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Irreducible character restrictions to maximal subgroups of low-rank classical groups of types B and C

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Representations are special functions on groups that give us a way to study abstract groups using matrices, which are often easier to understand. In particular, we are often interested in irreducible representations, which can be thought of as the building blocks of all representations. Much of the information about these representations can then be understood by instead looking at the trace of the matrices, which we call the character of the representation. This paper will address restricting characters to subgroups by shrinking the domain of the original representation to just the subgroup. In particular, we will discuss the problem of determining when such restricted characters remain irreducible for certain low-rank classical groups.

1. Introduction

Given a finite group G, a (*complex*) *representation* of G is a homomorphism $\Psi\colon G \to \mathrm{GL}_n(\mathbb{C})$. By summing the diagonal entries of the images $\Psi(g)$ for $g \in G$ (that is, taking the trace of the matrices), we obtain the corresponding *character*, $\chi = \mathrm{Tr}\circ\Psi$ of G. The *degree* of the representation Ψ or character χ is $n = \chi(1)$. It is well known that any character of G can be written as a sum of so-called *irreducible* characters of G. In this sense, irreducible characters are of particular importance in representation theory, and we write $\mathrm{Irr}(G)$ to denote the set of irreducible characters of G.

Given a subgroup H of G, we may view Ψ as a representation of H as well, simply by restricting the domain. As such, we will write $\chi|_H$ to denote the corresponding character of H, called the restricted character or character restriction. In this paper, we are interested in the general problem of classifying triples (G, H, χ), where G is a finite group, H is a maximal subgroup, and χ is an irreducible character of G

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whose restriction to H remains irreducible. There is a large body of work on this topic, see [Brundan and Kleshchev 2003; Kleshchev and Sheth 2002; Kleshchev and Tiep 2004; 2010; Liebeck 1985; Seitz 1987; Seitz and Testerman 1990; Nguyen and Tiep 2008; Himstedt et al. 2009; Nguyen 2008; Seitz 1990; Schaeffer Fry 2013], but several interesting cases remain unsolved.

We remark that although the general problem of classifying irreducible restrictions is of interest for representations over general fields, we work in this paper only with complex representations, and therefore the term "character" will refer specifically to complex characters here.

In this paper, we are concerned with the case that G is a classical group and H is a maximal subgroup of G. (For a brief introduction to classical groups, see Section 2A below.) In [Schaeffer Fry 2013], the faculty author classified all triples as above in the case $G = \mathrm{Sp}_4(q)$ or $\mathrm{Sp}_6(q)$, where q is a power of 2. There, and in many of the other articles on the topic, there are relatively few maximal subgroups that need to be considered using advanced techniques. In [Schaeffer Fry 2013], the process of reducing to these more difficult cases is referred to as the "initial reduction". Since $\mathrm{Sp}_6(2^a) \cong \Omega_7(2^a)$, the natural next step is to address the cases $G = \mathrm{Sp}_6(q)$ or $\Omega_7(q)$ with q odd.

Hence, here we work with symplectic groups $\mathrm{Sp}_{2n}(q)$ and orthogonal groups $\Omega_{2n+1}(q)$ with $1 \leq n \leq 3$ and q a power of an odd prime, which corresponds to the groups of Lie type B and C. Specifically, the goal of this paper is to provide the "initial reduction" for these groups, which leaves a short list of more difficult subgroups to be addressed. Our main results, providing this "initial reduction", are found in Theorems 4.1, 5.1, 6.1, and 7.1. Further, we provide complete classifications of irreducible restrictions for small values of q, which is found in Section 8.

The organization of the paper is as follows. In Section 2, we introduce some background material regarding finite classical groups and representations. In Section 3, we discuss the code used in the computer algebra system GAP for the cases that q is small. The remainder of the paper is dedicated to the main results.

1A. *Notation.* Here we introduce some basic notation for products and extensions of groups, which will be found throughout the paper. If H is a subgroup of G, we denote by $[G : H]$ the index of H in G. Given two groups X and Y, we denote the direct product of X and Y by $X \times Y$. The notation $X \circ Y$ will denote any central product of X and Y as defined in [Gorenstein 1968, Theorem 5.3]. Such a group is defined with respect to a subgroup Z of $Z(X)$ that may be identified under an isomorphism with a subgroup of $Z(Y)$. Then X and Y generate the group $X \circ Y$ and centralize each other, and $Z = X \cap Y \subseteq Z(X \circ Y)$.

If X acts on Y, we denote the semidirect product of Y with X by $Y : X$, defined as in [Gorenstein 1968, Theorem 5.1]. Here Y and X may be viewed as subgroups of $Y : X$ satisfying that Y is normal in $Y : X$ and $Y \cap X = \{1\}$. More generally, if Y is a normal subgroup of G with quotient $G/Y \cong X$, we write $G = Y.X$ or $Y^{\cdot}X$, where we use the latter if we specifically know that Y has no complement in G. If r and m are positive integers, we may simply write r^m for the direct product, $(C_r)^m$, of m copies of a cyclic group of order r.

If $q = p^a$ is a power of a prime, an elementary abelian group C_p^a of order q will be denoted by E_q. We will use S_n and A_n to denote the symmetric and alternating groups, respectively, on n letters. The *wreath product* of a group X and S_n will be denoted by $X \wr S_n$. This can be thought of as a semidirect product $X^n : S_n$, where X^n denotes the direct product of n copies of X. Further, D_n will denote the dihedral group of order $2n$. Given two integers r and m, we will write (r, m) for the gcd of the two integers.

2. Background material

2A. ***The finite classical groups.*** In this section, we introduce the main groups of study in this paper. Readers familiar with the construction of the finite classical groups may feel free to disregard this section. We will view the classical groups here as groups of matrices, although they may also be viewed as groups of Lie type or as certain groups of linear transformations. For a more in-depth discussion of these groups, we refer the reader to [Grove 2002; Kleidman and Liebeck 1990, Section 2].

Let q be a power of a prime p, and let $\mathbb{F}_q$ denote a finite field of size q. In general, the finite classical groups can be viewed as subgroups or subquotients of the *general linear group* $\mathrm{GL}_n(q)$, which is composed of all invertible $n \times n$ matrices with entries in $\mathbb{F}_q$. The *special linear group* is the normal subgroup, $\mathrm{SL}_n(q)$, of matrices with determinant 1. We obtain the *projective special linear group* as the quotient $\mathrm{PSL}_n(q) = \mathrm{SL}_n(q)/Z(\mathrm{SL}_n(q))$. The sizes of these groups are

$$|\mathrm{GL}_n(q)| = q^{\frac{1}{2}n(n-1)} \prod_{k=1}^{n} (q^k - 1),$$
$$|\mathrm{SL}_n(q)| = \frac{|\mathrm{GL}_n(q)|}{q-1}, \quad |\mathrm{PSL}_n(q)| = \frac{|\mathrm{SL}_n(q)|}{(n, q-1)}.$$

The *general unitary group* is a subgroup of $\mathrm{GL}_n(q^2)$, and can be defined as

$$\mathrm{GU}_n(q) := \{A \in \mathrm{GL}_n(q^2) : \bar{A}^T A = I_n\},$$

where $\bar{A}^T$ is the matrix obtained from A by raising each entry to the q-th power and taking the transpose. The *special unitary group*, $\mathrm{SU}_n(q)$, is the subgroup of $\mathrm{GU}_n(q)$ of matrices with determinant 1, and the *projective special unitary group* is the quotient $\mathrm{PSU}_n(q) = \mathrm{SU}_n(q)/Z(\mathrm{SU}_n(q))$. The corresponding sizes are

$$|\mathrm{GU}_n(q)| = q^{\frac{1}{2}n(n-1)} \prod_{k=1}^{n} (q^k - (-1)^k),$$

$$|\mathrm{SU}_n(q)| = \frac{|\mathrm{GU}_n(q)|}{q+1}, \qquad |\mathrm{PSU}_n(q)| = \frac{|\mathrm{SU}_n(q)|}{(n, q+1)}.$$

The *symplectic group* can be viewed as the subgroup

$$\mathrm{Sp}_{2n}(q) = \{g \in \mathrm{GL}_{2n}(q) : g^T J g = J\},$$

where J is the matrix

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

I_n is the $n \times n$ identity matrix, and g^T is the transpose of g. Note here that the dimension must be even. The *projective symplectic group* is then $\mathrm{PSp}_{2n}(q) = \mathrm{Sp}_{2n}(q)/Z(\mathrm{Sp}_{2n}(q))$. We have $|\mathrm{Sp}_{2n}(q)| = q^{n^2} \prod_{k=1}^{n}(q^{2k} - 1)$ and $|\mathrm{PSp}_{2n}(q)| = \frac{1}{2}|\mathrm{Sp}_{2n}(q)|$ when q is odd. For most values of n, q, the groups $\mathrm{PSL}_n(q)$, $\mathrm{PSU}_n(q)$, and $\mathrm{PSp}_{2n}(q)$ are simple.

The last type of finite classical group comes from the *orthogonal groups*. We will be primarily interested in odd-dimensional orthogonal groups. In this case, we can define $O_{2n+1}(q) := \{g \in \mathrm{GL}_{2n+1}(q) : g^T M g = M\}$, where M is the matrix

$$M = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The size of this group is $|O_{2n+1}(q)| = 2q^{n^2} \prod_{k=1}^{n}(q^{2k} - 1)$.

Taking the subgroup of elements with determinant 1, we get the *special orthogonal group*, denoted by $\mathrm{SO}_{2n+1}(q)$. Now, for $n \geq 1$ and q odd, $\mathrm{SO}_{2n+1}(q)$ contains a unique subgroup of index 2, which we denote by $\Omega_{2n+1}(q)$. The size of $\Omega_{2n+1}(q)$ is $\frac{1}{2}q^{n^2} \prod_{k=1}^{n}(q^{2k} - 1)$.

We remark that in even dimension, there are similar constructions, but this leads to two isomorphism classes: $\Omega_{2n}^-(q)$ and $\Omega_{2n}^+(q)$. We do not discuss these groups further, as for our purposes the isomorphisms mentioned below will suffice.

When the rank of the matrices is small, there are "accidental" isomorphisms between classical groups. The next theorem, found as part of [Kleidman and Liebeck 1990, Proposition 2.9.1] lists several of these isomorphisms relevant to the current work.

Theorem 2.1. *The following isomorphisms hold*:

- $\mathrm{SL}_2(q) \cong \mathrm{Sp}_2(q) \cong \mathrm{SU}_2(q)$.
- $\mathrm{PSL}_2(q) \cong \Omega_3(q)$ *for q odd.*

- $\Omega_4^-(q) \cong \mathrm{PSL}_2(q^2)$.
- $\Omega_4^+(q) \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q) \cong 2.(\mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q))$.
- $\mathrm{PSp}_4(q) \cong \Omega_5(q)$ *for q odd.*

2A1. *Maximal subgroups of finite classical groups.* For the purposes of this paper, we are interested in restricting irreducible characters of finite classical groups to maximal subgroups. Understanding and classifying the maximal subgroups of the finite classical groups has been a topic of particular importance in group theory and representation theory. We encourage the interested reader to explore the texts discussed here.

Aschbacher [1984] showed that a maximal subgroup of a finite classical group lies either in the class $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_8$ composed of eight naturally defined subclasses of subgroups or a collection $\mathcal{S}$ of almost quasisimple groups satisfying certain properties. Kleidman and Liebeck [1990] classify which groups in $\mathcal{C}$ are indeed maximal. For low-rank classical groups, all maximal subgroups have been classified by Bray, Holt, and Roney-Dougal [Bray et al. 2013].

2B. ***Preliminary observations on characters.*** Throughout, we denote by

$$b(G) := \max\{\chi(1) : \chi \in \mathrm{Irr}(G)\}$$

the largest irreducible character degree of G. It is well known that an upper bound for $b(G)$ is given by $\sqrt{|G|}$, which follows from the fact that $|G|$ can be expressed as the sum of the squares of the irreducible character degrees.

Now, note that if $\chi \in \mathrm{Irr}(G)$ restricts irreducibly to a subgroup H, then $\chi(1) = \chi|_H(1)$ must be at most $b(H)$. As we will use this fact throughout the paper, we record it here:

Lemma 2.2. *Let $H \le G$ and $\chi \in \mathrm{Irr}(G)$. If $\chi(1) > b(H)$, we must have $\chi|_H$ is reducible.*

This yields the following corollary, which will be essential throughout the following sections.

Corollary 2.3. *Let $H \le G$ and $\chi \in \mathrm{Irr}(G)$. Then if $\chi(1) \ge \sqrt{|H|}$, we must have $\chi|_H$ is reducible.*

One of our primary tools will be to use Lemma 2.2 or Corollary 2.3. It will be useful to have more efficient bounds for $b(H)$, however. The following well-known results from character theory will be crucial in this regard.

Theorem 2.4 (Itô's theorem, [Isaacs 1976, Theorem 6.15]). *If $H \lhd G$ is a normal abelian subgroup with $[G : H] = n$, then $\chi(1)$ divides n for all $\chi \in \mathrm{Irr}(G)$.*

Theorem 2.5 (Clifford's theorem, [James and Liebeck 2001, Theorem 20.8]). *If $H \lhd G$ are groups with H normal, and $\chi \in \mathrm{Irr}(G)$, then the restriction $\chi|_H$ satisfies*:

(1) $\chi|_H = e\big(\sum_{i=1}^{m} \psi_i\big)$ *for some irreducible characters* $\psi_i \in \mathrm{Irr}(H)$, $1 \leq i \leq m$, *and some positive integers* e *and* m.

(2) *All constituents* ψ_i *of* $\chi|_H$ *have the same degree.*

Theorem 2.6 [Isaacs 1976, Corollary 11.29]. *Let* $N \lhd G$ *and* $\chi \in \mathrm{Irr}(G)$. *Let* $\theta \in \mathrm{Irr}(N)$ *be a constituent of* $\chi|_N$. *Then*, $\chi(1)/\theta(1)$ *divides the index* $[G : N]$.

We remark that Theorem 2.4 can be viewed as a corollary to Theorem 2.6 and the fact that irreducible characters of an abelian group are always linear. (That is, every irreducible character of an abelian group has degree 1.)

It will also be beneficial to understand the characters of certain products of groups.

Theorem 2.7 [James and Liebeck 2001, Theorem 19.18]. *Let* G_1 *and* G_2 *be groups with corresponding irreducible characters* $\chi \in \mathrm{Irr}(G_1)$ *and* $\psi \in \mathrm{Irr}(G_2)$. *Then the function* $\chi \times \psi : G_1 \times G_2 \to \mathbb{C}$, *defined by* $(\chi \times \psi)(g, h) = \chi(g)\psi(h)$ *for* $g \in G_1$, $h \in G_2$, *is an irreducible character of the direct product* $G_1 \times G_2$. *Moreover, every irreducible character of* $G_1 \times G_2$ *is of this form.*

We remark that given two groups G_1, G_2, a central product $G_1 \circ G_2$ is, in a sense, lateral to other types of products we have come to understand, since the groups G_1 and G_2 do *not* have a trivial intersection. However, central products do have the property that all the elements in common commute with all other elements of the larger group. For a discussion of the representation theory of these objects, we refer the reader to [Gorenstein 1968, Chapter 3.7]. As discussed there, the irreducible characters of a central product $G_1 \circ G_2$ can be viewed as irreducible characters of a factor group when a suitable normal subgroup is chosen from the kernel of the representation. For our purposes, this means that irreducible characters of a central product $G_1 \circ G_2$ can again be taken to be products of characters of G_1 and G_2.

We end this section with the following lemma recording a divisibility property for the first several cyclotomic polynomials.

Lemma 2.8. *Let* $q \geq 3$ *be an odd number*:

(a) *If* $\ell \geq 5$ *is prime, then* ℓ *divides at most one of* q, $q-1$, $q+1$, q^2+1, q^2+q+1, *and* q^2-q+1.

(b) 3 *divides at most one of* q, $q-1$, *and* $q+1$, *and does not divide* q^2+1.

(c) *If* 3 *divides* $q-\epsilon$ *with* $\epsilon \in \{\pm 1\}$, *then*
- 3 *divides* $q^2+\epsilon q+1$;
- 9 *does not divide both* $q-\epsilon$ *and* $q^2+\epsilon q+1$; *and*
- 3 *does not divide* $q^2-\epsilon q+1$.

Proof. Let ℓ be a prime. Suppose first that ℓ divides q. We see easily that this implies that the other listed values are congruent to $\pm 1 \pmod{\ell}$, and hence cannot be divisible by ℓ. Similarly, if ℓ divides $q \pm 1$, then the remaining values are congruent to $1, \pm 2$, or $3 \pmod{\ell}$, and hence cannot be divisible by ℓ unless $\ell = 2$ or 3. When $\ell = 3$, we see from this that the only possibilities are that 3 divides $q - 1$ and $q^2 + q + 1$ or that 3 divides $q + 1$ and $q^2 - q + 1$. If ℓ divides $q^2 + 1$, then it cannot divide q, $q - 1$, or $q + 1$ from above, and the remaining two values are congruent to $\pm q \pmod{\ell}$. Since ℓ does not divide q, the latter are also not divisible by ℓ. If ℓ divides one of the last two values listed and is larger than 3, then it cannot divide any of the first four by the previous arguments. Further, the remaining value is congruent to $\pm 2q \pmod{\ell}$, and hence again cannot be divisible by ℓ. Finally, if 9 divides $q \pm 1$, then $q^2 \mp q + 1 \equiv 3 \pmod{9}$ and hence is not divisible by 9. $\square$

3. Using GAP

GAP is a computer algebra system ("Groups, algorithms, and programming") that is extremely useful for computing with finite groups and their characters. For the purposes of this paper, we especially make use of the character table library package [Breuer 2013], which builds on the results in the ATLAS [Conway et al. 1985] and contains several character tables, lists of maximal subgroups, and other useful information about certain small groups. Our goal in this section is to describe some of the functions and commands that will be useful for our results.

We can obtain the character table and corresponding irreducible character values for groups stored in the character table library by using the commands `CharacterTable` and `Irr`, respectively. For many groups stored in the library, the list of maximal subgroups is also available, which can be obtained using the `Maxes` command. Given the character table for a maximal subgroup H of G stored in GAP, the library also has the fusion of classes stored (that is, the way conjugacy classes of H embed into those of G), which is necessary for comparing the characters of H to those of G for the purposes of understanding the restrictions. This is obtained using the command `GetFusionMap`.

Below is the code used to generate our results in Section 8, given the character tables stored in the library, `ctg` and `cth` for G and H, respectively. This gives the indices of the nonlinear irreducible characters of G that restrict irreducibly to H:

```
irrg:=Irr(ctg);
irrh:=Irr(cth);
fus:=GetFusionMap(cth,ctg);
PositionsProperty(irrg, x -> x[1] > 1 and x{fus} in irrh);
```

4. Restrictions from $G = \Omega_3(q) \cong \mathrm{PSL}_2(q)$

In this section, we let G be the finite group $\Omega_3(q) \cong \mathrm{PSL}_2(q)$, where $q \geq 5$ is a power of an odd prime p. The character table for G is well-known, and the set of nontrivial character degrees is $\left\{\frac{1}{2}(q+\epsilon), q-1, q, q+1\right\}$, where $\epsilon \in \{\pm 1\}$ is such that $q \equiv \epsilon \pmod 4$.

From [Bray et al. 2013, Table 8.7], we see that a maximal subgroup H of G is isomorphic to one of the following:

(1) A_5 for $q = p \equiv \pm 1 \pmod{10}$ or $q = p^2$, with $p \equiv \pm 3 \pmod{10}$.

(2) S_4 for $q = p \equiv \pm 1 \pmod 8$.

(3) A_4 for $q = p \equiv \pm 3, 5, \pm 11, \pm 13, \pm 19 \pmod{40}$.

(4) $D_{q\pm 1}$ for $q > 5$.

(5) $\Omega_3(q_0)$ for $q = q_0^r$, with r an odd prime.

(6) $\mathrm{SO}_3(q_0)$ for $q = q_0^2$.

(7) $E_q : \left(\frac{1}{2}(q-1)\right)$.

The goal of this section is to prove the following theorem:

Theorem 4.1. *Let $q \geq 13$ and $G \cong \Omega_3(q)$. Let H be a maximal subgroup and $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) \neq 1$ and $\chi|_H$ is irreducible. Then $q \equiv 3 \pmod 4$, $H \cong E_q : \left(\frac{1}{2}(q-1)\right)$, and $\chi(1) = \frac{1}{2}(q-1)$.*

We prove Theorem 4.1 in Lemmas 4.2–4.6 below by addressing the cases (1)–(7) individually. We address the case $5 \leq q \leq 11$ in Section 8. Throughout the remainder of the section, let $\mathcal{L}$ denote the real-valued function

$$\mathcal{L}(x) = \tfrac{1}{2}(x-1).$$

Note that $\chi(1) \geq \mathcal{L}(q)$ for any $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$ and that $\mathcal{L}$ is increasing for all real x.

Lemma 4.2. *Let $H \cong A_5$, if $q = p \equiv \pm 1 \pmod{10}$, or $q = p^2$, with $p \equiv \pm 3 \pmod{10}$ and let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$. Then $\chi|_H$ is reducible, except possibly when $q = 11$ or $q = 9$.*

Proof. First, note that $|H| = 60$ and that $q > 19$ unless $q = 9$ or 11. Since $\mathcal{L}(19) = 9 > \sqrt{60} = \sqrt{|H|}$, we have that $\chi|_H$ is reducible by Corollary 2.3, except possibly for the stated exceptions of q. □

Lemma 4.3. *Let $H \cong A_4$ with $q = p \equiv \pm 3, 5, \pm 11, \pm 13, \pm 19 \pmod{40}$ or $H \cong S_4$ with $q = p \equiv \pm 1 \pmod 8$, and let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$. Then $\chi|_H$ is reducible, except possibly when $q \leq 7$.*

Proof. Note that $|H| \leq 24$ and that $q \geq 11$ unless q is 3, 5, or 7. Hence, since $\mathcal{L}(11) = 5 > \sqrt{24} \geq \sqrt{|H|}$, arguing as in the proof of Lemma 4.2, we see that $\chi|_H$ is reducible for all $\chi \in \mathrm{Irr}(G)$, except possibly in the case of the stated exceptions. □

Lemma 4.4. *Let $H \cong D_{q\pm 1}$ with $q \geq 11$, and let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$. Then $\chi|_H$ is reducible.*

Proof. Note that $|H| = 2(q \pm 1)$. We claim that $\mathcal{L}(q) \geq \sqrt{2(q+1)} > \sqrt{2(q-1)}$ for all $q \geq 11$, which will prove the statement by Corollary 2.3. Since the second inequality clearly holds, we will work with the first. This is equivalent to solving the inequality $\frac{1}{4}(x-1)^2 \geq 2(x+1)$, and hence to solving $x^2 - 10x - 7 \geq 0$ or $(x-5)^2 - 32 \geq 0$. Since $x^2 - 10x - 7$ is increasing for $x > 5$, we see that $\mathcal{L}(x) \geq \sqrt{2(x+1)}$ whenever $x \geq 5 + 4\sqrt{2}$, which is satisfied by $x \geq 11$. □

Lemma 4.5. *Let $H \cong \Omega_3(q_0)$, where $q = q_0^r$ and r is an odd prime, or let $H \cong \mathrm{SO}_3(q_0)$, where $q = q_0^2$. Then $\chi|_H$ is reducible for every $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) \neq 1$.*

Proof. First consider the case that $q_0 = 3$. If $q \geq 27$, then $\mathcal{L}(q) \geq 13 > \sqrt{24} \geq \sqrt{|H|}$. If $q = 9$, then $\mathcal{L}(9) = 4$, but we have $H \cong \mathrm{SO}_3(3)$, which is isomorphic to the symmetric group S_4. The character degrees of S_4 are $\{1, 2, 3\}$, so the claim holds in this case.

Hence we may assume that $q_0 \geq 5$. We have $|H| \leq q_0(q_0^2 - 1)$ and $\frac{1}{2}(q-1) \geq \frac{1}{2}(q_0^2 - 1)$, so by Corollary 2.3, it suffices to show that

$$\tfrac{1}{2}(q_0^2 - 1) \geq \sqrt{q_0(q_0^2 - 1)}.$$

We will do this by showing that the quotient $(x^2-1)/(2\sqrt{x(x^2-1)})$ is at least 1 for $x \geq 5$. We have

$$\left(\frac{(x^2-1)}{2\sqrt{x(x^2-1)}}\right)^2 = \frac{1}{4}\left(x - \frac{1}{x}\right) \geq \frac{1}{4}(x-1) \geq 1$$

for all such x, completing the proof. □

Lemma 4.6. *Let $H \cong E_q : \left(\frac{1}{2}(q-1)\right)$. Then $\chi|_H$ is reducible for every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$, unless $q \equiv 3 \pmod 4$ and $\chi(1) = \frac{1}{2}(q-1)$.*

Proof. Since E_q is normal and abelian in H and $[H : E_q] = \frac{1}{2}(q-1)$, Theorem 2.4 implies $b(H) \leq \frac{1}{2}(q-1)$. Hence by Lemma 2.2, all irreducible nonlinear characters χ of G restrict reducibly to H, except possibly if $\chi(1) = \frac{1}{2}(q-1)$. □

5. Restrictions from $G = \Omega_5(q) \cong \mathrm{PSp}_4(q)$

Throughout this section, let q be a power of an odd prime p and let G be the group $\mathrm{PSp}_4(q)$, which is isomorphic to $\Omega_5(q)$. In this section, we prove the following:

Theorem 5.1. *Let $G = \Omega_5(q)$ with $q \geq 7$ odd. Let H be a maximal subgroup of G and $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) \neq 1$ and $\chi|_H$ is irreducible. Then one of the following holds:*

- *H is isomorphic to $\Omega_4^{\pm}(q).2$ or a maximal parabolic subgroup of G.*
- *H is isomorphic to $\mathrm{SO}_5(q^{1/2})$.*

maximal $H \cong$	condition on q	treated in Lemma
A_6	$q \neq 7$ and $q = p \equiv \pm 5 \pmod{12}$	5.2
A_7	$q = 7$	5.2
S_6	$q = p \equiv \pm 1 \pmod{12}$	5.2
$\mathrm{PSL}_2(q)$	$q \geq 7$ and $p \geq 5$	5.3
$2^4 : A_5$	$p = q \equiv \pm 3 \pmod 8$	5.4
$2^4 : S_5$	$p = q \equiv \pm 1 \pmod 8$	5.4
$\mathrm{PSp}_4(q_0)$,	$q = q_0^r$ and r is an odd prime	5.5
$\bigl(\frac{1}{2}(q \pm 1) \times \mathrm{PSL}_2(q)\bigr).2^2$	$q \geq 5$	5.6

Table 1

We note that by [Bray et al. 2013], the groups excluded by the first item of Theorem 5.1 are the groups in Aschbacher class $\mathcal{C}_1$ but not isomorphic to $\bigl(\frac{1}{2}(q \pm 1) \times \mathrm{PSL}_2(q)\bigr).2^2$. We also remark that the exceptions listed in Theorem 5.1 are beyond the scope of this work, and that the cases $q = 3, 5$ are addressed in Section 8. By [Bray et al. 2013], to prove Theorem 5.1, we must show that $\chi|_H$ is reducible for $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$ and H isomorphic to one of the groups shown in Table 1.

By the main theorem of [Landazuri and Seitz 1974], we have a lower bound on the degree of the nonlinear irreducible characters for G given by the function

$$\mathcal{L}(q) := \tfrac{1}{2}(q^2 - 1).$$

Note that the continuous function $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ given by $\mathcal{L}(x) = \frac{1}{2}(x^2 - 1)$ is everywhere differentiable and $\mathcal{L}'(x) = x$, which we know to be greater than zero on the interval $(0, \infty)$. Hence we see that $\mathcal{L}$ is increasing on $(0, \infty)$.

As in the previous section, our main strategy is to determine an upper bound for $b(H)$ and to show that $\mathcal{L}(q)$ is larger than this bound, implying that the nontrivial characters of $G \cong \mathrm{PSp}_4(q)$ restrict reducibly to H.

Lemma 5.2. *Let $H \cong A_6$ with $q = p \equiv \pm 5 \pmod{12}$ and $q \neq 7$, $H \cong A_7$ with $q = 7$, or $H \cong S_6$ with $q = p \equiv \pm 1 \pmod{12}$. Then $\chi|_H$ is reducible for every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. First, if $q \neq 5, 7$, note that since $\mathcal{L}$ is increasing, $\mathcal{L}(q) \geq \mathcal{L}(11) = 60 > \sqrt{720} \geq \sqrt{|H|}$. Hence in these cases, the statement follows from Corollary 2.3.

When $q = 5$, we have $H \cong A_6$ and $\mathcal{L}(q) = 12$. However, the largest irreducible character degree of A_6, as seen in the ATLAS and the GAP character table library [Conway et al. 1985; Breuer 2013], is 10. When $q = 7$, we can see using the character tables for A_7 and G in GAP [Breuer 2013] that none of the nonlinear irreducible character degrees match. Hence in any case, $\chi|_H$ is reducible for any $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$. $\square$

Lemma 5.3. *Let q be a power of a prime p with $p \geq 5$ and $q \geq 7$, and let $H \cong \mathrm{PSL}_2(q)$. Then $\chi|_H$ is reducible for every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. Set $u(x) = \sqrt{\frac{1}{2}x(x^2-1)}$, so $u(q) = \sqrt{|H|}$. We claim that $\mathcal{L}(x) \geq u(x)$ for the relevant values of x, implying the statement by Corollary 2.3. Note that

$$\left(\frac{\mathcal{L}(x)}{u(x)}\right)^2 = \left(\frac{(x^2-1)}{2\sqrt{(1/2)x(x^2-1)}}\right)^2 = \frac{1}{2}\left(x - \frac{1}{x}\right) \geq \frac{1}{2}(x-1) \geq 1$$

for all $x \geq 3$, which proves the claim. $\square$

Lemma 5.4. *Suppose $q \geq 7$. Let $q = p \equiv \pm 1 \pmod 8$ and $H \cong 2^4 : S_5$, or let $p = q \equiv \pm 3 \pmod 8$ and $H \cong 2^4 : A_5$. Then $\chi|_H$ is reducible for every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. Since $\mathcal{L}$ is increasing, we see $\mathcal{L}(q) \geq \mathcal{L}(11) = 60$ unless $q = 7$. Since $\sqrt{|H|} \leq \sqrt{2^4 \cdot 120} < 60$, we see that the nontrivial characters of $\mathrm{PSp}_4(q)$ restrict reducibly in this case by Corollary 2.3. From the character tables available in GAP [Breuer 2013] for $2^4 : S_5$ and $\mathrm{PSp}_4(7)$, we see further that when $q = 7$, there are no nontrivial irreducible character degrees for G that are also degrees for H. $\square$

Lemma 5.5. *Let $H \cong \mathrm{PSp}_4(q_0)$, where $q = q_0^r$ and r is an odd prime. Then $\chi|_H$ is reducible for every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. In this case, $|H| = \frac{1}{2}q_0^4(q_0^2-1)(q_0^4-1)$. We define real-valued functions l and u by $l(x) = \frac{1}{2}(x^{2r}-1)$ and $u(x) = \left(\frac{1}{2}x^4(x^2-1)(x^4-1)\right)^{1/2}$. We will show that $l(x) > u(x)$ whenever $r \geq 3$ and $x \geq 3$, which will establish the statement by Corollary 2.3.

Indeed, notice that for $x > 1$,

$$u(x)^2 = \tfrac{1}{2}x^4(x^2-1)(x^4-1) < x^4(x^2-1)(x^4-1) = x^{10} - (x^8 + x^6 - x^4) < x^{10},$$

where the last inequality follows from the fact that $x^8 + x^6 - x^4 > 0$ for $x > 1$. Further, for $r \geq 3$, we have $l(x) \geq \frac{1}{2}(x^6 - 1)$, which is larger than x^5 for $x \geq 3$. This shows that $l(x)^2 > u(x)^2$ for $x \geq 3$ and $r \geq 3$, which completes the claim. $\square$

In the final case, $q \geq 5$ and $H \cong \left(\frac{1}{2}(q \pm 1) \times \mathrm{PSL}_2(q)\right).2^2$. Then H is an extension of $K := \frac{1}{2}(q \pm 1) \times \mathrm{PSL}_2(q)$ by the direct product $C_2 \times C_2$. That is, $H/K \cong C_2 \times C_2$.

Lemma 5.6. *Let $q \geq 7$ and $H \cong \left(\frac{1}{2}(q \pm 1) \times \mathrm{PSL}_2(q)\right).2^2$. Then $\chi|_H$ is reducible for every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. First, recall that for abelian groups, the degree of every irreducible character is exactly 1. Let C be the cyclic group of size $\frac{1}{2}(q \pm 1)$. Thus for all $\lambda \in \mathrm{Irr}(C)$, we have $\lambda(1) = 1$. Since K is the direct product of C with $\mathrm{PSL}_2(q)$, we know that all elements of $\mathrm{Irr}(K)$ are of the form $\lambda \times \varphi$ by Theorem 2.7, where $\lambda \in \mathrm{Irr}(C)$ and $\varphi \in \mathrm{Irr}(\mathrm{PSL}_2(q))$. In particular, the degree of each of these characters is simply given by $\varphi(1)$.

Further, since $[H:K]=4$ and $K \lhd H$, we see using Theorem 2.6 that if $\chi \in \mathrm{Irr}(H)$ and $\chi|_H$ contains $\theta \in \mathrm{Irr}(K)$ as a constituent, then $\chi(1)/\theta(1)$ divides 4. However, since C is abelian and we know the maximal degree given by the generic character table of $\mathrm{PSL}_2(q)$ (see, for example, [Geck et al. 1996]) is $q+1$, we have an upper bound for $b(H)$ given by $b(H) \le 4(q+1)$.

Letting $u(x) = 4(x+1)$, notice that

$$\frac{\mathcal{L}(x)}{u(x)} = \frac{(x^2-1)}{8(x+1)} = \frac{1}{8}(x-1) > 1$$

whenever $x > 9$, and hence $\mathcal{L}(x) > u(x)$, proving the statement for $q > 9$ by Lemma 2.2. Further, using the GAP character table library [Breuer 2013], we see that $\mathrm{PSp}_4(9)$ has smallest nontrivial degree $41 > u(9)$, finishing the case $q = 9$.

Finally, consider the case $q = 7$. Note that the character degrees of K must be in the set $\{d, 2d, 4d\}$, where d ranges over the irreducible character degrees of $\mathrm{PSL}_2(7)$. Utilizing GAP, we see that none of these numbers occur as character degrees of G larger than 1, completing the proof. □

6. Restrictions from $G = \mathrm{Sp}_6(q)$

In this section, let G be the symplectic group $\mathrm{Sp}_6(q)$, where q is a power of an odd prime p. We prove the following:

Theorem 6.1. *Let $G = \mathrm{Sp}_6(q)$, where $q \ge 5$ is a power of an odd prime, and let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \ne 1$. Suppose $H \le G$ is a maximal subgroup such that the restriction $\chi|_H$ is irreducible. Then one of the following holds*:

- *H is isomorphic to $\mathrm{Sp}_2(q) \times \mathrm{Sp}_4(q)$ or a maximal parabolic subgroup.*
- *$q = 5$, $H \cong 2^{\cdot}J_2$, and $\chi(1) = 63$.*
- *$q = 5$, $H \cong \mathrm{GL}_3(5).2$, and $\chi(1) = 62$.*
- *$H \cong \mathrm{Sp}_2(q^3):3$ and $\chi(1) = \frac{1}{2}(q^3 \pm 1)$.*
- *$H \cong \mathrm{Sp}_6(q_0).2$, where $q = q_0^2$, and $q_0 = 5$ or $\chi(1) = \frac{1}{2}(q^3 \pm 1)$.*

As in the case of Theorem 5.1, the groups excluded by the first item of Theorem 6.1 are those found in Aschbacher class $\mathcal{C}_1$. We remark that the case $q = 3$ will be considered in Section 8 and that addressing the exceptions listed in Theorem 6.1, aside from $2^{\cdot}J_2$ addressed in Lemmas 6.3 and 6.4 below, will require methods

maximal $H \cong$	condition on q	restriction behavior	treated in Lemma
$2^{\cdot}A_5$	$q = p \equiv \pm3, \pm11, \pm13, \pm19 \pmod{40}$	always red.	6.2
$2^{\cdot}S_5^-$	$q = p \equiv \pm1 \pmod 8$	always red.	6.2
$2^{\cdot}\mathrm{PSL}_2(7)^{\cdot}2^+$	$q = p \equiv \pm1 \pmod{16}$	always red.	6.2
$2^{\cdot}\mathrm{PSL}_2(7)$	$q = p \equiv \pm7 \pmod{16}$, $q \neq 7$	always red.	6.2
$2^{\cdot}\mathrm{PSL}_2(7)$	$q = p^2$, $p \equiv \pm3, \pm5 \pmod{16}$	always red.	6.2
$2^{\cdot}\mathrm{PSL}_2(13)$	$q = p \equiv \pm1, \pm3, \pm4 \pmod{13}$	always red.	6.2
$2^{\cdot}\mathrm{PSL}_2(13)$	$q = p^2$, $p \equiv \pm2, \pm5, \pm6 \pmod{13}$	always red.	6.2
$2^{\cdot}A_7$	$q = 9$	always red.	6.2
$2 \times \mathrm{PSU}_3(3)$	$q = p \equiv \pm7, \pm17, \pm19, \pm29 \pmod{60}$	always red.	6.2
$(2 \times \mathrm{PSU}_3(3)).2$	$q = p \equiv \pm1 \pmod{12}$	always red.	6.2
$2^{\cdot}J_2$	$q = p \equiv \pm1 \pmod 5$	always red.	6.3
$2^{\cdot}J_2$	$q = 5$	red. unless $\chi(1) = 63$	6.3, 6.4
$2^{\cdot}J_2$	$q = p^2$, $p \equiv \pm2 \pmod 5$	always red.	6.3
$2^{\cdot}\mathrm{PSL}_2(q)$	$p \geq 7$	always red.	6.5

Table 2

beyond the scope of this article. The remainder of this section is devoted to proving Theorem 6.1.

By the proof of [Tiep and Zalesskii 1996, Theorem 5.2], a lower bound for the nontrivial character degrees of G is

$$\mathcal{L}(q) := \tfrac{1}{2}(q^3 - 1).$$

As in the previous sections, our new lower bound $\mathcal{L}$ is an increasing function for $x > 1$.

We will first investigate the character restrictions to the subgroups listed in Table 2, which according to [Bray et al. 2013] are the maximal subgroups in the Aschbacher class $\mathcal{S}$. Recall here that we are assuming $q \geq 5$.

We may treat the first several cases using the strategies from the previous sections.

Lemma 6.2. *Let $q \geq 5$ and let H be one of the maximal subgroups listed above, aside from $2^{\cdot}J_2$ or $2^{\cdot}\mathrm{PSL}_2(q)$. Then $\chi|_H$ is reducible for every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. In each case, $\mathcal{L}(q) > \sqrt{|H|}$, which can be seen using the same arguments as in the previous sections. Hence the statement follows from Corollary 2.3. □

Now, we consider the second Janko group, J_2, which is one of the sporadic finite simple groups. Also called the Janko–Hall group, J_2 was one of the first simple sporadic groups discovered and its order is $|J_2| = 604800$. The group $H \cong 2^{\cdot}J_2$ is the so-called Schur cover or universal covering group of J_2.

Lemma 6.3. *Let $H \cong 2^{\cdot}J_2$, where $p = q \equiv \pm 1 \pmod 5$, $q = 5$, or $q = p^2$ and $p \equiv \pm 2 \pmod 5$. Let $\chi \in \operatorname{Irr}(G)$ with $\chi(1) \neq 1$. Then $\chi|_H$ is reducible, except in the case $q = 5$ and $\chi(1) = 63$.*

Proof. Note that $b(H) < \sqrt{2 \cdot 604800} \approx 1099.8$. Since $\mathcal{L}$ is increasing and $\mathcal{L}(19) > \sqrt{2 \cdot 604800}$, the statement follows by Corollary 2.3 as long as $q \neq 5, 9$, or 11. Further, the character table of H is available in the ATLAS or the GAP character table library [Conway et al. 1985; Breuer 2013], from which we see that $b(H) = 448$. Since $\mathcal{L}(11) = 665$, we are finished in this case.

Now, let $q = 9$. Using the character degrees for G available from [Lübeck 2007], we see that the only degrees below the maximal degree of $2^{\cdot}J_2$ are 364 and 365, but neither of these appear in the list of degrees from $2^{\cdot}J_2$, so they must restrict reducibly.

When $q = 5$, again using [Lübeck 2007], we must consider characters of G of degrees 62 and 63. However, only 63 occurs as a character degree for $2^{\cdot}J_2$, completing the proof. □

Lemma 6.4. *The irreducible characters of $G = \mathrm{Sp}_6(5)$ with degree 63 restrict irreducibly to $H \cong 2^{\cdot}J_2$.*

Proof. From [Lübeck 2007], we see there are two characters of G of degree 63. Further, from the character table of $\mathrm{PSp}_6(5)$ available in GAP, we see that $G/Z(G)$ also has two irreducible characters of degree 63. That is, the two irreducible characters of G of degree 63 are trivial on the center. We also see, using the character tables available in GAP, that the character of H of degree 63 is trivial on the center. Hence the characters of degree 63 of G and H can be considered as characters of $\mathrm{PSp}_6(5)$ and J_2, respectively.

Implementing the algorithm described in Section 3, we see that the characters χ_2 and χ_3 of degree 63 of $\mathrm{PSp}_6(5)$ restrict irreducibly to the character χ_7 of J_2. Hence the inflations to G will restrict irreducibly to H as well. □

Lemma 6.5. *Let $p \geq 7$ and let $H \cong 2^{\cdot}\mathrm{PSL}_2(q)$. Then $\chi|_H$ is reducible for every irreducible character χ of $G = \mathrm{Sp}_6(q)$ with $\chi(1) \neq 1$.*

Proof. From the character table for $H \cong \mathrm{SL}_2(q)$, we know $b(H) = q + 1$. Since $x+1 < \frac{1}{2}(x^3 - 1) = \mathcal{L}(x)$ whenever $x > 2$, the statement follows from Lemma 2.2. □

We now consider the maximal subgroups of $G = \mathrm{Sp}_6(q)$ from Aschbacher class $\mathcal{C}$ given in Table 3. Recall here that $q \geq 5$.

We remark that these are all of the maximal subgroups in $\mathcal{C}$, with the exception of those in $\mathcal{C}_1$ and $\mathcal{C}_8$. Addressing these omitted groups and those for which we only attain partial results will require methods beyond the scope of this article.

Aschbacher class	maximal $H \cong$	condition on q	restriction behavior	treated in Lemma
$\mathcal{C}_2$	$\mathrm{Sp}_2(q) \wr S_3$		always red.	6.6
$\mathcal{C}_2$	$\mathrm{GL}_3(q).2$		red. unless $q = 5$, $\chi(1) = 62$	6.7
$\mathcal{C}_3$	$\mathrm{Sp}_2(q^3) : 3$		partial results	6.8
$\mathcal{C}_3$	$\mathrm{GU}_3(q).2$		always red.	6.9
$\mathcal{C}_4$	$\mathrm{Sp}_2(q) \circ \mathrm{GO}_3(q)$		always red.	6.10
$\mathcal{C}_5$	$\mathrm{Sp}_6(q_0)$	$q = q_0^r$, r odd prime	always red.	6.11
$\mathcal{C}_5$	$\mathrm{Sp}_6(q_0).2$	$q = q_0^2$	partial results	6.12

Table 3

For the remainder of the section recall that $q \geq 5$ and define d_i to be the i-th irreducible character degree of $\mathrm{Sp}_6(q)$ as obtained from the list generated by [Lübeck 2007]. In particular, we have

$$d_2 := \tfrac{1}{2}(q^3 - 1), \quad d_4 := \tfrac{1}{2}q(q-1)(q^3-1),$$
$$d_3 := \tfrac{1}{2}(q^3 + 1), \quad d_5 := \tfrac{1}{2}(q-1)(q^2+q+1)(q^2-q+1).$$

Certainly $d_2 < d_3$ and $d_4 < d_5$, since $q^3 - 1 < q^3 + 1$ and $q^2 - q < q^2 - q + 1$. Further, using the upper bound $\sum_{i=0}^{n-1} |a_i|$ for the positive roots of a polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, with real coefficients, we see that for $q > 5$, $2(d_4 - d_3) = q^5 - q^4 - q^3 - q^2 + q - 1$ must be positive. Since $d_4 > d_3$ when $q = 5$ by checking directly, we therefore see that $d_5 > d_4 > d_3 > d_2$ for all $q \geq 5$. Further, similar arguments using the polynomials in q in the list generated by [Lübeck 2007] yield that $d_i \geq d_5$ for each $i \geq 5$.

Lemma 6.6. *If $q \geq 5$ and $H \cong \mathrm{Sp}_2(q) \wr S_3$, then $\chi|_H$ is reducible for each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. Recall that H can be viewed as the semidirect product $\mathrm{Sp}_2(q)^3 : S_3$. Theorem 2.7, combined with the fact that q is odd and $\mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$, yields that the irreducible characters of $\mathrm{Sp}_2(q)^3$ have degree at most $(q+1)^3$. Then, Theorem 2.6 implies $b(H) \leq 6(q+1)^3$.

Now, note that $b(H) < d_4$ for $q > 5$ and that $b(H) < d_5$ for $q \geq 5$. Further, when $q = 5$, we have $d_4 = 1240$, which has 31 as a prime factor. Since 31 does not divide $|H|$ in this case, we see d_4 cannot be a character degree for H. Hence it suffices to show neither d_2 nor d_3 can be a character degree for H when $q \geq 5$.

Assume by way of contradiction that d_2, respectively d_3, is the degree of some irreducible character of H. Note that this means $d := d_2$, respectively d_3, must divide $|H| = 6q^3(q-1)^3(q+1)^3$. Letting $\epsilon = 1$ in the case $d = d_2$ and $\epsilon = -1$ in the case $d = d_3$, recall that $d = \frac{1}{2}(q-\epsilon)(q^2+\epsilon q+1)$. In particular, we see that any prime dividing $q^2+\epsilon q+1$, must also divide $|H|$. Applying Lemma 2.8, it follows that d must be a product of powers of 2 and 3. Since $q \geq 5$ is odd, this means that the odd number $q^2+\epsilon q+1$ is of the form 3^r with $r > 3$. However, by Lemma 2.8(c), this means that 3^5 divides d but that the largest power of 3 dividing $|H|$ is 3^4, a contradiction. □

Lemma 6.7. *If $q \geq 5$ and $H \cong \mathrm{GL}_3(q).2$, then $\chi|_H$ is reducible for each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$, except possibly if $q = 5$ and $\chi(1) = 62$.*

Proof. From the generic character table available for $\mathrm{GL}_3(q)$ in CHEVIE [Geck et al. 1996], we see that the irreducible character degrees for $\mathrm{GL}_3(q)$ are

$$\{1,\ q(q+1),\ q^3,\ q^2+q+1,\ q(q^2+q+1),\ (q\pm1)(q^2+q+1),\ (q-1)^2(q+1)\} \quad (1)$$

and that the largest of these is $(q+1)(q^2+q+1)$. Then $b(H) \leq 2(q+1)(q^2+q+1)$ by Theorem 2.6.

Recall that $d_4 = \frac{1}{2}q(q^2+q+1)(q-1)^2$ and that for $i \geq 4$, we have $d_i \geq d_4$. Notice also that d_4 is an increasing function and $d_4 > b(H)$ whenever $q \geq 5$. This shows that every irreducible character of degree larger than d_3 must restrict reducibly to H, by Lemma 2.2.

Now, applying Theorem 2.6, we see that every member of $\mathrm{Irr}(H)$ has degree of the form m or $2m$ for some m in the set (1). Arguing as in Lemma 6.6, using Lemma 2.8, we see that for each member $m \neq 1$ in this list, there is some odd divisor of $d_3 = \frac{1}{2}(q+1)(q^2-q+1)$ that does not divide m, and hence no character of degree d_3 restricts irreducibly to H. The same argument yields the same statement for $d_2 = \frac{1}{2}(q-1)(q^2+q+1)$, except possibly if m is one of the numbers in the list with divisor q^2+q+1. But since $q \geq 5$, we further have $\frac{1}{2}(q-1)$ cannot be in the set $\{1,\ 2,\ q,\ 2q,\ q\pm1,\ 2(q\pm1)\}$, and hence d_2 also cannot coincide with any character degree of H, unless $q = 5$ and $d_2 = 62 = 2(q^2+q+1)$. □

We remark that the character degree 62 does not appear for the simple group $\mathrm{PSp}_6(5)$, so the unsolved exception in Lemma 6.7 is irrelevant for the problem of determining irreducible restrictions from the simple group $G/Z(G)$.

Lemma 6.8. *If $q \geq 5$ is odd and $H \cong \mathrm{Sp}_2(q^3):3$, then $\chi|_H$ is reducible for each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$, with the possible exception of those with degree equal to d_2 or d_3.*

Proof. Recall from above that $\mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$ and whenever q is odd, the maximum degree is $q+1$. A quick application of Theorem 2.6 gives us that the characters of H have degree at most $3(q^3+1)$. Since $q > 3$, it is easy to see that the inequality

$$3(q^3+1) < \tfrac{1}{2}q(q^2+q+1)(q-1)^2$$

is true. Since the degree d_4 is an increasing function and for $i \geq 4$ we have $d_i \geq d_4$, we get that the characters of $\mathrm{Sp}_6(q)$ with degrees greater than or equal to d_4 will restrict reducibly to H. □

Lemma 6.9. *Let $q \geq 5$ and $H \cong \mathrm{GU}_3(q).2$. Then $\chi|_H$ is reducible for each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. The set of irreducible character degrees for $\mathrm{GU}_3(q)$ is

$$\{1,\, q(q-1),\, q^3,\, q^2-q+1,\, q(q^2-q+1),\, (q\pm1)(q^2-q+1),\, (q+1)^2(q-1)\}, \quad (2)$$

which can be seen from the generic character table for $\mathrm{GU}_3(q)$ available in CHEVIE [Geck et al. 1996]. Then the maximum character degree of $\mathrm{GU}_3(q)$ is $(q+1)^2(q-1)$, so Theorem 2.6 implies that $b(H)$ is at most $2(q+1)^2(q-1)$.

Since the inequality

$$2(q+1)^2(q-1) < d_4 = \tfrac{1}{2}q(q^2+q+1)(q-1)^2$$

is true for $q \geq 3$, we see by Lemma 2.2 that the statement is true, except possibly for characters of G with degree d_2 or d_3.

Arguing exactly as in Lemma 6.7 with the roles of d_2 and d_3 reversed, we see that no character of degree d_2 or d_3 may restrict irreducibly to H. Note that in this case, we do not need to make an exception like that in Lemma 6.7, since when $q \geq 5$, $\frac{1}{2}(q+1)$ cannot be in the set $\{1,\ 2,\ q,\ 2q,\ q\pm1,\ 2(q\pm1)\}$. □

Lemma 6.10. *If $q \geq 5$ and $H \cong \mathrm{Sp}_2(q) \circ \mathrm{GO}_3(q)$, then $\chi|_H$ is reducible for each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. Let $q \geq 5$ be odd and note that $\mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$ and $\Omega_3(q) \cong \mathrm{PSL}_2(q)$ and that the largest irreducible character degree of either of these groups is at most $q+1$. Further, note that $\mathrm{GO}_3(q)$ contains a normal subgroup of index 4 isomorphic to the latter group. Using this information and Theorem 2.6, we see that $b(\mathrm{GO}_3(q)) \leq 4(q+1)$.

Now, recalling that the irreducible characters of H are products of the irreducible characters of the groups $\mathrm{GO}_3(q)$ and $\mathrm{Sp}_2(q)$ since it is a central product, we obtain an upper bound on the character degrees of $H \cong \mathrm{Sp}_2(q) \circ \mathrm{GO}_3(q)$ given by $b(H) \leq 4(q+1)^2$.

Solving the inequality computationally, we get that $4(q+1)^2 < \mathcal{L}(q)$ whenever $q \geq 11$, proving the statement for $q \geq 11$ by Lemma 2.2. Further, note that if $q \geq 3$, the inequality

$$4(q+1)^2 < \tfrac{1}{2}q(q^2+q+1)(q-1)^2$$

is satisfied, so any irreducible character of G of degree at least d_4 must restrict reducibly to H.

Let d be d_2 or d_3. Note that the character degrees for $\mathrm{Sp}_2(q)$ and $\mathrm{GO}_3(q)$ are $1, q, q-1$, and $q+1$, up to multiplying or dividing by powers of 2. Since H is a central product, its irreducible character degrees are composed of products of two of these values, up to multiplying or dividing by powers of 2. Using Lemma 2.8 and arguing exactly as in Lemma 6.6, we again see that either some prime $\ell \geq 5$ or some power of 3 divides d but not any of the irreducible character degrees of H. Hence we see that d cannot be the degree of any irreducible character of H, and each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$ must therefore restrict reducibly to H. □

We next turn our attention to the subgroups of the form $\mathrm{Sp}_6(q_0).(2, r)$, where $q = q_0^r$ and r is prime. Recall that $|\mathrm{Sp}_6(q_0)| = q_0^9 \prod_{i=1}^3 (q_0^{2i} - 1)$. We begin with the case that r is odd.

Lemma 6.11. *Let q_0 be a prime power such that $q = q_0^r$, where r is an odd prime. Let $H \cong \mathrm{Sp}_6(q_0)$. Then $\chi|_H$ is reducible for each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. From the list available at [Lübeck 2007], we see that the largest irreducible character degree for $\mathrm{Sp}_6(q_0)$ is at most $(q_0^2+1)(q_0^2+q_0+1)(q_0^2-q_0+1)(q_0+1)^3$, which is smaller than $\mathcal{L}(q) = \frac{1}{2}(q_0^{3r} - 1)$ when $r \geq 5$ and $q_0 \geq 3$. Then by Corollary 2.3, we are done in the case $r > 3$.

Now let $r = 3$ and notice that

$$d_4 = \tfrac{1}{2}(q^5 - q^4 - q^2 + q) = \tfrac{1}{2}(q_0^{15} - q_0^{12} - q_0^6 + q_0^3).$$

Then, we see computationally that $b(H) < d_4$ for all $q_0 \geq 3$. Hence it suffices to show that the character degrees d_2 and d_3 for G do not appear as irreducible character degrees for H.

Notice that

$$d_2 = \tfrac{1}{2}(q_0^9 - 1) = \tfrac{1}{2}(q_0 - 1)(q_0^2 + q_0 + 1)(q_0^6 + q_0^3 + 1),$$
$$d_3 = \tfrac{1}{2}(q_0^9 + 1) = \tfrac{1}{2}(q_0 + 1)(q_0^2 - q_0 + 1)(q_0^6 - q_0^3 + 1).$$

Further, observing Lübeck's list of character degrees, see [Lübeck 2007], we see that, up to dividing by 2, every degree for H is a product of the cyclotomic polynomials dividing $q_0(q_0^6 - 1)$, which are those listed in Lemma 2.8 in terms of q_0. Using Lemma 2.8 applied to q_0^3 and the arguments used before, we see that $(q_0^6 + q_0^3 + 1)$ and $(q_0^6 - q_0^3 + 1)$ have odd divisors that do not divide $q_0(q_0^6 - 1) = q_0(q_0^3 - 1)(q_0^3 + 1)$, and therefore the same is true for d_2 and d_3. Hence these do not appear as character degrees for $\mathrm{Sp}_6(q_0)$, completing the proof. □

Finally, we address the more delicate case that $H \cong \mathrm{Sp}_6(q_0).2$, where $q = q_0^2$. In this case, we only achieve partial results.

Lemma 6.12. *Let $q = q_0^2$ be odd such that $q_0 \geq 7$ and let $H \cong \mathrm{Sp}_6(q_0).2$. Then $\chi|_H$ is reducible for each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$, except possibly those with degree d_2 or d_3.*

Proof. First, consider a character $\chi \in \mathrm{Irr}(\mathrm{Sp}_6(q))$ with degree greater than or equal to

$$d_4 = \tfrac{1}{2}q(q^2+q+1)(q-1)^2 = \tfrac{1}{2}q_0^2(q_0^4+q_0^2+1)(q_0^2-1)^2.$$

Again, from the list available at [Lübeck 2007], we see that the largest irreducible character degree for $\mathrm{Sp}_6(q_0)$ is at most $(q_0^2+1)(q_0^2+q_0+1)(q_0^2-q_0+1)(q_0+1)^3$. Hence by Theorem 2.6, we see that

$$b(H) \leq 2(q_0^2+1)(q_0^2+q_0+1)(q_0^2-q_0+1)(q_0+1)^3.$$

When $q_0 \geq 7$, we therefore have $d_4 > b(H)$, which completes the proof by Lemma 2.2. □

7. Restrictions from $G = \Omega_7(q)$

In this section, let G be the group $\Omega_7(q)$, where q is the power of an odd prime p. We prove the following:

Theorem 7.1. *Let $G = \Omega_7(q)$, where $q \geq 5$ is a power of an odd prime, and let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$. Suppose $H \leq G$ is a maximal subgroup such that the restriction $\chi|_H$ is irreducible. Then one of the following holds*:

- *H is isomorphic to $\Omega_6^{\pm}(q).2$, $(\Omega_2^{\pm}(q) \times \Omega_5(q)).2^2$, or a maximal parabolic subgroup*;
- *H is isomorphic to $\mathrm{SO}_7(q^{1/2})$; or*
- *H is isomorphic to the exceptional group of Lie type $G_2(q)$.*

The groups listed in the first item of Theorem 7.1 are the maximal subgroups in Aschbacher class $\mathcal{C}_1$ other than $(\Omega_3(q) \times \Omega_4^{\pm}(q)).2^2$, by [Bray et al. 2013]. As before, addressing the exceptions listed in Theorem 7.1 is beyond the scope of this article. We further remark that the case $H \cong G_2(q) \leq \Omega_7(q)$ is pointed out in [Seitz 1990] as one of very few embeddings of groups of Lie type into finite classical groups defined in the same characteristic that produce examples of irreducible restrictions, and is the topic of a forthcoming paper by the faculty author. The reader may also note that the exceptions listed are similar to those that must be carefully treated in [Schaeffer Fry 2013] in the case $p = 2$. The remainder of this section is devoted to proving Theorem 7.1.

Note that the smallest nontrivial irreducible character degree of G is $\mathcal{L}(q) = q^4+q^2+1$, see [Tiep and Zalesskii 1996, Theorem 1.1], and that the real-valued function $\mathcal{L}(x)$ is an increasing function for positive x. Our methods in this section will largely be similar to those in previous sections.

Aschbacher class	maximal $H \cong$	condition on q	treated in Lemma
$\mathcal{S}$	$\Omega_7(2)$	$q = p$	7.2
$\mathcal{C}_2$	$2^6 : A_7$	$p = q \equiv \pm 3 \pmod 8$	7.3
$\mathcal{C}_2$	$2^6 : S_7$	$p = q \equiv \pm 1 \pmod 8$	7.3
$\mathcal{C}_5$	$\Omega_7(q_0)$	$q = q_0^r$, r odd prime	7.4
$\mathcal{C}_1$	$(\Omega_3(q) \times \Omega_4^{\pm}(q)).2^2$		7.5

Table 4

From [Bray et al. 2013], we see that Table 4 lists all maximal subgroups when $q \geq 5$, aside from those excepted in Theorem 7.1. Note that we will treat the case $q = 3$ in Section 8 below.

Lemma 7.2. *Let $H \cong \Omega_7(2)$ and let $q \geq 5$ be an odd prime. Then $\chi|_H$ is reducible for each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. Using the character table for $H \cong \Omega_7(2) \cong \mathrm{Sp}_6(2)$ available in GAP [Breuer 2013], we see that $b(H) = 512$. The statement follows since $\mathcal{L}(5) = 651 > b(H)$ and $\mathcal{L}$ is increasing. □

Lemma 7.3. *Let $q \geq 5$ be an odd prime and let H be a maximal subgroup of G isomorphic to $2^6 : A_7$, where $q = p \equiv \pm 3 \pmod 8$ or $2^6 : S_7$, where $q = p \equiv \pm 1 \pmod 8$. Then $\chi|_H$ is reducible for each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. Since C_2^6 is abelian and normal in H, we may use Theorem 2.4 to see that $b(H) \leq [H : C_2^6] \leq |S_7| = 5040$. Note that $\mathcal{L}(9) = 6643 > b(H)$, so that the statement follows for $q \geq 9$ since $\mathcal{L}$ is increasing. When $q = 5$, we may obtain the character table for G using GAP [Breuer 2013], from which we see that the character degrees of G that are less than $[H : C_2^6] = |A_7| = 2520$ do not divide 2520. Similarly, using GAP and [Lübeck 2007], we see that the only character degree of G when $q = 7$ that is less than 5040 is 2451, which does not divide 5040. Hence applying Theorem 2.4 again yields that these cannot be character degrees of H. □

Lemma 7.4. *Let $q = q_0^r$ for a power of an odd prime q_0 and an odd prime r and let $H \cong \Omega_7(q_0)$. Then $\chi|_H$ is reducible for each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. We have

$$|H| = |\Omega_7(q_0)| = q_0^9(q_0^2 - 1)(q_0^4 - 1)(q_0^6 - 1),$$
$$\mathcal{L}(q) = q^4 + q^2 + 1 = q_0^{4r} + q_0^{2r} + 1 \geq q_0^{12} + q_0^6 + 1.$$

Consider the real-valued functions

$$g(x) = x^{12} + x^6 + 1,$$
$$h(x) = \sqrt{x^9(x^2 - 1)(x^4 - 1)(x^6 - 1)}.$$

Since both functions are positive for $x > 1$ and

$$h(x)^2 = x^{21} - (x^{19} + x^{17} - x^{13} - x^{11} - x^{13} + x^9) < x^{24} < g(x)^2,$$

we see that $g(x) > h(x)$ for all $x \geq 1$, and hence $\mathcal{L}(q) > b(H)$. $\square$

Lemma 7.5. *Let $q \geq 5$ and let $H \cong (\Omega_3(q) \times \Omega_4^{\pm}(q)).2^2$. Then $\chi|_H$ is reducible for each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$.*

Proof. Let $N \lhd H$ be the normal subgroup $N \cong \Omega_3(q) \times \Omega_4^{-}(q)$ and let $\chi \in \mathrm{Irr}(H)$. From Clifford's theorem, Theorem 2.5, $\chi|_N = e(\psi_1 + \cdots + \psi_m)$ for some positive integer e and some $\psi_1, \ldots, \psi_m \in \mathrm{Irr}(N)$ such that $\psi_1(1) = \cdots = \psi_m(1)$. Then

$$\chi(1) = \chi|_N(1) = e(\psi_1(1) + \cdots + \psi_m(1)) \leq e \cdot m \cdot b(N) \leq 4b(N),$$

where the last inequality follows from Theorem 2.6.

Now, by Theorem 2.7, each character ψ_i of N is of the form $\phi_i \times \varphi_i$, where ϕ_i is an irreducible character of $\Omega_3(q) \cong \mathrm{PSL}_2(q)$ and φ_i is an irreducible character of $\Omega_4^{\pm}(q)$, which is isomorphic to $\mathrm{PSL}_2(q^2)$ in the case "$-$" and to $\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q) \cong 2.(\mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q))$ in the case "$+$".

Then in the case "$-$", we have

$$b(N) = b(\mathrm{PSL}_2(q))b(\mathrm{PSL}_2(q^2)) = (q+1)(q^2+1) = q^3 + q^2 + q + 1,$$

and so $\chi(1) \leq 4(q^3 + q^2 + q + 1) < \mathcal{L}(q)$, for $q \geq 5$, where the last inequality follows by analyzing the corresponding real-valued functions. This completes the proof in the case "$-$".

In the case "$+$", we have $b(N) = (q+1)^3$, using the discussion after Theorem 2.7, so that $\chi(1) \leq 4(q+1)^3 < \mathcal{L}(q)$ whenever $q \geq 7$. Now, when $q = 5$, the only nontrivial character degree of G that is at most $4(5+1)^3 = 864$ is 651, using the character table available in GAP [Breuer 2013]. However, note that 651 is not divisible by 2, and that $651 > (5+1)^3 = 216$. This yields that 651 is not a character degree of N or of H, again using Theorem 2.6, and completes the proof. $\square$

8. Results for small values of q

Here we address the case that q is "small". That is, we consider the exceptional values of q from Theorems 4.1, 5.1, 6.1, and 7.1. We do this using GAP [Breuer 2013] and our algorithm discussed in Section 3. For $G \cong \Omega_3(q) \cong \mathrm{PSL}_2(q)$, we obtain the results for $5 \leq q \leq 11$, which are summarized in Table 5. For $G \cong \Omega_5(q) \cong \mathrm{PSp}_4(q)$, we summarize the results for $q = 3, 5$ in Table 6. The results for $G = \mathrm{PSp}_6(3)$ and $G = \Omega_7(3)$ are summarized in Tables 7 and 8, respectively.

In the tables, Maxes$[i]$ means the i-th maximal subgroup of G found using the `Maxes` command in GAP and the labeling for the characters is as in the GAP character table for G.

value of q	maximal subgroup with irreducible restrictions	irreducible restrictions	degree
$q=5$	Maxes[1] $\cong A_4$	χ_2, χ_3	3
$q=7$	Maxes[1] $\cong S_4$	χ_2, χ_3	3
$q=7$	Maxes[2] $\cong S_4$	χ_2, χ_3	3
$q=7$	Maxes[3] $\cong 7:3$	χ_2, χ_3	3
$q=9$	Maxes[1] $\cong A_5$	χ_3	5
$q=9$	Maxes[2]	χ_2	5
$q=11$	Maxes[1] $\cong A_5$	χ_2, χ_3	5
$q=11$	Maxes[2] $\cong A_5$	χ_2, χ_3	5
$q=11$	Maxes[3] $\cong 11:5$	χ_2, χ_3	5

Table 5. Irreducible restrictions for $\mathrm{PSL}_2(q)$ for small q.

value of q	maximal subgroup with irreducible restrictions	irreducible restrictions	degree
$q=3$	Maxes[1]	χ_2, χ_3 χ_5, χ_6	5 10
$q=3$	Maxes[2] $\cong A_6.2_1$	χ_2, χ_3 χ_5, χ_6	5 10
$q=3$	Maxes[4] $\cong 3^3 : S_4'$	χ_4	6
$q=5$	Maxes[2] $\cong 5^3 : (2\times A_5).2$	χ_4	40
$q=5$	Maxes[3] $\cong \mathrm{PSL}_2(25).2_2$	χ_2, χ_3	13

Table 6. Irreducible restrictions for $\mathrm{PSp}_4(q)$ for small q.

maximal subgroup with irreducible restrictions	irreducible restrictions	degree
$3^6 : \mathrm{PSL}_3(3)$	χ_2, χ_3 χ_4	13 78
$\mathrm{PSL}_2(27).3$	χ_2, χ_3 χ_4	13 78
$\mathrm{PSL}_3(3).2$	χ_2, χ_3	13
Maxes[9] $\cong \mathrm{PSL}_2(13)$	χ_2, χ_3	13
Maxes[10] $\cong \mathrm{PSL}_2(13)$	χ_2, χ_3	13

Table 7. Irreducible restrictions from $\mathrm{PSp}_6(3)$ to maximal subgroups.

maximal subgroup with irreducible restrictions	irreducible restrictions	degree
Maxes[3] $\cong \mathrm{PSL}_4(3).2$	χ_8, χ_9	260
Maxes[4] $\cong G_2(3)$	χ_2 χ_3 χ_5 χ_6 χ_{10} χ_{11}	78 91 168 182 273 546
Maxes[5]	χ_2 χ_3 χ_5 χ_6 χ_{10} χ_{11}	78 91 168 182 273 546
Maxes[6] $\cong (C_3^3.C_3^3) : \mathrm{PSL}_3(3)$	χ_2	78
Maxes[7] $\cong \mathrm{Sp}_6(2)$	χ_4	105
Maxes[8]	χ_4	105
Maxes[10] $\cong A_9.2$	χ_4	105
Maxes[11]	χ_4	105

Table 8. Irreducible restrictions for $\Omega_7(3)$.

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