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Log-concavity of Hölder means and an application to geometric inequalities

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The log-concavity of the Hölder mean of two numbers, as a function of its index, is presented first. The notion of α-cevian of a triangle is introduced next, for any real number α. We use this property of the Hölder mean to find the smallest index $p(\alpha)$ such that the length of an α-cevian of a triangle is less than or equal to the $p(\alpha)$-Hölder mean of the lengths of the two sides of the triangle that are adjacent to that cevian.

1. Introduction

All parts of mathematics are interconnected, including two important branches, geometry and analysis. Continuity, which is a fundamental notion in real analysis, is used in Euclidean geometry as one axiom in Hilbert axiomatization, and in proving Thales' theorem for irrational ratios. On the other hand, geometry helps real analysis by providing pictures that help us understand certain theorems. For example, Euler's theorem, which says that in any parallelogram the sum of the squares of the lengths of its sides is equal to the sum of the squares of its diagonals, provides a visual representation for the parallelogram identity that characterizes the norms of inner product spaces.

There is an abundant literature of geometric inequalities concerning important line segments in a triangle; see [Bottema et al. 1969; Mitrinović et al. 1989], for example. Some of these inequalities improve previously existing inequalities.

In this paper we present an application of the log-concavity of the Hölder mean with positive index, of two numbers, to find sharp inequalities relating lengths of cevians and sides of a triangle. Using these inequalities we find the best possible index for the Hölder mean, in a certain sense.

The paper is divided as follows:

In Section 2, we prove that the Hölder mean of two positive numbers, viewed as a function of its index, is logarithmically concave on $[0, \infty)$. In Section 3,

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we define the notion of an α-cevian in a triangle, and find the smallest index $p(\alpha)$ such that the length of every α-cevian is less than or equal to the $p(\alpha)$-Hölder mean of the lengths of the two sides of the triangle that are adjacent to that cevian.

2. Log-concavity of Hölder means

Let a and b be two positive numbers. For any $p \in [-\infty, \infty]$, we define the p-Hölder mean of a and b, as

$$H_p(a,b) := \begin{cases} \left(\frac{1}{2}a^p + \frac{1}{2}b^p\right)^{1/p} & \text{if } p \in \mathbb{R} \setminus \{0\}, \\ \lim_{p\to 0} H_p(a,b) = \sqrt{ab} & \text{if } p = 0, \\ \lim_{p\to -\infty} H_p(a,b) = \min\{a,b\} & \text{if } p = -\infty, \\ \lim_{p\to \infty} H_p(a,b) = \max\{a,b\} & \text{if } p = \infty. \end{cases} \tag{2-1}$$

It follows from Jensen's inequality that for all $-\infty \le p < q \le \infty$, we have

$$H_p(a,b) \le H_q(a,b), \tag{2-2}$$

and this inequality is strict if $a \neq b$; see [Bullen 1998; Bullen et al. 1988; Pólya and Szegő 1972].

We prove now that the Hölder mean of two positive numbers, viewed as a function of its index, is logarithmically concave on $[0, \infty)$.

Lemma 2.1. *For all positive numbers a and b, the function $f : [0,\infty) \to \mathbb{R}$, defined by*

$$f(x) := \ln(H_x(a,b)), \tag{2-3}$$

is concave downward.

Proof. If $a = b$, then the lemma is obvious since f is a constant function, and its value is $f(x) = \ln(a)$ for all x in $[0, \infty)$.

Let us assume now that $0 < a < b$. Then, defining $c := \frac{b}{a} > 1$ for all $x \ge 0$, we have

$$f(x) = \ln(H_x(a,b)) = \ln\left(aH_x\left(1, \frac{b}{a}\right)\right) = \ln(H_x(1,c)) + \ln(a).$$

Thus the graph of f is just a vertical translation by $\ln(a)$ of the graph of $g : [0,\infty) \to \mathbb{R}$, defined by

$$g(x) = \ln(H_x(1,c)). \tag{2-4}$$

Therefore, it suffices to show that g is concave downward on $[0, \infty)$.

We know that g is continuous on $[0, \infty)$, and so to achieve our goal we need to prove that the second derivative of g is negative on $(0, \infty)$.

Indeed, if $'$ denotes the derivative with respect to x, then we have

$$g'(x) = \frac{d}{dx}\left[\frac{1}{x}\ln(1+c^x) - \frac{1}{x}\ln(2)\right]$$
$$= -\frac{1}{x^2}\ln(1+c^x) + \frac{1}{x}\frac{c^x\ln(c)}{1+c^x} + \frac{\ln(2)}{x^2}. \tag{2-5}$$

Differentiating one more time, we obtain

$$g''(x) = \frac{2}{x^3}\ln(1+c^x) - \frac{2}{x^2}\frac{c^x\ln(c)}{1+c^x} + \frac{1}{x}\frac{c^x\ln^2(c)}{(1+c^x)^2} - \frac{2\ln(2)}{x^3}. \tag{2-6}$$

We make now the change of variable

$$y := c^x \in (1,\infty), \tag{2-7}$$

which means

$$x = \frac{\ln(y)}{\ln(c)}. \tag{2-8}$$

Substituting back in the formula of $g''(x)$, we obtain

$$g''(x) = \frac{2\ln^3(c)}{\ln^3(y)}\ln(1+y) - \frac{2\ln^2(c)}{\ln^2(y)}\frac{y\ln(c)}{1+y} + \frac{\ln(c)}{\ln(y)}\frac{y\ln^2(c)}{(1+y)^2} - \frac{2\ln(2)\ln^3(c)}{\ln^3(y)}. \tag{2-9}$$

Thus, to show that, for all $x>0$, we have $g''(x)<0$, by multiplying both sides by the positive number $(1+y)^2\ln^3(y)/\ln^3(c)$, we have to prove that for all $y>1$

$$h(y) := 2(1+y)^2\ln(1+y) - 2y(1+y)\ln(y) + y\ln^2(y) - 2(1+y)^2\ln(2) \tag{2-10}$$

is negative.

The function h is defined even for $y=1$, and we have $h(1)=0$.

We will study the sign of the first, second, and third derivatives of h on $[1,\infty)$.

Using the product rule of differentiation, the derivative of h with respect to y is

$$h'(y) = 4(1+y)\ln(1+y) + 2(1+y)^2\frac{1}{1+y} - 2(1+y)\ln(y) - 2y\ln(y)$$
$$- 2y(1+y)\frac{1}{y} + \ln^2(y) + 2y\ln(y)\frac{1}{y} - 4(1+y)\ln(2)$$
$$= 4(1+y)\ln(1+y) - 4y\ln(y) + \ln^2(y) - 4(1+y)\ln(2). \tag{2-11}$$

Let us observe that $h'(1)=0$.

Differentiating again, we obtain

$$h''(y) = 4\ln(1+y) + 4(1+y)\frac{1}{1+y} - 4\ln(y) - 4y\frac{1}{y} + 2\frac{1}{y}\ln(y) - 4\ln(2)$$
$$= 4\ln(1+y) - 4\ln(y) + \frac{2\ln(y)}{y} - 4\ln(2). \tag{2-12}$$

We observe that $h''(1)=0$.

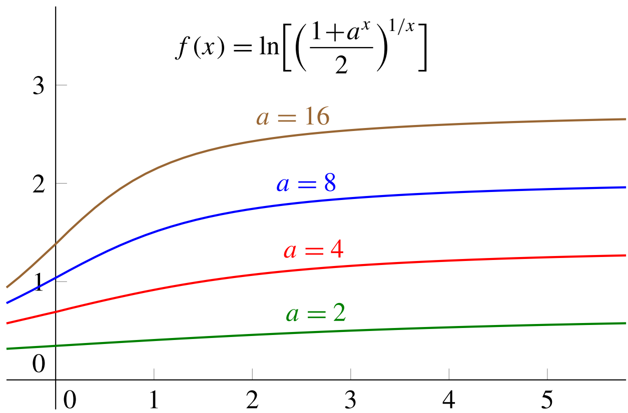

Figure 1. Graph of $y = \ln[((1+a^x)/2)^{1/x}]$ for various values of a.

Finally, differentiating one more time, we obtain

$$h'''(y) = 2\left[\frac{2}{1+y} - \frac{2}{y} + \frac{1}{y^2} - \frac{\ln(y)}{y^2}\right] = 2\left[\frac{1-y}{y^2(y+1)} - \frac{\ln(y)}{y^2}\right] < 0 \tag{2-13}$$

for all $y > 1$, since $1 - y < 0$ and $-\ln(y) < 0$.

Thus, we conclude that h'' is strictly decreasing on $[1, \infty)$. This implies that for all $y > 1$, we have $h''(y) < h''(1) = 0$. Hence, h' is strictly decreasing on $[1, \infty)$. This implies that for all $y > 1$, we have $h'(y) < h'(1) = 0$. Therefore, h is strictly decreasing on $[1, \infty)$. Finally, from this assertion we conclude that $h(y) < h(1) = 0$ for all $y > 1$. The last statement is equivalent to the fact that $g''(x) < 0$ for all $x > 0$, and this proves that f is strictly concave on $[0, \infty)$. Therefore, the Hölder mean function of two positive, distinct numbers is strictly logarithmically concave downward on $[0, \infty)$. $\square$

A graphical illustration of the logarithmic concavity of the Hölder means of two positive numbers 1 and a, for various values of a, is presented in Figure 1.

We make now the following simple observation.

Observation 2.2. The Hölder mean of two positive numbers is logarithmically symmetric about the geometric mean of the two numbers. That means, if a and b are positive numbers, then for all $x \in [-\infty, \infty]$, we have

$$H_x(a, b)H_{-x}(a, b) = H_0^2(a, b). \tag{2-14}$$

Proof. Indeed, if $x = \infty$, then

$$\begin{aligned} H_\infty(a, b)H_{-\infty}(a, b) &= \max\{a, b\} \min\{a, b\} \\ &= ab = H_0^2(a, b). \end{aligned}$$

On the other hand, for all $x \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned} H_x(a,b)H_{-x}(a,b) &= \left(\frac{a^x+b^x}{2}\right)^{1/x}\left(\frac{a^{-x}+b^{-x}}{2}\right)^{-1/x} \\ &= \frac{(a^x+b^x)^{1/x}}{2^{1/x}}\frac{(2a^xb^x)^{1/x}}{(a^x+b^x)^{1/x}} = ab = H_0^2(a,b). \end{aligned} \qquad \square$$

Corollary 2.3. *Since for any two positive numbers a and b, the function $x \mapsto \ln(H_x(a,b))$ is concave downward on $[0,\infty)$, and its graph is symmetric about the point $(0, \ln(\sqrt{ab}))$, this function is concave upward on $(-\infty, 0]$.*

3. Sharp inequalities concerning α-cevians in a triangle

In this section we use the logarithmic concavity property of the Hölder mean, of two positive numbers, as a function of the index, to prove a sharp inequality for the length of an α-cevian in a triangle.

We give first some definitions.

Definition 3.1. Given a triangle ABC in the plane, for any point M on the side BC, we call AM a *cevian*.

If $M \in BC$, meaning M is between B and C, then we say that AM is an *interior* cevian.

We say that sides AB and AC of the triangle ABC are *adjacent* to the cevian AM.

Definition 3.2. Given a triangle ABC in the plane and α a real number, if $M_\alpha \in BC$, then we say that AM_α is an *α-interior cevian* if

$$\frac{\overline{BM_\alpha}}{\overline{CM_\alpha}} = \left(\frac{\overline{AB}}{\overline{AC}}\right)^\alpha. \tag{3-1}$$

Here $\overline{PQ}$ denotes the length of the segment PQ for any two points P and Q in the plane. See Figure 2.

Observation 3.3. For any real number α, the three α-interior cevians AM_α, BN_α, and CP_α of a triangle ABC are concurrent.

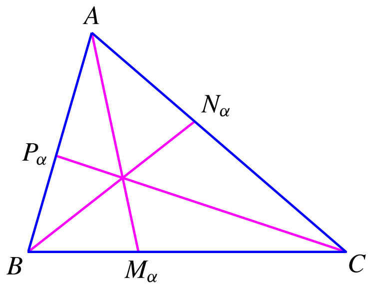

Figure 2. A triangle and its three α-cevians.

Proof. Indeed, we have (see Figure 2)

$$\frac{\overline{BM_\alpha}}{\overline{CM_\alpha}} \cdot \frac{\overline{CN_\alpha}}{\overline{AN_\alpha}} \cdot \frac{\overline{AP_\alpha}}{\overline{BP_\alpha}} = \frac{\overline{AB}^\alpha}{\overline{AC}^\alpha} \cdot \frac{\overline{BC}^\alpha}{\overline{BA}^\alpha} \cdot \frac{\overline{CA}^\alpha}{\overline{CB}^\alpha} = 1.$$

It follows now from Ceva's theorem that AM_α, BN_α, and CP_α are concurrent. □

Observation 3.4. We make the following observations:

- For $\alpha = 0$, AM_0, BN_0, and CP_0 are the medians of the triangle ABC and they are concurrent in the *centroid* of the triangle ABC. The centroid of a triangle is denoted by $X(2)$ in [Kimberling 1994].

- For $\alpha = 1$, AM_1, BN_1, and CP_1 are the inner bisectors of the angles of the triangle ABC and they are concurrent in the *incenter* of the triangle ABC. The incenter of a triangle is denoted by $X(1)$ in [Kimberling 1994].

- For $\alpha = 2$, AM_2, BN_2, and CP_2 are the symmedians (symmetric to the medians about the corresponding bisectors) of the triangle ABC and they are concurrent in the *Lemoine point*, also called the *Grebe point*, of the triangle ABC. The Lemoine (Grebe) point of a triangle is denoted by $X(6)$ in [Kimberling 1994].

Let us observe that if AM is an interior cevian of a triangle ABC, then at least one of the angles $\sphericalangle AMB$ and $\sphericalangle AMC$ is obtuse or right. If the angle $\sphericalangle AMB$ is obtuse or right, then in the triangle AMB, the side AB opposite to this angle, with say $\overline{AB} = c$, is the largest side of the triangle. Thus, we have $\overline{AM} < c$.

Similarly, if the angle $\sphericalangle AMC$ is obtuse or right, then $\overline{AM} < b$.

Therefore, in both cases we conclude that

$$\overline{AM} < \max\{b, c\} = H_\infty(b, c).$$

Starting from this simple inequality, we can ask the question:

Question 3.5. *Given a real number α, what is the smallest number $p = p(\alpha) \in [-\infty, \infty]$ such that for all triangles ABC, if AM_α is an α-interior cevian, we have*

$$\overline{AM_\alpha} \le H_p(\overline{AB}, \overline{AC})? \tag{3-2}$$

We have the following proposition:

Proposition 3.6. *Let b and c be two fixed positive numbers. We denote by $\mathcal{T}_{b,c}$ the set of all triangles ABC in the plane such that $\overline{AB} = c$ and $\overline{AC} = b$. Then, we have*

$$\sup_{ABC \in \mathcal{T}_{b,c}} \{\overline{AM_\alpha} \mid AM_\alpha \textit{is an } \alpha\textit{-interior cevian in } ABC\} = bc\,\frac{b^{\alpha-1} + c^{\alpha-1}}{b^\alpha + c^\alpha}. \tag{3-3}$$

Proof. We give a vectorial proof.

In triangle ABM_α we have

$$\overrightarrow{AM_\alpha} = \overrightarrow{AB} + \overrightarrow{BM_\alpha}. \tag{3-4}$$

In triangle ACM_α we have

$$\overrightarrow{AM_\alpha} = \overrightarrow{AC} + \overrightarrow{CM_\alpha}. \tag{3-5}$$

Let us first multiply both sides of (3-4) by b^α, and both sides of (3-5) by c^α, and then add the two resulting equations. We obtain

$$(b^\alpha + c^\alpha)\overrightarrow{AM_\alpha} = b^\alpha \overrightarrow{AB} + c^\alpha \overrightarrow{AC} + (b^\alpha \overrightarrow{BM_\alpha} + c^\alpha \overrightarrow{CM_\alpha}). \tag{3-6}$$

Since AM_α is an α-interior cevian, we have

$$\frac{\overline{BM}_\alpha}{\overline{CM}_\alpha} = \frac{c^\alpha}{b^\alpha}.$$

This is equivalent to

$$b^\alpha \overrightarrow{BM_\alpha} + c^\alpha \overrightarrow{CM_\alpha} = 0. \tag{3-7}$$

It follows now from (3-6) that

$$\overrightarrow{AM_\alpha} = \frac{1}{b^\alpha + c^\alpha}(b^\alpha \overrightarrow{AB} + c^\alpha \overrightarrow{AC}). \tag{3-8}$$

Applying the triangle inequality in (3-8), we conclude that

$$\begin{aligned} \overline{AM}_\alpha &\le \frac{1}{b^\alpha + c^\alpha}(b^\alpha \overline{AB} + c^\alpha \overline{AC}) \\ &= \frac{1}{b^\alpha + c^\alpha}(b^\alpha c + c^\alpha b) = bc\,\frac{b^{\alpha-1} + c^{\alpha-1}}{b^\alpha + c^\alpha}. \end{aligned} \tag{3-9}$$

Since this happens for all triangles ABC such that $\overline{AB} = c$ and $\overline{AC} = b$, we conclude that

$$\mathcal{S} \le bc\,\frac{b^{\alpha-1} + c^{\alpha-1}}{b^\alpha + c^\alpha}, \tag{3-10}$$

where

$$\mathcal{S} = \sup_{ABC \in \mathcal{T}_{b,c}} \{\overline{AM}_\alpha \mid AM_\alpha \text{ is an } \alpha\text{-interior cevian in } ABC\}.$$

On the other hand, we have

$$\begin{aligned} \mathcal{S} &\ge \lim_{m(\sphericalangle BAC)\to 0^+} \overline{AM}_\alpha \\ &= \lim_{m(\sphericalangle BAC)\to 0^+} \left[\frac{1}{b^\alpha + c^\alpha}\left|b^\alpha \overrightarrow{AB} + c^\alpha \overrightarrow{AC}\right|\right] \\ &= \left[\frac{1}{b^\alpha + c^\alpha}(b^\alpha \overline{AB} + c^\alpha \overline{AC})\right] = bc\,\frac{b^{\alpha-1} + c^{\alpha-1}}{b^\alpha + c^\alpha}, \end{aligned} \tag{3-11}$$

where $|\vec{v}|$ denotes the length of the vector $\vec{v}$ for any vector $\vec{v}$ in $\mathbb{R}^2$.

The result of our proposition follows now from inequalities (3-10) and (3-11). $\square$

We can write

$$bc\,\frac{b^{\alpha-1}+c^{\alpha-1}}{b^{\alpha}+c^{\alpha}} = bc\,\frac{(b^{\alpha-1}+c^{\alpha-1})/2}{(b^{\alpha}+c^{\alpha})/2} = H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_{\alpha}^{\alpha}(b,c)}. \tag{3-12}$$

Thus, we obtain

$$\mathcal{S} = H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_{\alpha}^{\alpha}(b,c)}. \tag{3-13}$$

Now, Question 3.5 becomes:

Question 3.7. *Given a real number α, what is the smallest number $p = p(\alpha) \in [-\infty, \infty]$ such that for all b and c positive, we have*

$$H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_{\alpha}^{\alpha}(b,c)} \le H_p(b,c)? \tag{3-14}$$

Before answering this question, we present the following necessary condition for an inequality between two functions, whose graphs touch at one point, to hold.

Lemma 3.8. *Let $I \subseteq R$ be an interval, and let*

$$\mathring{I} := \{x \in I \mid \text{there exists } r > 0 \text{ such that } (x-r, x+r) \subset I\}$$

be the set of the interior points of I. Suppose f and g are two real-valued functions such that:

(1) *$f(x) \le g(x)$ for all $x \in I$.*

(2) *f and g are continuous on I.*

(3) *f and g are twice-differentiable on $\mathring{I}$.*

(4) *There exists $x_0 \in \mathring{I}$ such that $f(x_0) = g(x_0)$.*

(5) *f'' is continuous at x_0.*

Then, we must have $f'(x_0) = g'(x_0)$ and $f''(x_0) \le g''(x_0)$.

Proof. Let $h(x) := g(x) - f(x)$. Then, for all $x \in I$, we have

$$h(x) \ge 0 = h(x_0).$$

Thus, h has an absolute minimum value at x_0, and since x_0 is a point in the interior of I, Fermat's theorem implies $h'(x_0) = 0$. This is equivalent to $f'(x_0) = g'(x_0)$.

Since $x_0 \in \mathring{I}$, there exists $r > 0$ such that $(x_0 - r, x_0 + r) \subset I$. Because the function f is dominated by function g, for all $0 < h < r$, we have

$$\begin{aligned} f(x_0+h) &\le g(x_0+h),\\ f(x_0-h) &\le g(x_0-h),\\ -2f(x_0) &= -2g(x_0). \end{aligned}$$

Adding these three relations and dividing both sides by the positive number h^2, we obtain

$$\frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2} \le \frac{g(x_0+h)+g(x_0-h)-2g(x_0)}{h^2}.$$

Passing to the limit as $h \to 0^+$, we obtain

$$\lim_{h\to 0^+} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2} \le \lim_{h\to 0^+} \frac{g(x_0+h)+g(x_0-h)-2g(x_0)}{h^2}. \tag{3-15}$$

Applying L'Hôpital's rule in the $\frac{0}{0}$ case, twice, or using Taylor's formula with Lagrange's remainder, it is not hard to see that due to the continuity of f'' at x_0, the last inequality becomes

$$f''(x_0) \le g''(x_0). \qquad \square$$

To answer Question 3.7, we will analyze four cases.

Case 1. If $\alpha \ge 1$, then the answer of Question 3.7 is given by the following proposition.

Proposition 3.9. *If $\alpha \ge 1$, then the smallest number $p = p(\alpha) \in [-\infty, \infty]$ such that for all positive numbers b and c, we have*

$$H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} \le H_p(b,c) \tag{3-16}$$

is

$$p(\alpha) = 1-2\alpha. \tag{3-17}$$

Proof. Step 1: We prove first the inequality $p(\alpha) \le 1-2\alpha$.

Indeed, using Observation 2.2, we have

$$\begin{aligned} H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} &= (H_{1-2\alpha}(b,c)H_{2\alpha-1}(b,c))\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} \\ &= H_{1-2\alpha}(b,c)\left[\frac{H_{\alpha-1}^{(\alpha-1)/\alpha}(b,c)H_{2\alpha-1}^{1/\alpha}(b,c)}{H_\alpha(b,c)}\right]^\alpha \\ &\le H_{1-2\alpha}(b,c)\cdot 1^\alpha = H_{1-2\alpha}(b,c), \end{aligned} \tag{3-18}$$

since $0 \le \alpha-1 < \alpha \le 2\alpha-1$ (due to the fact that $\alpha \ge 1$),

$$\frac{\alpha-1}{\alpha}(\alpha-1) + \frac{1}{\alpha}(2\alpha-1) = \alpha, \tag{3-19}$$

and so, because $x \mapsto H_x(b,c)$ is logarithmically concave on $[0,\infty)$, we have

$$H_{\alpha-1}^{(\alpha-1)/\alpha}(b,c)H_{2\alpha-1}^{1/\alpha}(b,c) \le H_\alpha(b,c). \tag{3-20}$$

Step 2: We prove now that if p is a positive number such that for all positive numbers b and c, we have

$$H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} \le H_p(b,c),$$

then $p \ge 1-2\alpha$.

Choosing $b=1$ and $c=x$, where x is an arbitrary positive number, the above inequality becomes

$$x\,\frac{1+x^{\alpha-1}}{1+x^\alpha} \le \left(\frac{1+x^p}{2}\right)^{1/p}. \tag{3-21}$$

We can see now that the hypotheses of Lemma 3.8 are satisfied for the functions

$$f(x) := \frac{x+x^\alpha}{1+x^\alpha} = 1+\frac{x-1}{1+x^\alpha} = 1+\frac{1}{2}(x-1)+(x-1)\left(\frac{1}{1+x^\alpha}-\frac{1}{2}\right) \tag{3-22}$$

and

$$g(x) := \left(\frac{1+x^p}{2}\right)^{1/p}, \tag{3-23}$$

and the point

$$x_0 := 1. \tag{3-24}$$

Thus, we obtain

$$f''(1) \le g''(1). \tag{3-25}$$

Using Leibniz's rule of differentiation and keeping only the nonzero terms, we obtain

$$\begin{aligned} f''(1) &= \frac{d^2}{dx^2}\left[1+\frac{1}{2}(x-1)+(x-1)\left(\frac{1}{1+x^\alpha}-\frac{1}{2}\right)\right]\Bigg|_{x=1} \\ &= \frac{d^2}{dx^2}\left[(x-1)\left(\frac{1}{1+x^\alpha}-\frac{1}{2}\right)\right]\Bigg|_{x=1} \\ &= \binom{2}{1}\frac{d}{dx}(x-1)\bigg|_{x=1}\frac{d}{dx}\left(\frac{1}{1+x^\alpha}-\frac{1}{2}\right)\bigg|_{x=1} \\ &= 2\left(\frac{-\alpha x^{\alpha-1}}{(1+x^\alpha)^2}\right)\bigg|_{x=1} = -\frac{\alpha}{2}. \end{aligned} \tag{3-26}$$

On the other hand, we have

$$\begin{aligned} g'(x) &= \frac{1}{2^{1/p}}\frac{d}{dx}[(1+x^p)^{1/p}] \\ &= \frac{1}{2^{1/p}}\frac{1}{p}(1+x^p)^{(1/p)-1}\,px^{p-1} \\ &= \frac{1}{2^{1/p}}\left(\frac{1+x^p}{x^p}\right)^{(1-p)/p} = \frac{1}{2^{1/p}}(x^{-p}+1)^{(1-p)/p}. \end{aligned}$$

Thus, we obtain

$$g''(x) = \frac{1}{2^{1/p}} \frac{1-p}{p} (x^{-p}+1)^{(1-2p)/p} (-p) x^{-p-1}$$
$$= \frac{p-1}{2^{1/p}} (x^{-p}+1)^{(1-2p)/p} x^{-p-1}.$$

Hence, we have

$$g''(1) = \frac{p-1}{4}. \tag{3-27}$$

Therefore, inequality (3-25) becomes

$$-\frac{\alpha}{2} \le \frac{p-1}{4}. \tag{3-28}$$

This inequality is equivalent to

$$p \ge 1 - 2\alpha, \tag{3-29}$$

and so, our proof is complete. □

Case 2. If $\frac{1}{2} < \alpha < 1$, then the answer to Question 3.7 is given by the following proposition.

Proposition 3.10. *If $\frac{1}{2} < \alpha < 1$, then the smallest number $p = p(\alpha) \in [-\infty, \infty]$ such that for all positive numbers b and c, we have*

$$H_0^2(b,c) \frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} \le H_p(b,c)$$

is

$$p(\alpha) = 0. \tag{3-30}$$

Proof. Step 1: We prove first the inequality $p(\alpha) \le 0$. That means, we show that for all positive numbers b and c we have

$$H_0^2(b,c) \frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} \le H_0(b,c).$$

Indeed, using the symmetry of the function $x \mapsto \ln(H_x(b,c))$ with respect to the origin

$$H_x(b,c) H_{-x}(b,c) = H_0^2(b,c), \tag{3-31}$$

for $x = \alpha - 1$, we obtain

$$H_{\alpha-1}(b,c) = \frac{H_0^2(b,c)}{H_{1-\alpha}(b,c)}. \tag{3-32}$$

Thus, we have

$$\begin{aligned}
H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} &= H_0^2(b,c)\left[\frac{H_0^2(b,c)}{H_{1-\alpha}(b,c)}\right]^{\alpha-1}\frac{1}{H_\alpha^\alpha(b,c)}\\
&= \frac{H_0^{2\alpha}(b,c)H_{1-\alpha}^{1-\alpha}(b,c)}{H_\alpha^\alpha(b,c)}\\
&= H_0(b,c)\left[\frac{H_0(b,c)}{H_\alpha(b,c)}\right]^{2\alpha-1}\left[\frac{H_{1-\alpha}(b,c)}{H_\alpha(b,c)}\right]^{1-\alpha}\\
&\le H_0(b,c)\cdot 1^{2\alpha-1}\cdot 1^{1-\alpha} = H_0(b,c), \qquad (3\text{-}33)
\end{aligned}$$

since $0<\alpha$, $1-\alpha<\alpha$, the function $x\mapsto H_x(b,c)$ is increasing, $2\alpha-1>0$, and $1-\alpha>0$.

Step 2: We show now that if $p<0$, then the inequality

$$H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} \le H_p(b,c)$$

cannot hold for all positive numbers b and c.

Indeed, if we assume by contradiction that it holds for all positive numbers b and c, then choosing $b=1$ and $c=x$, where x is an arbitrary positive number, we obtain

$$x\,\frac{1+x^{\alpha-1}}{1+x^\alpha} \le \left(\frac{1+x^p}{2}\right)^{1/p}. \qquad (3\text{-}34)$$

Passing to the limit as $x\to\infty$, we get

$$\lim_{x\to\infty}\frac{x+x^\alpha}{1+x^\alpha} \le \lim_{x\to\infty}\left(\frac{1+x^p}{2}\right)^{1/p}. \qquad (3\text{-}35)$$

Since $\alpha<1$ and $p<0$, the last inequality becomes

$$\infty \le \left(\tfrac{1}{2}\right)^{1/p},$$

which is a contradiction.

Thus the smallest number p for which inequality (3-14) holds is $p(\alpha)=0$. ☐

Case 3. If $0\le\alpha\le\frac{1}{2}$, then the answer to Question 3.7 is given by the following proposition.

Proposition 3.11. *If* $0\le\alpha\le 1$, *then the smallest number* $p=p(\alpha)\in[-\infty,\infty]$ *such that for all positive numbers b and c we have*

$$H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} \le H_p(b,c)$$

is

$$p(\alpha)=1-2\alpha. \qquad (3\text{-}36)$$

Proof. Step 1: We show first that $p(\alpha) \le 1-2\alpha$. Using the logarithmic symmetry of the function $x \mapsto H_x(b,c)$, we have

$$H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} = H_0^2(b,c)\left[\frac{H_0^2(b,c)}{H_{1-\alpha}(b,c)}\right]^{\alpha-1}\frac{1}{H_\alpha^\alpha(b,c)} = \frac{H_0^{2\alpha}(b,c)H_{1-\alpha}^{1-\alpha}(b,c)}{H_\alpha^\alpha(b,c)}. \tag{3-37}$$

Since $0 \le \alpha \le \frac{1}{2}$, we have $0 \le \alpha \le 1-\alpha$, and α can be written as a convex combination of 0 and $1-\alpha$ in the following way:

$$\alpha = \left(1 - \frac{\alpha}{1-\alpha}\right)\cdot 0 + \frac{\alpha}{1-\alpha}\cdot(1-\alpha). \tag{3-38}$$

Since $x \mapsto H_x(b,c)$ is logarithmically concave on $[0,\infty)$, applying Jensen's inequality, we obtain

$$H_\alpha \ge H_0^{1-\alpha/(1-\alpha)} H_{1-\alpha}^{\alpha/(1-\alpha)}. \tag{3-39}$$

Thus, using (3-37) and (3-39), we have

$$\begin{aligned} H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} &= \frac{H_0^{2\alpha}(b,c)H_{1-\alpha}^{1-\alpha}(b,c)}{H_\alpha^\alpha(b,c)} \\ &\le \frac{H_0^{2\alpha}(b,c)H_{1-\alpha}^{1-\alpha}(b,c)}{[H_0^{1-\alpha/(1-\alpha)}(b,c)H_{1-\alpha}^{\alpha/(1-\alpha)}(b,c)]^\alpha} \\ &= H_0^{\alpha/(1-\alpha)}(b,c)H_{1-\alpha}^{(1-2\alpha)/(1-\alpha)}(b,c). \end{aligned} \tag{3-40}$$

Let us observe that $\alpha/(1-\alpha) \in [0,1]$, $(1-2\alpha)/(1-\alpha) \in [0,1]$, and

$$\frac{\alpha}{1-\alpha} + \frac{1-2\alpha}{1-\alpha} = 1. \tag{3-41}$$

Applying, Jensen's inequality again, we obtain

$$\begin{aligned} H_0^2(b,c)\,\frac{H_{\alpha-1}^{\alpha-1}(b,c)}{H_\alpha^\alpha(b,c)} &\le H_0^{\alpha/(1-\alpha)}(b,c)H_{1-\alpha}^{(1-2\alpha)/(1-\alpha)}(b,c) \\ &\le H_{[\alpha/(1-\alpha)]\cdot 0+[(1-2\alpha)/(1-\alpha)]\cdot(1-\alpha)}(b,c) \\ &= H_{1-2\alpha}(b,c). \end{aligned} \tag{3-42}$$

Step 2: We can prove now in exactly the same way as in the proof of Proposition 3.9 that if p is real number such that inequality (3-14) holds for all positive numbers b and c, then

$$p \ge 1-2\alpha. \qquad \square$$

Case 4. If $\alpha < 0$, then the answer to Question 3.7 is given by the following proposition.

Proposition 3.12. *If $\alpha < 0$, then the smallest (only) number $p = p(\alpha) \in [-\infty, \infty]$ such that for all positive numbers b and c, we have*

$$H_0^2(b, c)\,\frac{H_{\alpha-1}^{\alpha-1}(b, c)}{H_\alpha^\alpha(b, c)} \le H_p(b, c)$$

is

$$p(\alpha) = \infty. \tag{3-43}$$

Proof. Indeed, we saw geometrically at the beginning of the paper that for all triangles ABC, and all interior cevians AM, we have

$$\overline{AM} \le \max\{\overline{AB}, \overline{AC}\} = H_\infty(b, c),$$

where $b := \overline{AC}$ and $c := \overline{AB}$.

To show that $p(\alpha) = \infty$, we must prove that for all $p < \infty$, inequality (3-14) cannot hold for all positive numbers b and c.

Supposing that for some $p < \infty$ (we may assume $p > 0$) inequality (3-14) holds for all positive numbers b and c, we can choose $b = 1$ and $c = x$, where x is an arbitrary positive number. That means, for all $x > 0$, we have

$$\frac{x + x^\alpha}{1 + x^\alpha} \le \left(\frac{1 + x^p}{2}\right)^{1/p}.$$

Passing to the limit in this inequality as $x \to 0^+$, we obtain

$$\lim_{x\to 0^+} \frac{x + x^\alpha}{1 + x^\alpha} \le \lim_{x\to 0^+} \left(\frac{1 + x^p}{2}\right)^{1/p}.$$

Since $\alpha < 0$, the last inequality is equivalent to

$$1 \le \left(\tfrac{1}{2}\right)^{1/p}.$$

This inequality is impossible, since $0 < \frac{1}{2} < 1$ and $\frac{1}{p} > 0$. □

Therefore, the function $\alpha \mapsto p(\alpha)$ that gives the smallest p such that in any triangle ABC the α-interior cevian starting from A, AM_α, has a length less than or equal to the p-Hölder mean of $\overline{AB}$ and $\overline{AC}$ is $P : \mathbb{R} \to [-\infty, \infty]$, defined by

$$P(\alpha) = \begin{cases} \infty & \text{if } \alpha < 0, \\ 1 - 2\alpha & \text{if } 0 \le \alpha \le \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < \alpha < 1, \\ 1 - 2\alpha & \text{if } \alpha > 1. \end{cases} \tag{3-44}$$

See the graph of P in Figure 3.

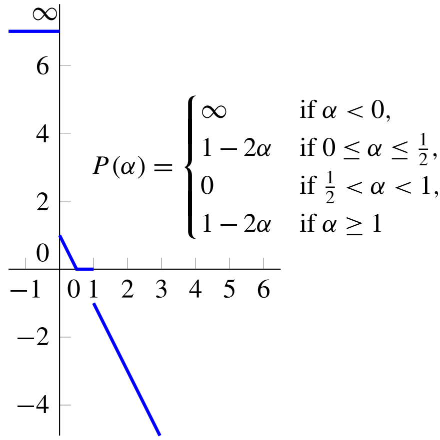

Figure 3. The graph of function $y = P(\alpha)$.

We observe that the function P is nonincreasing and lower semicontinuous.

The branching point $\alpha = 0$ of the piecewise-defined function P corresponds to the median AM_0 of the triangle ABC.

The branching point $\alpha = 1$ corresponds to the bisector AM_1 of the angle $\sphericalangle BAC$.

The branching point $\alpha = \frac{1}{2}$ corresponds to a cevian $AM_{1/2}$ that is concurrent with the corresponding cevians $BN_{1/2}$ and $CP_{1/2}$ in the point $X(366)$ from [Kimberling 1994]. The point $X(366)$ is the isogonal conjugate of $X(365)$, the square root point, which is the intersection point of the three $\frac{3}{2}$-interior cevians of the triangle ABC.

We summarize below our results, in the case of some classic cevians:

Proposition 3.13. *Let ABC be a triangle with sides, starting from A, of lengths $\overline{AC} = b$ and $\overline{AB} = c$. Let M be a point on the side BC of this triangle. Then*:

(1) *If AM is the median corresponding to the vertex A, then its length satisfies*

$$\overline{AM} < \frac{b+c}{2}. \tag{3-45}$$

Moreover, for every $p < 1$, there exists a triangle ABC (depending on p) such that

$$\overline{AM} > \left(\frac{b^p + c^p}{2}\right)^{1/p}. \tag{3-46}$$

(2) *If AM is the interior bisector of the angle $\sphericalangle(BAC)$, then its length satisfies*

$$\overline{AM} < \frac{2}{\frac{1}{a} + \frac{1}{b}}. \tag{3-47}$$

Moreover, for every $p < -1$, there exists a triangle ABC such that

$$\overline{AM} > \left(\frac{b^p + c^p}{2}\right)^{1/p}. \tag{3-48}$$

(3) *If AM is the symmedian corresponding to the vertex A, then its length satisfies*

$$\overline{AM} < \left(\frac{b^{-3}+c^{-3}}{2}\right)^{-1/3}. \tag{3-49}$$

Moreover, for every $p < -3$, there exists a triangle ABC such that

$$\overline{AM} > \left(\frac{b^{p}+c^{p}}{2}\right)^{1/p}. \tag{3-50}$$

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