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in a forest-building process

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(Communicated by Glenn Hurlbert)

Given a graph, we can form a spanning forest by first sorting the edges in a random order, and then only keeping edges incident to a vertex which is not incident to any previous edge. The resulting forest is dependent on the ordering of the edges, and so we can ask, for example, how likely is it for the process to produce a graph with k trees.

We look at all graphs which can produce at most two trees in this process and determine the probabilities of having either one or two trees. From this we construct infinite families of graphs which are nonisomorphic but produce the same probabilities.

1. Introduction

We consider the following *forest-building process*:

- (1) Take all of the edges of the graph, remove them and sort them in a random order.
- (2) Go through the edges in this order and only put those edges back in which connect to some vertex not previously seen by any edge.

From this, we must end up with a forest (a graph without cycles) since we can never add an edge that closes a cycle. As an example, in Figure 1 we list all 24 different ways to order the edges and group them based on the resulting forest formed.

We will consider this problem: how many different edge orderings produce a given number, say k , of trees in the resulting graph? Equivalently, what is the probability that if we take a random ordering of the edges, we produce a forest with k trees? We will let $P(G, k)$ denote this probability. As an example when G is the paw graph, we see that $P(G, 1) = \frac{5}{6}$ and $P(G, 2) = \frac{1}{6}$ (see Figure 1).

This process was implicitly used in [Butler et al. 2015] for the complete graph, and explicitly introduced in [Berikkyzy et al. 2018], where some basic properties

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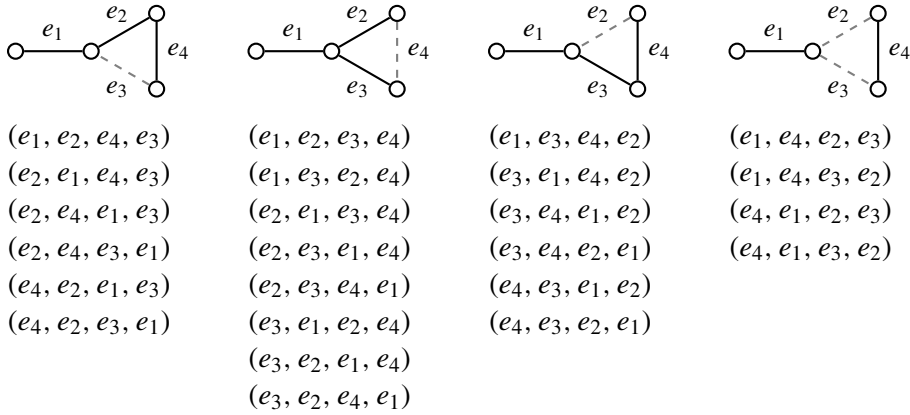


Figure 1. The results from different edge orderings of the paw graph.

were established and the probabilities for complete bipartite graphs were determined. We summarize these results here.

Theorem 1 [Butler et al. 2015]. *We have*

$$P(K_n, k) = \frac{\binom{n-1}{n-2k, k, k-1} 2^{n-2k}}{\binom{2n-2}{n}}.$$

Theorem 2 [Berikkyzy et al. 2018]. *We have*

$$P(K_{s,t}, k) = \frac{(s+t) \binom{s}{k} \binom{t}{k}}{st \binom{s+t}{s}}.$$

For small graphs (at most five vertices), the probabilities are given in [Berikkyzy et al. 2018]. There are a few instances where two graphs would have the same probabilities for all k , and most of those are edge-transitive graphs. More generally, the following was observed.

Lemma 3 [Berikkyzy et al. 2018]. *If G is an edge-transitive graph with minimum degree of at least 2 and e is any edge, then we have $P(G, k) = P(G - e, k)$ for all k .*

In essence, this follows by noting that the last edge in an ordering is never kept, and by symmetry every edge is the last edge in an ordering equally often.

The goal of this note is to compute the probabilities for more families of graphs, namely graphs which can produce at most two trees in the forest-building process. Using this, we will produce infinitely many examples of nonisomorphic graphs G and H where the probabilities agree and neither G or H are edge-transitive.

2. Graphs with at most two trees

We are interested in exploring the graphs which can produce at most two trees in the forest-building process. Equivalently, this means that there are at most two disjoint edges in the graph (disjoint in the sense that they share no vertex).

Proposition 4. *The only nonempty graphs without isolated vertices, which contain no pair of disjoint edges, are star graphs ($K_{1,n}$) and the triangle graph (K_3).*

Proof. If the graph is not connected, then taking one edge from two different components gives two disjoint edges. So we may assume the graph is connected.

If the graph has two disjoint edges, then we can connect these together by a path, creating a path with at least four vertices. Conversely, if the graph has a path with at least four vertices, then it must contain two disjoint edges. So we can conclude the longest path is a path with at most three vertices. If the longest path has two vertices, then the graph is a K_2 .

If the longest path has three vertices and the ends of the path are not leaves, then it must be that the ends connect and form a triangle. Since the paw graph has two disjoint edges, this can only happen if the graph is a K_3 .

Finally, if the graph is not a triangle and doesn't have a path of length 4 (and hence has no cycles), then it must be a star. $\square$

Proposition 5. *If a graph without isolated vertices has a vertex v of degree at least 5 and contains no set of three disjoint edges, then deleting v and all incident edges, and removing any isolated vertices results in either an empty graph, a star, or a K_3 .*

Proof. Let v be a vertex of degree at least 5 in the graph. Suppose that the graph resulting from deleting the vertex v and all incident edges, and removing any isolated vertices, is not the empty graph, a star, or a K_3 . Proposition 4 states that the only nonempty graphs without isolated vertices, that contain no set of two disjoint edges, are the star graph and K_3 . Thus the resulting graph will contain at least two disjoint edges. Call these edges e_1 and e_2 ; note neither of these edges are incident to v .

At most four edges incident to v are also incident to the edges e_1 and e_2 . Since v has degree at least 5, this leaves at least one edge, e_3 , that is connected to v and not incident to e_1 or e_2 . Thus the original graph contains a set of three disjoint edges. $\square$

Finally, we observe that if all the degrees are bounded by at most 4 and the graph is connected, then as n gets large, the diameter must also grow — which forces three disjoint edges. In particular there are finitely many graphs with maximum degree at most 4 which produce at most two trees.

Putting this all together, we see that for n sufficiently large ($n \geq 6$; verified computationally) the only connected graphs which produce at most two trees, and are not stars, are the following five families, shown in Figure 2:

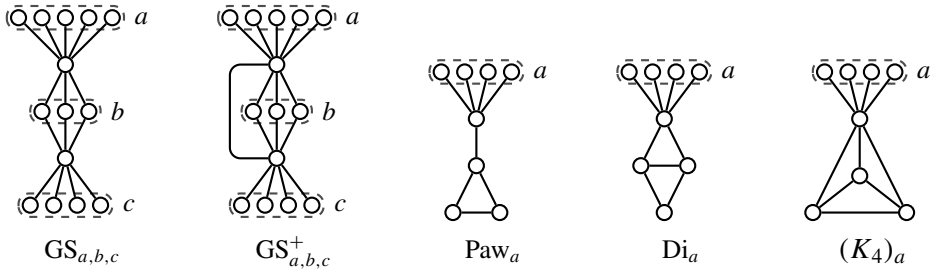


Figure 2. The five families of graphs.

- $GS_{a,b,c}$: the stars $K_{1,a+b}$ and $K_{1,b+c}$ which have b leaves glued together (glued stars).
- $GS_{a,b,c}^+$: the stars $K_{1,a+b}$ and $K_{1,b+c}$ which have b leaves glued together and the centers joined by an edge (glued stars with an edge).
- Paw_a : the paw graph with a leaves appended to the vertex of degree 1.
- Di_a : the diamond graph (a four-cycle with an extra edge) with a leaves appended to one of the vertices of degree 2.
- $(K_4)_a$: the complete graph on four vertices with a leaves appended to one of the vertices.

Note that when the degree is at least 5, the first two of these correspond to Proposition 5 where the remaining graph is a star and the last three of these correspond to Proposition 5 where the remaining graph is a K_3 .

3. Computing probabilities for the families

We now turn our attention to computing the probabilities that a graph ends in one or two trees in the forest-building process. We can find these probabilities by noting that if there are m edges in the graph, then the probability that we end with two trees is

$$P(G, 2) = \frac{|\{\text{rearrangements with two trees}\}|}{m!}.$$

We will focus on counting the rearrangements which produce two trees. Particularly, we want to count rearrangements where an edge occurs exactly once, not at the start, and involves two vertices which have not been previously seen.

Since we will be counting rearrangements, we will find it useful to know how to manipulate binomial coefficients. Recall that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the number of ways to choose k elements (in our case this will usually be locations) out of an n element set. There are many binomial coefficient identities (see the book of Graham, Knuth, and Patashnik [Graham et al. 1994, Chapter 5] for a good introduction); we will need to make repeated use of the following well-known result.

Proposition 6. *We have*

$$\sum_j \binom{\ell-j}{m} \binom{q+j}{n} = \binom{\ell+q+1}{m+n+1}, \quad (1)$$

where the sum ranges over all values where the summands are nonzero.

Theorem 7. *We have the following probabilities:*

$$P(\text{GS}_{a,b,c}, 1) = \frac{b}{(b+c+1)(b+c)} + \frac{b}{(a+b+1)(a+b)},$$

$$P(\text{GS}_{a,b,c}^+, 1) = \frac{2b+c+2}{(b+c+1)(b+c+2)} + \frac{2b+a+2}{(b+a+1)(b+a+2)} - \frac{1}{a+2b+c+1}.$$

Proof. Since $P(\text{GS}_{a,b,c}, 1) + P(\text{GS}_{a,b,c}, 2) = 1$, we can focus on computing the probability of resulting in two trees. We now claim

$$\begin{aligned} P(\text{GS}_{a,b,c}, 2) &= \sum_{i=0}^b \sum_{j=0}^a \frac{\binom{a}{j} \binom{b}{i} (i+j)! (b+c-i)(a+2b+c-i-j-1)!}{(a+2b+c)!} - \frac{b+c}{a+2b+c} \\ &\quad + \sum_{i=0}^b \sum_{j=0}^c \frac{\binom{c}{j} \binom{b}{i} (i+j)! (a+b-i)(a+2b+c-i-j-1)!}{(a+2b+c)!} - \frac{a+b}{a+2b+c}. \end{aligned} \quad (2)$$

This comes from the two cases, namely where our first edge initially comes from the “top half” (i.e., edges coming from the star $K_{1,a+b}$), and where our first edge initially comes from the “bottom half” (i.e., edges coming from the star $K_{1,b+c}$). We focus on the top-half case, as the bottom half follows by an identical argument by interchanging the roles of a and c .

Determining if we have two trees comes down to what happens when we pick our first edge from the star $K_{1,b+c}$. We look at all ways that this occurs by first picking edges from $K_{1,a+b}$ and then considering what happens when we pick our edge from $K_{1,b+c}$. In particular, we will pick j edges from the a leaf vertices and i edges from the b gluing vertices. We now run over all possibilities for i and j .

For each choice of edges we now consider all possible orderings as follows:

- $\binom{a}{j}$ corresponds to which of the j edges among the a were chosen.
- $\binom{b}{i}$ corresponds to which of the i edges among the b were chosen.

- $(i + j)!$ indicates how many ways to order these $i + j$ edges (note that these $i + j$ edges are all of the initial edges).
- $(b + c - i)$ indicates how many edges disjoint from the ones above are available to choose, if we want to create two trees.
- $(a + 2b + c - i - j - 1)!$ is the number of ways to rearrange the remaining edges.

This gives all orderings of edges possible; to get the probability we now divide by the total number of orderings, which is $(a + 2b + c)!$.

Note that in the summation we need to correct for $i = 0, j = 0$, which does not fall into the case where the first edge is from the top. So we subtract this term, which gives the $-(b + c)/(a + 2b + c)$ at the end.

To now simplify these sums we can repeatedly apply (1). So we have

$$\begin{aligned}
 & \sum_{i=0}^b \sum_{j=0}^a \frac{\binom{a}{j} \binom{b}{i} (i + j)! (b + c - i) (a + 2b + c - i - j - 1)!}{(a + 2b + c)!} \\
 &= \sum_{i=0}^b \sum_{j=0}^a \frac{\frac{a!}{j! (a-j)!} \frac{b!}{i! (b-i)!} (i + j)! (b + c - i) (a + 2b + c - i - j - 1)!}{(a + 2b + c)!} \\
 &= \sum_{i=0}^b \frac{a! b! (b + c - i)}{(a + 2b + c)! (b - i)!} \sum_{j=0}^a \frac{(i + j)! (a + 2b + c - i - j - 1)!}{i! j! (a - j)!} \\
 &= \sum_{i=0}^b \frac{a! b! (b + c - i) (2b + c - i - 1)!}{(a + 2b + c)! (b - i)!} \sum_{j=0}^a \frac{(i + j)! (a + 2b + c - i - j - 1)!}{i! j! (a - j)! (2b + c - i - 1)!} \\
 &= \sum_{i=0}^b \frac{a! b! (b + c - i) (2b + c - i - 1)!}{(a + 2b + c)! (b - i)!} \sum_{j=0}^a \binom{i + j}{i} \binom{a + 2b + c - i - 1 - j}{2b + c - i - 1} \\
 &= \sum_{i=0}^b \frac{a! b! (b + c - i) (2b + c - i - 1)!}{(a + 2b + c)! (b - i)!} \binom{a + 2b + c}{2b + c} \\
 &= \sum_{i=0}^b \frac{a! b! (b + c - i) (2b + c - i - 1)! (a + 2b + c)!}{(a + 2b + c)! (b - i)! (2b + c)! a!} \\
 &= \sum_{i=0}^b \frac{b! (b + c - i) (2b + c - i - 1)!}{(b - i)! (2b + c)!} \\
 &= \frac{b! (b + c - 1)!}{(2b + c)!} \sum_{i=0}^b \frac{(2b + c - i - 1)!}{(b - i)! (b + c - 1)!} (b + c - i)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{b!(b+c-1)!}{(2b+c)!} \left((b+c) \sum_{i=0}^b \binom{2b+c-1-i}{b+c-1} \binom{i}{0} - \sum_{i=0}^b \binom{2b+c-1-i}{b+c-1} \binom{i}{1} \right) \\
 &= \frac{b!(b+c-1)!}{(2b+c)!} \left((b+c) \binom{2b+c}{b+c} - \binom{2b+c}{b+c+1} \right) \\
 &= \frac{b!(b+c-1)!}{(2b+c)!} \left((b+c) \frac{(2b+c)!}{(b+c)!b!} - \frac{(2b+c)!}{(b+c+1)!(b-1)!} \right) \\
 &= 1 - \frac{b}{(b+c+1)(b+c)}.
 \end{aligned}$$

By a similar process, the other double sum becomes

$$\sum_{i=0}^b \sum_{j=0}^c \frac{\binom{c}{j} \binom{b}{i} (i+j)! (a+b-i)(a+2b+c-i-j-1)!}{(a+2b+c)!} = 1 - \frac{b}{(b+a+1)(b+a)}.$$

Now replacing the double sums by these simplified expressions we have

$$\begin{aligned}
 P(\text{GS}_{a,b,c}, 2) &= \left(1 - \frac{b}{(b+c+1)(b+c)} \right) - \frac{b+c}{a+2b+c} + \left(1 - \frac{b}{(b+a+1)(b+a)} \right) - \frac{a+b}{a+2b+c} \\
 &= 1 - \frac{b}{(b+c+1)(b+c)} - \frac{b}{(b+a+1)(b+a)}.
 \end{aligned}$$

Finally we note

$$P(\text{GS}_{a,b,c}, 1) = 1 - P(\text{GS}_{a,b,c}, 2) = \frac{b}{(b+c+1)(b+c)} + \frac{b}{(b+a+1)(b+a)},$$

establishing the result for $\text{GS}_{a,b,c}$.

The result for $P(\text{GS}_{a,b,c}^+, 2)$ follows by a similar argument, the only difference being the additional edge which *cannot* be used in order to result in two trees. So (2) would now become

$$\begin{aligned}
 P(\text{GS}_{a,b,c}^+, 2) &= \sum_{i=0}^b \sum_{j=0}^a \frac{\binom{a}{j} \binom{b}{i} (i+j)! (b+c-i)(a+2b+c-i-j)!}{(a+2b+c+1)!} - \frac{b+c}{a+2b+c+1} \\
 &\quad + \sum_{i=0}^b \sum_{j=0}^c \frac{\binom{c}{j} \binom{b}{i} (i+j)! (a+b-i)(a+2b+c-i-j)!}{(a+2b+c+1)!} - \frac{a+b}{a+2b+c+1}.
 \end{aligned}$$

The rest of the argument works in the same way as before. $\square$

Our approach was to focus on the probability of producing two trees. It is possible to establish the result by focusing on one tree, and the following proof was communicated to us by a referee of the paper.

Alternative proof of Theorem 7. To compute the probability of producing a single tree in the graph $\text{GS}_{a,b,c}$ we focus on finding the probability of producing a “bridge”, a path of length 2 connecting the vertices v_a and v_c (the vertices with a and b leaves respectively). Label the b vertices which connect to both v_a , and v_c as $v_1, \dots, v_b$.

The probability that we first pick an edge incident to a and form a bridge going through v_i is

$$\frac{1}{(b+c+1)(b+c)}.$$

To see this, once we have picked the first edge there are $b+c+1$ important edges remaining, namely the $b+c$ edges incident to v_c and the edge $\{v_a, v_i\}$. In order to form the bridge among these $b+c+1$ edges we must first choose $\{v_a, v_i\}$ (probability $1/(b+c+1)$) and must second choose $\{v_i, v_c\}$ (probability $1/(b+c)$), establishing the above probability. This is independent of v_i and so going over all b of the v_i we have that the probability that we first pick an edge incident to a and form a bridge is

$$\frac{b}{(b+c+1)(b+c)}.$$

A symmetrical argument gives that the probability that we first pick an edge incident to c and form a bridge is

$$\frac{b}{(b+a+1)(b+a)}.$$

Finally these events are disjoint and cover all ways to form a bridge. So we can conclude that the probability of forming a bridge, and hence the probability of having one component, is

$$\frac{b}{(b+a+1)(b+a)} + \frac{b}{(b+c+1)(b+c)}.$$

A similar approach works for $\text{GS}_{a,b,c}^+$ once we also account for the edge $\{v_a, v_c\}$ being a bridge. We leave the details of this to the interested reader. $\square$

Theorem 8. *We have the following probabilities:*

$$\begin{aligned} P(\text{Paw}_a, 1) &= \frac{1}{6} - \frac{1}{a+3} + \frac{1}{a+1}, \\ P(\text{Di}_a, 1) &= \frac{3}{10} - \frac{2}{a+4} + \frac{2}{a+2}, \\ P((K_4)_a, 1) &= \frac{2}{5} - \frac{3}{a+5} + \frac{3}{a+3}. \end{aligned}$$

Proof. We will again compute the probability that there are two trees in the process. However, in these cases there are many more possibilities to consider. To simplify the situation, we make the following observation: every edge which is a leaf in the original graph will always be kept in the forest-building process. This indicates if there

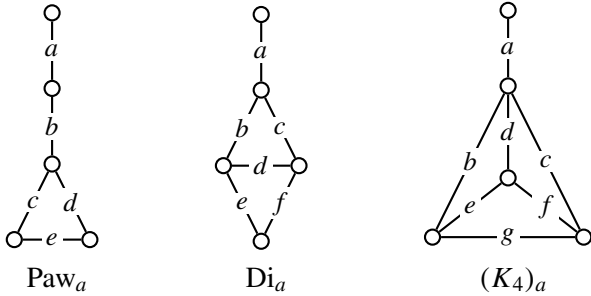


Figure 3. The remaining three graphs with the leaves collapsed to a single edge a .

are multiple leaves off of a single vertex v , then we only need to know when the *first* leaf was chosen. This is because, after first leaf, v will have been seen by some edge.

So we now represent the remaining graphs from Figure 2, as shown in Figure 3, where a corresponds to all of the a leaves condensed down, and the remaining edges are labeled as indicated with each label other than a corresponding to a single edge.

For each graph, we now look at all possible ways to start selecting edges and end with a pair of disjoint edges. We also find the probability of starting our selection in a particular way. Recall that an edge marked a corresponds to a different edges, and so until we select that edge, we assume all a of them haven't been seen and are available for picking; after selection, by the observation we can assume they have all been seen. (In other words, it is only the relative ordering of the different types of edges that matter.)

For the paw graph, we have the possibilities shown in Table 1 (the first column indicates every possible sequence of choices of edges until two trees are formed, while the second column indicates the probability of any one of those sequence of

start of edge orderings resulting in two trees	probabilities of an ordering
ac, ad, ae	$\frac{a}{a+4} \cdot \frac{1}{4}$
be, eb	$\frac{1}{a+4} \cdot \frac{1}{a+3}$
ca, da, ea	$\frac{1}{a+4} \cdot \frac{a}{a+3}$
abe	$\frac{a}{a+4} \cdot \frac{1}{4} \cdot \frac{1}{3}$
bae	$\frac{1}{a+4} \cdot \frac{a}{a+3} \cdot \frac{1}{3}$
$cda, cea, dca, dea, eda, eca$	$\frac{1}{a+4} \cdot \frac{1}{a+3} \cdot \frac{a}{a+2}$
$ceda, cdea, dcea, dec a, ecda, edca$	$\frac{1}{a+4} \cdot \frac{1}{a+3} \cdot \frac{1}{a+2} \cdot \frac{a}{a+1}$

Table 1. Probabilities associated with Paw_a .

start of edge orderings resulting in two trees	probabilities of an ordering
ad, ae, af	$\frac{a}{a+5} \cdot \frac{1}{5}$
bf, ce, ec, fb	$\frac{1}{a+5} \cdot \frac{1}{a+4}$
da, ea, fa	$\frac{1}{a+5} \cdot \frac{a}{a+4}$
abf, ace	$\frac{a}{a+5} \cdot \frac{1}{5} \cdot \frac{1}{4}$
baf, cae	$\frac{1}{a+5} \cdot \frac{a}{a+4} \cdot \frac{1}{4}$
$dea, dfa, eda, efa, fda, fea$	$\frac{1}{a+5} \cdot \frac{1}{a+4} \cdot \frac{a}{a+3}$
$defa, dfea, edfa, efda, fdea, feda$	$\frac{1}{a+5} \cdot \frac{1}{a+4} \cdot \frac{1}{a+3} \cdot \frac{a}{a+2}$

Table 2. Probabilities associated with Di_a .

start of edge orderings resulting in two trees	probabilities of an ordering
ae, af, ag	$\frac{a}{a+6} \cdot \frac{1}{6}$
bf, ce, dg, ec, fb, gd	$\frac{1}{a+6} \cdot \frac{1}{a+5}$
ea, fa, ga	$\frac{1}{a+6} \cdot \frac{a}{a+5}$
abf, adg, ace	$\frac{a}{a+6} \cdot \frac{1}{6} \cdot \frac{1}{5}$
baf, dag, cae	$\frac{1}{a+6} \cdot \frac{a}{a+5} \cdot \frac{1}{5}$
$efa, ega, fea, fga, gea, gfa$	$\frac{1}{a+6} \cdot \frac{1}{a+5} \cdot \frac{a}{a+4}$
$efga, egfa, fega, fgea, gefa, gfea$	$\frac{1}{a+6} \cdot \frac{1}{a+5} \cdot \frac{1}{a+4} \cdot \frac{a}{a+3}$

Table 3. Probabilities associated with $(K_4)_a$.

choices being made). If we now sum all of these probabilities together, we get

$$P(\text{Paw}_a, 2) = \frac{5}{6} + \frac{1}{a+3} - \frac{1}{a+1},$$

establishing the result (recall that $P(\text{Paw}_a, 1) + P(\text{Paw}_a, 2) = 1$).

The results for the remaining two graphs are established in the same way and the corresponding probabilities are given in Tables 2 and 3. $\square$

4. Examples of graphs with the same probabilities

Using the formulas from the theorems in the preceding section, we can now compute the probabilities for a large number of these graphs efficiently. In particular, we examined all graphs up through 500 vertices in these families, and discovered several examples of families of nonisomorphic graphs which produce the same probabilities.

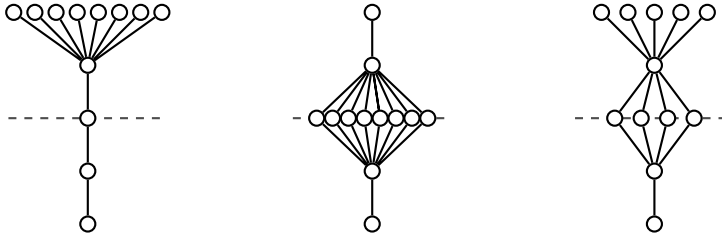


Figure 4. Examples of a set of three graphs with the same probabilities from Proposition 9. In this case $s = t = 1$.

Proposition 9. *Given $s, t \geq 1$ with s dividing $2t(t + 1)$, let $r = 2t(t + 1)/s$. Then we have for all k*

$$P(\text{GS}_{r+3t+1,s,t}, k) = P(\text{GS}_{t,r+s+2t+1,t}, k) = P(\text{GS}_{3t+s+1,r,t}, k).$$

This immediately follows by applying the formulas for the probabilities from Theorem 7.

Proposition 10. *Given $t \geq 1$, we have for all k*

$$P(\text{GS}_{5t+3,t,2t}, k) = P(\text{GS}_{5t+1,t+1,2t+1}, k).$$

This also immediately follows by applying the formulas for probabilities from Theorem 7. We note that there were many other examples of pairs of glued star graphs which are not explained by Propositions 9 and 10. A complete characterization of all such pairs of glued stars remains elusive.

Looking beyond glued stars, we found very few pairs of graphs with the same probabilities and the results do not seem to fit any patterns. As an example, all pairs of graphs from the $\text{GS}_{a,b,c}^+$ family up through 500 vertices with the same probabilities are listed below (it is possible that this is a complete list for this family; showing this would relate to solving a system of diophantine equations).

$$\begin{aligned} P(\text{GS}_{17,3,9}^+, k) &= P(\text{GS}_{10,9,10}^+, k), & P(\text{GS}_{28,5,9}^+, k) &= P(\text{GS}_{26,8,8}^+, k), \\ P(\text{GS}_{103,15,48}^+, k) &= P(\text{GS}_{63,71,32}^+, k), & P(\text{GS}_{95,23,53}^+, k) &= P(\text{GS}_{53,66,52}^+, k). \end{aligned}$$

5. Conclusion

We found the probabilities for all connected graphs which can form at most two trees in this forest-building process. A natural next step is to consider graphs with at most three trees. As an example, two pairs of graphs with at most three trees and matching probabilities are given in Figure 5. This is suggestive that these are the start of an infinite family of such graphs, but we have not yet established this. One difficulty is that unlike the situation for two trees where only one probability

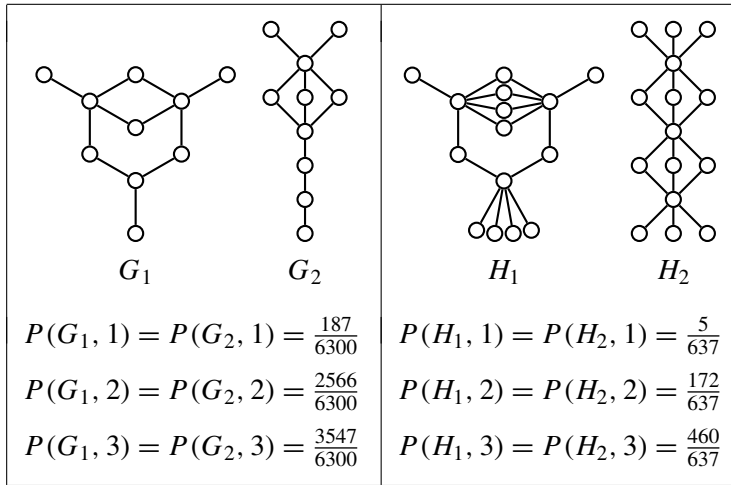


Figure 5. Two pairs of graphs with at most three trees and producing the same probabilities.

needed to be computed (since the probabilities sum to 1), this requires that two probabilities be computed.

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