs $(i, \sigma(i)) \in \mathbb{R}^2$ and will be denoted by $P(\sigma)$. The plot of 342516 is shown in Figure 1. Let

$$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, r_{-1}, r_0, r_1, r_\infty\},$$

where R_θ is counterclockwise rotation of a plot by an angle of θ degrees and r_m is reflection across a line of slope m. A couple of these rigid motions have easy descriptions in terms of the one-line notation for permutations. If $\pi = a_1a_2\cdots a_k$ then its *reversal* is $\pi^r = a_k\cdots a_2a_1 = r_\infty(\pi)$, and its *complement* is $\pi^c = (k+1-a_1)(k+1-a_2)\cdots(k+1-a_k) = r_0(\pi)$.

Note that $\sigma \in \mathrm{Av}_n(\pi)$ if and only if $f(\sigma) \in \mathrm{Av}_n(f(\pi))$ for any $f \in D_4$; hence $\pi \equiv f(\pi)$. For this reason, the equivalences induced by the dihedral action on a square are often referred to as the *trivial Wilf equivalences*.

Using these techniques, it is easy to show that 123 and 321 are trivially Wilf equivalent, as are all of 132, 213, 231, and 312. It is less obvious, however, whether 123 and 132 are Wilf equivalent. This question was settled by independent results due to [MacMahon 1960] and [Knuth 1969], whose combined work showed that $\mathrm{Av}_n(132)$ and $\mathrm{Av}_n(123)$ are enumerated by the *n-th Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Catalan numbers are famous for appearing in a multitude of combinatorial situations; see [Stanley 2015] for many of them.

One of the most well-known combinatorial objects enumerated by the Catalan numbers are Dyck paths. A *Dyck path of length* $2n$ is a lattice path in $\mathbb{R}^2$ starting at $(0, 0)$ and ending at $(2n, 0)$, using steps $(1, 1)$ and $(1, -1)$, which never goes below the x-axis. See Figure 2 for an example Dyck path of length 8.

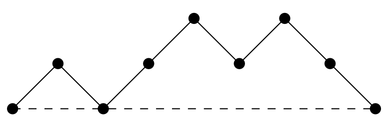
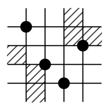


Figure 2. A Dyck path of length 8.

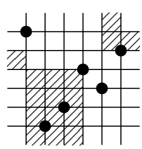
Nonclassical patterns. In this section, we will define two classes of nonclassical patterns and describe what it means for a permutation to contain or avoid them. The definitions of Wilf equivalence and des-Wilf equivalence then extend to these patterns in the same way as classical patterns, so their precise definitions will be omitted.

A *mesh pattern* is a pair (π, M), where $\pi \in \mathfrak{S}_k$ and $M \subseteq [0, k]^2$. Mesh patterns are a vast generalization of classical patterns and were first introduced by Brändén and Claesson [2011]. It is convenient to represent a mesh pattern as a grid which plots π and shades in the unit squares whose bottom-left corners are the elements of M. For example, one may represent the mesh pattern $(\pi_0, M_0) = (4213, \{(0, 2), (1, 0), (1, 1), (3, 3), (3, 4), (4, 3)\})$ as follows:

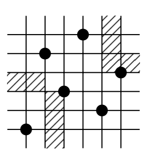
$(\pi_0, M_0) =$  .

Containment of mesh patterns is most easily understood by an informal statement and illustrative examples; the formal definition, given in [Brändén and Claesson 2011], shows that the intuition developed this way behaves as expected. We say that $\sigma \in \mathfrak{S}_n$ *contains the mesh pattern* (π, M) if σ contains an occurrence of π and the shaded regions of $P(\pi)$ corresponding to this occurrence contain no other elements of $P(\sigma)$. If σ does not contain (π, M), then we say σ *avoids* (π, M).

For the illustrative examples, first consider $\sigma = 612435$. Notice that while 6435 is an occurrence of 4213 in σ, it is not an occurrence of the mesh pattern (π_0, M_0) given above, since the shaded regions in $P(\sigma)$ dictated by M_0 yield



Now consider $\sigma' = 153624$. In this case, 5324 is an occurrence of both 4213 and (π_0, M_0) in σ', since the shading in this case is



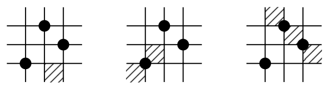
In certain cases, determining which permutations avoid a mesh pattern (π, M) with M nonempty is equivalent to determining which permutations avoid π as a classical pattern. When this happens, we say that (π, M) has *superfluous mesh*, and Tenner [2013] identified when exactly a mesh pattern has superfluous mesh. To do this, we first define an *enclosed diagonal* of (π, M) to be a triple $((i, j), \varepsilon, \ell)$ where $\varepsilon \in \{-1, 1\}$, $\ell \geq 1$, and the following three properties hold:

(1) The plot of π contains the set $\{(i+d, j+\varepsilon d) \mid 1 \leq d < \ell\}$.

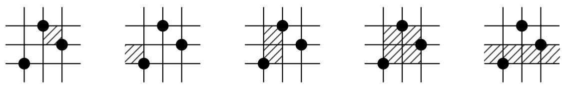
(2) The plot of π contains neither (i, j) nor $(i+\ell, j+\varepsilon\ell)$.

(3) $\{(i+d, j+\varepsilon d) \mid 0 \leq d < \ell\} \subseteq M$.

Note that an enclosed diagonal may consist of a single element, as long as the corresponding box in the mesh pattern contains no element of $P(\pi)$. To illustrate, the following three mesh patterns all have a unique enclosed diagonal:



However, none of the following five mesh patterns have any enclosed diagonals:



The following theorem gives the characterization of when a pattern has superfluous mesh. As a result, we will not focus on any patterns with superfluous mesh, but we will still use the theorem briefly.

Theorem 2.1 [Tenner 2013, Theorem 3.5′]. *A mesh pattern has superfluous mesh if and only if it has no enclosed diagonals.*

Mesh patterns also generalize 1-*barred patterns*, in which a classical pattern is allowed (but not required) to have a bar above one letter. This is a special case of *barred patterns*, in which each letter is allowed to have a bar above it. The bars above letters indicate that certain additional rules are required in order to define containment of the pattern. We will not give the precise definition of containment and avoidance of barred patterns in general, but will observe that if there are two or more bars in the pattern, there is not necessarily a simple translation of the barred pattern into a mesh pattern. In some instances, a barred pattern may be described as a *decorated mesh pattern* [Úlfarsson 2011/12], but this is not always possible. To avoid this difficulty in the statement and proof of Proposition 3.7, we will simply describe here what it means for a permutation to avoid two specific barred patterns.

We say that $\sigma = a_1 \cdots a_n$ *avoids* $\bar{1}\bar{2}43$ if, whenever $a_i a_j$ is an occurrence of 21, then there are some integers k, l such that $k < l < i$ and $a_k a_l a_i a_j$ is an occurrence of 1243. We also say that σ *avoids* $\bar{1}32\bar{4}$ if, whenever $a_i a_j$ is an occurrence of 21, then there are some integers k, l such that $k < i < j < l$ and $a_k a_i a_j a_l$ is an occurrence of 1324. As an example, $\sigma = 124635$ avoids $\bar{1}\bar{2}43$ since all occurrences of 21, which are 43, 63, and 65, extend to an occurrence of 1243 by placing 12 before them. However, σ contains $\bar{1}32\bar{4}$ since 63, which is an occurrence of 21, does not play the role of 32 in any occurrence of 1324 in σ.

3. Main results

We now have all of the tools we need to begin proving results. We begin with a simple application of several known theorems.

Proposition 3.1. *If* $(132, M_1)$ *and* $(312, M_2)$ *are mesh patterns, neither of which contain an enclosed diagonal, then*

$$(132, M_1) \overset{\text{des}}{\equiv} (312, M_2).$$

Proof. By Theorem 2.1, $\text{Av}_n((312, M_2)) = \text{Av}_n(312)$, so $(312, M_2) \overset{\text{des}}{\equiv} 312$. It then follows directly from [Reifegerste 2003, Remark 2.5(b)] that the number of elements in $\text{Av}_n(312)$ with exactly k descents is

$$N_{n,k} := \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

Since the sequence $\{N_{n,k}\}_{k=0}^{n-1}$ is symmetric for fixed n, and since

$$\text{des}(\sigma) = n - 1 - \text{des}(\sigma^c),$$

we have

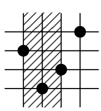
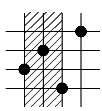
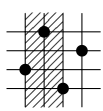
$$(312, M_2) \overset{\text{des}}{\equiv} 312 \overset{\text{des}}{\equiv} 132.$$

Again by Theorem 2.1, we have $\text{Av}_n(132) = \text{Av}_n((132, M_1))$, so these two patterns are des-Wilf equivalent as well. Connecting the equivalences, the claim follows. $\square$

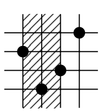
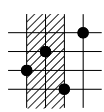
Characterizing the des-Wilf classes for mesh patterns (π, M) where $\pi \in \mathfrak{S}_4$ is difficult, and we will not attempt to fully characterize the des-Wilf equivalence classes of such patterns. In what follows, we merely wish to present a step toward understanding these in more depth, but first we need two more definitions.

If $A \subseteq [n]$, $f \in D_4$, and $\sigma \in \mathfrak{S}_A$, then we let $f^A(\sigma)$ denote the unique element of $\mathfrak{S}_A$ whose standardization is $f(\text{std}(\sigma))$. We say that f^A is a dihedral action *relative to* A. As a simple example, if $7461 \in \mathfrak{S}_{\{1,4,6,7\}}$, then $\text{std}(7461) = 4231$ and $R_{90}^{\{1,4,6,7\}}(\sigma) = 1647$.

Theorem 3.2. *We have*

 $\overset{\text{des}}{\equiv}$  $\overset{\text{des}}{\equiv}$  .

Proof. First consider

$(\pi_1, M_1) =$  and $(\pi_2, M_2) =$  .

To prove their des-Wilf equivalence, we will form a des-preserving bijection

$$\alpha : \mathfrak{S}_n \setminus \mathrm{Av}_n((\pi_1, M_1)) \to \mathfrak{S}_n \setminus \mathrm{Av}_n((\pi_2, M_2)),$$

that is, a des-preserving bijection between permutations in $\mathfrak{S}_n$ containing (π_1, M_1) and those containing (π_2, M_2).

Suppose $\sigma = a_1 \cdots a_n \in \mathfrak{S}_n$ contains (π_1, M_1). If σ contains (π_2, M_2), then set $\alpha(\sigma) = \sigma$. Otherwise, let j be the smallest index in which an occurrence of (π_1, M_1) begins, and consider $a_i a_{i+1} \cdots a_p$, where

$$p = \min\{m \mid m > j+2,\ a_m > a_j\},$$
$$i = \min\{m \mid m \leq j,\ a_m, a_{m+1}, \ldots, a_j < a_p\}.$$

Let $A = \{a_i, a_{i+1}, \ldots, a_p\}$, and set

$$R_{180}^{A}(a_i \cdots a_p) = b_i \cdots b_p,$$

and further set

$$\alpha(\sigma) = a_1 \cdots a_{i-1} b_i \cdots b_p a_{p+1} \cdots a_n.$$

Since R_{180}^{A} is a des-preserving map, we have that for any $k \in \{1, \ldots, p-1-i\}$, $i+k \in \mathrm{Des}(\sigma)$ if and only if $p-k \in \mathrm{Des}(\alpha(\sigma))$. Additionally, for any $k \in \{1, \ldots, i-1, p, p+1, \ldots, n-1\}$, $k \in \mathrm{Des}(\sigma)$ if and only if $k \in \mathrm{Des}(\alpha(\sigma))$. Thus, α is des-preserving.

To show that α is invertible, we will construct a map

$$\beta : \mathfrak{S}_n \setminus \mathrm{Av}_n((\pi_2, M_2)) \to \mathfrak{S}_n \setminus \mathrm{Av}_n((\pi_1, M_1))$$

and show that $\beta \circ \alpha$ is the identity map on $\mathfrak{S}_n \setminus \mathrm{Av}_n((\pi_1, M_1))$. If $\sigma' = a'_1 \cdots a'_n$ contains (π_2, M_2), then we create $\beta(\sigma)$ by first testing a construction similar to the one from the previous paragraph. Namely, let j' be the smallest index in which an occurrence of (π_2, M_2) begins, and consider $a'_i a'_{i+1} \cdots a'_p$, where

$$p' = \min\{m \mid m > j'+2,\ a'_m > a'_{j'+1}\},$$
$$i' = \min\{m \mid m \leq j',\ a'_m, a'_{m+1}, \ldots, a_{j'} < a'_p\}.$$

This time, let $A' = \{a'_i, a'_{i+1}, \ldots, a'_p\}$, and set

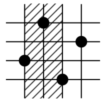
$$R_{180}^{A'}(a'_i \cdots a'_p) = b'_i \cdots b'_p.$$

If $a'_1 \cdots a'_{i-1} b'_i \cdots b'_{p'} a'_{p'+1} \cdots a'_n$ contains both (π_2, M_2) and (π_1, M_1), then set $\beta(\sigma') = \sigma'$. Otherwise, set

$$\beta(\sigma') = a'_1 \cdots a'_{i-1} b'_i \cdots b'_{p'} a'_{p'+1} \cdots a'_n.$$

The fact that $\beta \circ \alpha$ is the identity map on $\mathfrak{S}_n \setminus \mathrm{Av}_n((\pi_1, M_1))$ follows from construction.

Now consider (π_2, M_2) and

$(\pi_3, M_3) =$  .

Suppose $\sigma = a_1 a_2 \cdots a_n$ and $a_j a_{j+1} a_{j+2} a_p$ is the first copy of (π_3, M_3), as identified in the second paragraph in this proof. If a_p is the only a_l for which $l > j+2$ and $a_l > a_j$, then set $\alpha(\sigma)$ to be σ with a_{j+1} and a_p transposed. Otherwise, choose

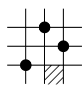
$$r = \min\{l \mid a_j < a_l < a_{j+1},\ l > j+2\}.$$

Let $S = \{a_r, a_{r+1}, \ldots, a_q\}$ where q is the maximum index for which $\{a_r, a_{r+1}, \ldots, a_q\}$ is increasing and $a_j < a_k < a_{j+1}$ for all $k \in S$. Set $\alpha(\sigma)$ to be σ with a_{j+1} and $\max S$ transposed. By choosing the maximum of S we are guaranteeing that α is des-preserving. By construction, $\alpha(\sigma)$ contains an occurrence of (π_2, M_2). Using an argument similar to the first part of this proof, α is invertible and is therefore a bijection. □

Recall that the *Stirling numbers of the second kind*, denoted by $S(n, k)$, record the number of ways to partition $[n]$ into k nonempty blocks. Here, we will begin to find useful the notation

$$\mathrm{Av}_n^{\mathrm{des},k}(\Pi) = \{\sigma \in \mathrm{Av}_n(\Pi) \mid \mathrm{des}(\sigma) = k\}.$$

Proposition 3.3. *Let*

$(\pi, M) =$  .

For all n, we have

$$F_n^{\mathrm{des}}((\pi, M); q) = \sum_{k=0}^{n-1} S(n, k+1) q^k.$$

Proof. Let $\Sigma_{n,k}$ denote the collection of set partitions of $[n]$ into exactly k nonempty blocks. We will create a bijection

$$f : \mathrm{Av}_n^{\mathrm{des},k}((\pi, M)) \to \Sigma_{n,k+1},$$

from which the conclusion follows.

First, let $\sigma = a_1 \cdots a_n \in \mathrm{Av}_n^{\mathrm{des},k}((\pi, M))$. It follows from [Burstein and Lankham 2005/07, Theorem 4.1] that any such permutation is the concatenation of substrings

$$\begin{gathered} a_1 < \cdots < a_{i_0}, \\ a_{i_0+1} < \cdots < a_{i_1}, \\ \vdots \\ a_{i_k+1} < \cdots < a_n, \end{gathered}$$

where $a_1 < a_{i_j+1} > a_{i_{j+1}+1}$ for all j. In particular, the values $a_{i_0}, \dots, a_{i_k}$ determine the entire permutation.

Associate to σ the set partition

$$f(\sigma) = \{\{a_1, \dots, a_{i_0}\}, \{a_{i_0+1}, \dots, a_{i_1}\}, \dots, \{a_{i_k+1}, \dots, a_n\}\}.$$

Note that if $\sigma = 12\cdots n$, then $k = 0$, so this partition consists of only one block. Thus, if σ has k descents, then the partition obtained has $k+1$ blocks. Because each choice of the a_{i_j} determines σ, we know that $f(\sigma) \neq f(\sigma')$ whenever $\sigma' \in \mathrm{Av}_n^{\mathrm{des},k}((\pi, M))$ and $\sigma \neq \sigma'$. That is, f is injective.

Now we will show that f is surjective. Consider a set partition $B = \{B_1, \dots, B_{k+1}\}$ of $[n]$ into $k+1$ blocks. We are free to write the B_i such that

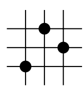
$$B_i = \{b_{i,1} < \cdots < b_{i,i_l}\} \quad \text{and} \quad \min B_i < \min B_{i+1}$$

for all i. Construct the permutation

$$b_{k+1,1} b_{k+1,2} \cdots b_{k+1,i_{k+1}} b_{k,1} b_{k,2} \cdots b_{k,i_k} \cdots b_{1,1} b_{1,2} \cdots b_{1,i_1}.$$

We claim that this permutation is an element of $\mathrm{Av}_n^{\mathrm{des},k}((\pi, M))$.

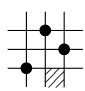
Any occurrence of

,

say, $b_\alpha b_\beta b_\gamma$, implies that $b_\alpha \in B_i$, $b_\beta \in B_j$, and $b_\gamma \in B_k$ for some $i \le j < k$. Since the sequence of minima of the blocks is decreasing, we know that $\min B_k < b_\alpha < b_\gamma$. Thus, the string

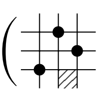
$$b_\alpha b_\beta (\min B_k) b_\gamma$$

is an occurrence of

.

Since the elements of the blocks strictly increase, the minima decrease, and since there are $k+1$ blocks, there are k descents in the permutation. Thus f is surjective, completing the proof. □

Example 3.4. Consider the permutation

$3427156 \in \mathrm{Av}_6^{\mathrm{des},2}\Big($  $\Big)$.

Our construction in the previous proof associates to this permutation the partition

$$\{\{3, 4\}, \{2, 7\}, \{1, 5, 6\}\}.$$

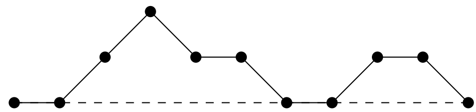


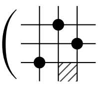
Figure 3. A Motzkin path of length 10 with 3 up-steps.

In the other direction, given the set partition

$$\{\{5\},\{3,1,4\},\{7,2,6\}\}=\{\{5\},\{2,6,7\},\{1,3,4\}\},$$

we obtain the permutation 5267134, which the reader may verify is indeed an element of

$$\mathrm{Av}_7^{\mathrm{des},2}\Bigg(\quad\Bigg).$$



A *Motzkin path of length n* is a lattice path from $(0,0)$ to $(n,0)$ using only *up-steps* $(1,1)$, *down-steps* $(1,-1)$, and *horizontal steps* $(1,0)$ such that the path does not go below the x-axis. An example is shown in Figure 3. We let $\mathcal{M}_{n,k}$ denote the set of Motzkin paths of length n with exactly k up-steps.

The next result we present was first proven in [Chen et al. 2002/03] by writing Motzkin paths according to a "strip decomposition" and by writing permutations according to canonical reduced decompositions. Here, we present a new, simpler proof. To do so, we only need a few more definitions.

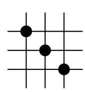
If i is a descent of $\sigma=a_1\cdots a_n$, then we call a_i a *descent top* and a_{i+1} a *descent bottom*. Let $\mathrm{Destop}(\sigma)$ denote the set of descent tops of σ and let $\mathrm{Desbot}(\sigma)$ denote the set of descent bottoms of σ. A *valley* in σ is an element i for which $a_{i-1}>a_i<a_{i+1}$.

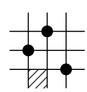
Theorem 3.5 [Chen et al. 2002/03, Theorem 3.1]. *Let*

$$\Pi=\{(\pi_4,M_4),(\pi_5,M_5)\},$$

where

$$(\pi_4,M_4)=\qquad,\qquad(\pi_5,M_5)=\qquad.$$





For all n,

$$F_n^{\mathrm{des}}(\Pi;q)=\sum_{k=0}^{n}|\mathcal{M}_{n,k}|q^k.$$

Proof. We will form a bijection

$$\mu:\mathrm{Av}_n^{\mathrm{des},k}(\Pi)\to\mathcal{M}_{n,k}.$$

For $\sigma=a_1\cdots a_n\in\mathrm{Av}_n^{\mathrm{des},k}(\Pi)$, let $\mu(\sigma)$ be the lattice path obtained by making step a_i a down-step if a_i is a descent bottom, an up-step if a_i is a descent top, and a horizontal step if a_i is neither.

First, we need to check that μ is well-defined. Note that no letter of σ can be both a descent top and a descent bottom, since this would imply σ contains an instance of π_4, which is forbidden. So, since the sets of descent tops and of descent bottoms are disjoint, and these appear in pairs, we can be certain that the path constructed by μ has length n and ends at $(n, 0)$. Moreover, since a descent top always appears before a descent bottom, at no step of the path can there have been more down-steps than up-steps. This establishes that $\mu(\sigma)$ is a Motzkin path of length n. Finally, since there are k descents, there are k descent tops, and $\mu(\sigma)$ will have k up-steps. Hence, $\mu(\sigma) \in \mathcal{M}_{n,k}$.

Next we will show that μ is injective. To do so, we will determine exactly the structure of the elements in $\mathrm{Av}_n(\Pi)$. Notice that the descent bottoms of σ must appear in increasing order in σ, since, otherwise, there would be an occurrence of π_4. For the same reason, the descent tops must appear in increasing order in σ.

Let $\sigma = a_1 \cdots a_n \in \mathrm{Av}_n^{\mathrm{des},k}(\Pi)$ and suppose that i is neither a descent top nor a descent bottom. Suppose for now that j is the first descent greater than i. If $a_{j+1} < a_i < a_j$, then $a_i a_j a_{j+1}$ is an occurrence of 231. Since σ avoids (π_5, M_5), there must be some l for which σ has the subsequence $a_i a_l a_j a_{j+1}$ and $a_l < a_{j+1}$. This implies that some integer $i+1, i+2, \ldots, l-1$ is a descent, which contradicts the fact that j is the first descent greater than i. So, it must be true that $a_i < a_{j+1} < a_j$. Since j is the first descent greater than i, it follows that $a_i a_{i+1} \cdots a_{j-1} a_{j+1}$ is an increasing sequence. It follows that the subsequence of σ consisting of all letters that are not descent tops is an increasing sequence.

Now we will show that μ is injective. If $\mu(\sigma_1) = \mu(\sigma_2)$ for $\sigma_1, \sigma_2 \in \mathrm{Av}_n(\Pi)$, then $\mathrm{Destop}(\sigma_1) = \mathrm{Destop}(\sigma_2)$ and $\mathrm{Desbot}(\sigma_1) = \mathrm{Desbot}(\sigma_2)$, since these are identified by the up-steps and down-steps in the Motzkin path. Our description of elements of $\mathrm{Av}_n(\Pi)$ shows that once the descent-top sets and descent-bottom sets have been identified, there is a unique σ in the avoidance class with those sets. Therefore, $\sigma_1 = \sigma_2$, and μ is injective.

Finally, we will show that μ is surjective. Let $A \in \mathcal{M}_{n,k}$, and label its steps $1, \ldots, n$ from left to right. We will construct its preimage in stages. First write down $1, \ldots, n$, but exclude the labels on the down-steps. Then insert the label on the i-th down-step immediately before the label of the i-th up-step. Call the resulting permutation σ_A. Using the description of elements of $\mathrm{Av}_n(\Pi)$ from earlier in this proof, we see that $\sigma_A \in \mathrm{Av}_n(\Pi)$. Additionally, it is clear that $\mu(\sigma_A) = A$ by our construction of σ_A and the definition of μ. Therefore, μ is surjective, completing the proof. □

Example 3.6. Let A be the Motzkin path in Figure 3. Steps 2, 3, and 8 are up-steps, and therefore will be descents bottoms. Steps 4, 6, and 10 are down-steps, so these will be descent tops. The remaining numbers will be neither descent tops nor bottoms.

When the descent tops are removed from $\mu^{-1}(A)$, the result will be an increasing string of numbers: 1235789. The descent tops are then placed immediately preceding the descent bottoms, to obtain 1426357(10)89.

For the final result of the section, we make two notes. First, recall that the *Eulerian polynomial* $A_n(q)$ is the polynomial

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{des}(\sigma)} = A_n(q).$$

It should be noted that some authors, e.g., in [Stanley 1997], define the Eulerian polynomials using $q^{\mathrm{des}(\sigma)+1}$ rather than the definition given here. So, one should take care when encountering Eulerian polynomials in the literature. Second, recall from the end of Section 2 what it means for a permutation to contain and avoid the barred patterns $\bar{1}\bar{2}43$ or $\bar{1}32\bar{4}$.

Proposition 3.7. *For all n,*

$$F_n^{\mathrm{des}}(\bar{1}\bar{2}43;q) = F_n^{\mathrm{des}}(\bar{1}32\bar{4};q) = \begin{cases} 1 & \text{if } n=0,1, \\ A_{n-2}(q) & \text{if } n\ge 2. \end{cases}$$

Proof. We will first show that $F_n(\bar{1}\bar{2}43;q)$ satisfies the right-hand side. The conclusion is clearly true for $n<2$, so we will restrict our attention to when $n\ge 2$. Choose $\sigma = a_1\cdots a_n \in \mathrm{Av}_n(\bar{1}\bar{2}43)$. Note first that $a_1<a_2$ since, if $a_1>a_2$, then a_1a_2 would be an occurrence of $u(\bar{1}\bar{2}43)=21$ but this cannot extend to an occurrence of 1243.

Now, suppose $a_2>2$. Setting $a_m=\min\{a_i \mid 3\le i\le n\}$ we have $a_2>a_m$, so a_2a_m is an occurrence of $u(\bar{1}\bar{2}43)$ in σ. However, there is only letter to the left of a_2, so this pattern does not extend to an instance of 1243. Thus, $a_2=2$. Together with the previous paragraph, we know $a_1=1$ as well. In particular, $a_1<a_2<a_i$ for all $i\ge 3$.

Now, take any occurrence a_ia_j of 21 in which $2<i<j$. Clearly, $a_1a_2a_ia_j$ is an extension to 1243. This holds for any possible permutation of $3,\dots,n$ as the final $n-2$ letters. Since 1 and 2 are never descents of these permutations, we have

$$F_n^{\mathrm{des}}(\bar{1}\bar{2}43;q) = A_{n-2}(q),$$

as claimed.

Now we will show that the same formula holds for $\bar{1}32\bar{4}$. This time, assume $\sigma\in\mathrm{Av}_n(\bar{1}32\bar{4})$. If $a_i=1$ for some $i>1$, then a_1a_i would be an occurrence of 21. However, this can never extend to 1324 since there is no letter to the left of a_1. Thus, $a_1=1$. An analogous argument shows $a_n=n$.

This allows $a_2\cdots a_{n-1}$ to be any arrangement of $2,3,\dots,n-1$, since, whenever a_ia_j is an occurrence of 21 for $2\le i,j\le n-1$, this extends to $1a_ia_jn$. So, we have the bijection

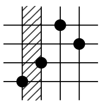
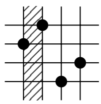
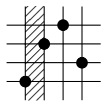
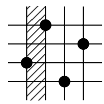
$$a_1a_2\cdots a_n \mapsto (a_2-1)(a_3-1)\cdots(a_{n-1}-1)$$

with elements of $\mathfrak{S}_{n-2}$. Since 1 and n are never descents in $\mathrm{Av}_n(\bar{1}32\bar{4})$, this is a des-preserving bijection. Therefore, $F_n^{\mathrm{des}}(\bar{1}\bar{2}43; q) = A_{n-2}(q)$. □

4. Conjectures and further directions

In this section, we provide a few conjectures, supporting data, and additional direction in which this work could proceed. In all cases, no closed forms for the functions $F_n^{\mathrm{des}}(\Pi; q)$ are known. We refer the reader to Table 1 for all known polynomials $F_n^{\mathrm{des}}(\Pi; q)$ for $4 \le n \le 8$, since, for these choices of Π, $F_n^{\mathrm{des}}(\Pi; q) = F_n^{\mathrm{des}}(\varnothing; q)$ for $n \le 3$.

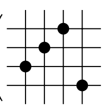
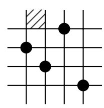
Conjecture 4.1. The following des-Wilf equivalences hold:

 $\overset{\mathrm{des}}{\equiv}$  and  $\overset{\mathrm{des}}{\equiv}$  .

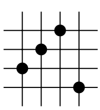
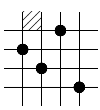
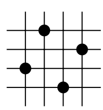
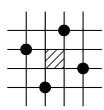
To state our next conjecture, we must discuss a particular sorting of permutations. Let $\sigma = a_1\cdots a_n \in \mathfrak{S}_n$ and suppose $a_i = n$. Let Γ be the operator defined recursively as

$$\Gamma(\sigma) = \Gamma(a_1\cdots a_{i-1})\Gamma(a_{i+1}\cdots a_n)n.$$

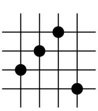
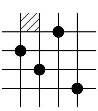
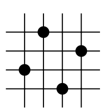
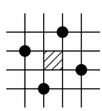
We say that σ is *West-t-stack-sortable* if $\Gamma^t(\sigma)$ is the identity permutation. Note that the 2-West-stack-sortable permutations [West 1990] are exactly those in

$\mathrm{Av}_n\Big($  ,  $\Big)$.

Conjecture 4.2. The following des-Wilf equivalence holds:

$\Big\{$  ,  $\Big\}$ $\overset{\mathrm{des}}{\equiv}$ $\Big\{$  ,  $\Big\}$.

If this conjecture is true, then from [Bóna 2002] it follows that

$F_n^{\mathrm{des}}\Big(\Big\{$  ,  $\Big\}; q\Big) = F_n^{\mathrm{des}}\Big(\Big\{$  ,  $\Big\}; q\Big)$

$$= \sum_{k=0}^{n-1} \frac{(n+k)!\,(2n-k-1)!}{(k+1)!\,(n-k)!\,(2k+1)!\,(2n-2k-1)!}q^k.$$

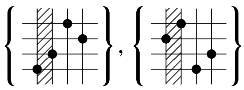
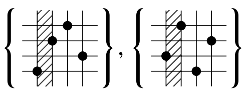
Π	n	$F_n^{\mathrm{des}}(\Pi; q)$
	4	$1+10q+11q^2+q^3$
	5	$1+20q+57q^2+26q^3+q^4$
	6	$1+35q+204q^2+252q^3+57q^4+q^5$
	7	$1+56q+581q^2+1500q^3+969q^4+120q^5+q^6$
	8	$1+84q+1414q^2+6588q^3+9117q^4+3426q^5+247q^6+q^7$
	4	$1+10q+11q^2+q^3$
	5	$1+20q+56q^2+26q^3+q^4$
	6	$1+35q+196q^2+241q^3+57q^4+q^5$
	7	$1+56q+546q^2+1361q^3+897q^4+120q^5+q^6$
	8	$1+84q+1302q^2+5675q^3+7739q^4+3060q^5+247q^6+q^7$

Table 1. The polynomials $F_n^{\mathrm{des}}(\Pi; q)$ for certain sets of patterns Π.

Instead of generalizing the patterns being avoided, one may generalize permutations themselves. One way to do this is to consider the *colored permutations*

$$G_{r,n} := \{(\varepsilon, \sigma) \mid \varepsilon \in \mathbb{Z}_r,\ \sigma \in \mathfrak{S}_n\}.$$

In this case, we say that $(\varepsilon, \sigma) \in G_{r,n}$ *contains* $(\zeta, \pi) \in G_{s,m}$ if there are elements $1 \le i_1 < i_2 < \cdots < i_s \le n$ such that $\mathrm{std}(\sigma_{i_1} \cdots \sigma_{i_s}) = \pi$ and $\varepsilon_{i_j} = \zeta_j$ for all j. If no such choice of i_j exist, then we say (ε, σ) *avoids* (ζ, π). For a set of colored permutations Π, let

$$\mathrm{Av}_{r,n}(\Pi) = \{(\varepsilon, \sigma) \in G_{r,n} \mid (\varepsilon, \sigma) \text{ avoids all } (\zeta, \pi) \in \Pi\}.$$

Question 4.3. What can be said about the polynomials

$$F_{r,n}^{\mathrm{st}}(\Pi; q) = \sum_{(\varepsilon,\sigma)\in \mathrm{Av}_{r,n}(\Pi)} q^{\mathrm{st}(\varepsilon,\sigma)}?$$

We close by noting that $G_{r,n}$ is the set of elements in the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$, a fact which may be useful when addressing the above questions.

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