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Applying prospect theory to multiattribute problems with independence assumptions

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We discuss a descriptive theory of decision making which has received much attention in recent decades: prospect theory. We specifically focus on applying the theory to problems with two attributes, assisted by different independence assumptions. We discuss a process for solving decision problems using the theory before applying it to a real life example of purchasing breakdown cover.

1. Introduction

In this paper, we apply prospect theory (PT) to multiattribute problems, specifically those with two attributes. We will consider levels of independence between the attributes in the problem and will split the corresponding value function into different parts. When discussing independence between attributes in multiattribute expected utility theory (EUT), Keeney and Raiffa [1976] use the term "utility independence". However, in this paper, we use the term "independence" to represent utility or value independence, dependent on whether we are in the EUT or PT case. This can be seen later in Definition 1.

Within PT, the reference point is chosen to be the point from which you consider gains and losses. As such, the different parts of the value function are all based on whether outcomes are better or worse than the reference point with respect to each attribute. Following this, we will design a process which can be used to effectively and efficiently solve a multiattribute decision problem. We show how this can be applied to a real-life problem of purchasing breakdown cover.

We will begin by covering some background information in Section 2, introducing notation and explaining how EUT deals with multiattribute problems. We also introduce PT and how it is applied to single attribute problems. In Section 3, we derive formulas for applying PT to multiattribute problems under different levels of independence. Following this, in Section 4, we introduce a standard process which

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can be used to solve multiattribute decision problems. We illustrate the approach with an example of purchasing breakdown cover in Section 5.

2. Background

We begin by introducing some definitions and notation. The following notation is based on [Starmer 2000, p. 334]. We define a prospect to consist of a set of outcomes (e.g., $x_1, x_2, \dots, x_n$) with probabilities corresponding to them (e.g., $p_1, p_2, \dots, p_n$). At this point, it should be made clear that prospects will be present for both EUT and PT. A prospect simply represents what it is that we are considering and should not be thought of as being linked exclusively to PT.

Notationally, we will consider prospects using capital letters (e.g., A, B, C) and will consider probabilities using p with subscripts (e.g., p_1, p_2, p_3). Therefore, an example of a prospect is A = ($x_1, p_1; x_2, p_2; \dots; x_n, p_n$). Here, ($p_1, p_2, \dots, p_n$) is a probability distribution (hence $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$) and ($x_1, x_2, \dots, x_n$) are the associated outcomes. So, with prospect A as described above, we would expect outcome x_i with probability p_i for $i = 1, \dots, n$. Interestingly, we can consider some of the outcomes within a prospect to be prospects themselves. So, for example, we could have prospects B, C and D with D = (B, 0.25; C, 0.75). This would yield prospect B with probability 0.25 and prospect C with probability 0.75.

When comparing prospects A and B:

- A $\prec$ B denotes B is preferred to A.
- A $\preccurlyeq$ B denotes B is at least as preferable as A.
- A $\sim$ B denotes indifference between A and B.

We now discuss prospect theory (PT) in the single-attribute case, which we later extend to be applicable to multiattribute decision problems. PT was introduced by Daniel Kahneman and Amos Tversky [1979]. It was designed to be a direct alternative to EUT, dealing with some of the issues where EUT fails to reflect the preferences of the majority of individuals. In reflecting these attitudes, it is clear that PT was designed as a descriptive theory.

PT is made up of two stages: an editing phase and an evaluation phase. The editing phase of PT is designed to simplify the decision-making process. Using several different rules and operations, the individual can change the prospects which they have to choose between and eliminate any that should never be picked. After the editing phase has been completed, the decision maker will then complete an evaluation phase in which they will decide what the best decision will be for them, based on their opinions and beliefs. Within the evaluation phase, there are two main functions which are considered and combined to help evaluate each prospect numerically: a decision-weighting function, represented by $\pi(p)$, and a value function, represented by $v(x)$.

The reference point in PT is of great importance as it allows us to choose a neutral point and then consider better points as gains and worse points as losses. This is because it was found in [Kahneman and Tversky 1979, pp. 268–269] that individuals are risk-averse when it comes to gains, whereas they are risk-seeking when dealing with losses. To reflect this, the value function is generally designed to be concave above the reference point and convex below the reference point.

Now that we have introduced PT, we will discuss the independence assumptions which we will later use in deriving the formulas for multiattribute PT. Note here that we are going to denote the values the attributes can take by Y and Z. In this paper, we will be focussing on the instances where we have certain levels of independence between the attributes Y and Z. This independence is used in multiattribute EUT for utility functions and later we will use it for multiattribute PT when we work with value functions. We define it as follows:

Definition 1 (adapted from [Keeney and Raiffa 1976, pp. 226, 229] to be applicable to PT).

- Y is independent of Z when conditional preferences for outcomes on Y given z do not depend on the particular level of z.
- Y and Z are mutually independent attributes if Y is independent of Z and Z is independent of Y.

We will now briefly discuss multiattribute EUT and explain how with certain levels of independence, it is possible to construct a utility function which helps to choose the best possible alternative for a decision problem. This focusses on [Keeney and Raiffa 1976].

To begin with, we discuss how Keeney and Raiffa show that independence between attributes can be represented effectively with an equation. This can be found in [loc. cit., p. 144] where it is suggested that two strategically equivalent utility functions can be linked by

$$u_1(x) = h + k u_2(x) \quad \text{for all } x. \tag{1}$$

Now, let us consider the instance where Y is independent of Z. By the definition of independence, this means that $u(y, z_0)$ and $u(y, z_1)$ are strategically equivalent (i.e., $u(y, z_0) \sim u(y, z_1)$) for any $z_0, z_1 \in \mathrm{Z}$. Due to this, we can represent the utility function of any $y \in \mathrm{Y}$ and any $z \in \mathrm{Z}$ as a transformation of another utility function of $y \in \mathrm{Y}$ and a different $z' \in \mathrm{Z}$. Namely

$$u(y, z) = g(z) + h(z)u(y, z') \quad \text{for any } y \in \mathrm{Y} \text{ and } z \in \mathrm{Z}, \tag{2}$$

with the functions g and h only functions of z (i.e., constant in y). Following this, we can say that an attribute Y is independent of Z if and only if (2) holds. This will later be applied to PT and form the basis of the derivation of the formulas for multiattribute PT.

If we have two attributes which are mutually independent, we can use Theorem 5.2 from [loc. cit., p. 234] to evaluate the utility of any alternative. Similarly, if the attributes only have Z independent of Y, we use Theorem 5.6 from [loc. cit., p. 244]. This briefly describes how EUT can be applied to multiattribute problems to assist in choosing the best alternative in a decision problem. We will now show how PT can be applied in a similar way.

3. Applying PT to multiattribute problems

As PT is the natural alternative to EUT and EUT has been applied to multiattribute problems, it seems logical to apply PT to multiattribute problems. One method which uses PT is the TODIM method [Gomes and Lima 1991]. This uses pairwise comparisons over each alternative based on each attribute, choosing one attribute to be the reference attribute. Other than that, a lot of the research in multiattribute problems using prospect theory involves combining the theory itself with the theory of fuzzy sets. The theory of fuzzy sets was introduced in [Zadeh 1979] and applied to multiattribute problems by [Bellman and Zadeh 1970]. Combining prospect theory with the theory of fuzzy sets will not be covered in this paper, research into this area can be found in [Krohling and de Souza 2012; Liu et al. 2011; Wang and Sun 2008].

There is limited research looking into applying PT in the same way as EUT is applied to multiattribute decision problems in [Keeney and Raiffa 1976]. The closest thing currently available is [Hu and Zhou 2009], which is described in [Liu et al. 2011] as being a "multiple criteria decision-making method for the risk decision-making problem based on prospect theory". However, it still does not produce a piecewise form of the value function and instead focusses on using the weighting function from cumulative PT, introduced in [Tversky and Kahneman 1992]. Further research which uses PT methods with multiattribute problems can be found in [Egozcue et al. 2014].

In introducing the theory, we begin by considering what the reference point will be for each attribute Y and Z. Denote these as (y_0, z_0) and let $v(y_0, z_0) = 0$. In this paper, we keep the reference point constant and as such, we do not include it in the notation. It should be emphasised here how important the choice of reference point is. Different choices of reference point have a significant impact on the outcome of a decision problem. This can be seen later in Section 5.

We begin by defining strategic equivalence of two value functions (similarly to [Keeney and Raiffa 1976, p. 144]) as follows:

Definition 2. Two value functions, v and v^*, are *strategically equivalent*, denoted as $v \sim v^*$, if and only if they imply the same preference ranking for any two prospects or outcomes.

Suppose we have two value functions, v and v^*, which are strategically equivalent. We assume that there are constants h_1, k_1, h_2, k_2 with $k_1 > 0$ and $k_2 > 0$ such that

$$v(x) = \begin{cases} h_1 + k_1 v^*(x) & \text{if } x \succcurlyeq x_0, \\ h_2 + k_2 v^*(x) & \text{if } x \preccurlyeq x_0. \end{cases} \tag{3}$$

This is a natural assumption for the value functions v and v^*, because multiplication by a positive constant will not affect the preference ordering over outcomes and neither will a transformation by an additive constant.

Similarly to the multiattribute EUT case, we can represent attribute Y being independent of Z with the equation

$$v(y, z) = \begin{cases} c_1^+(z) + c_2^+(z) v(y, z_0) & \text{if } y \succcurlyeq y_0, \\ c_1^-(z) + c_2^-(z) v(y, z_0) & \text{if } y \preccurlyeq y_0, \end{cases} \tag{4}$$

and attribute Z being independent of Y with

$$v(y, z) = \begin{cases} d_1^+(y) + d_2^+(y) v(y_0, z) & \text{if } z \succcurlyeq z_0, \\ d_1^-(y) + d_2^-(y) v(y_0, z) & \text{if } z \preccurlyeq z_0. \end{cases} \tag{5}$$

Notice here the similarities with (3). For example, $c_1^+(z)$ and $d_1^+(y)$ are similar to h_1 in (3). Note that $c_1^+(z)$ is a function of z meaning it is constant in y, as is required for the value function which is only a function of y. The same logic applies for the rest of the c and d values in (4) and (5) respectively.

3.1. ***Working with mutually independent attributes.*** To begin with, we are going to consider the case where we have attributes Y and Z which are mutually independent. Assuming this, we are going to loosely follow the proof for the multiattribute EUT formula in [Keeney and Raiffa 1976, p. 234–235] but apply it to PT.

Y and Z being mutually independent means we can represent the independence using (4) and (5). Substituting y_0 into (4) gives that $c_1^+(z) = c_1^-(z) = v(y_0, z)$. This is to retain a level of continuity and to ensure that the limit as y tends to y_0 from above or below is the same. Then, consider a value y_1 such that $y_1 \succ y_0$ and substitute this value into (4) to get

$$v(y_1, z) = v(y_0, z) + c_2^+(z) v(y_1, z_0) \quad \Longrightarrow \quad c_2^+(z) = \frac{v(y_1, z) - v(y_0, z)}{v(y_1, z_0)}. \tag{6}$$

Similarly, considering a value y_{-1} with $y_{-1} \prec y_0$ and substituting this into (4) gives

$$v(y_{-1}, z) = v(y_0, z) + c_2^-(z) v(y_{-1}, z_0) \quad \Longrightarrow \quad c_2^-(z) = \frac{v(y_{-1}, z) - v(y_0, z)}{v(y_{-1}, z_0)}. \tag{7}$$

Notice that both constants $c_2^+(z)$ and $c_2^-(z)$ are positive as is required in (3). The function $c_2^-(z)$ is positive as both the numerator and denominator are negative as $v(y_0, z_0) = 0$, so $v(y_{-1}, z_0) < 0$.

Now that we have evaluated the values of the constants $c_1^+(z)$, $c_1^-(z)$, $c_2^+(z)$ and $c_2^-(z)$, we can substitute these back into (4) to get

$$v(y,z)=\begin{cases} v(y_0,z)+\Big(\dfrac{v(y_1,z)-v(y_0,z)}{v(y_1,z_0)}\Big)v(y,z_0) & \text{if } y\succcurlyeq y_0,\\ v(y_0,z)+\Big(\dfrac{v(y_{-1},z)-v(y_0,z)}{v(y_{-1},z_0)}\Big)v(y,z_0) & \text{if } y\preccurlyeq y_0.\end{cases} \tag{8}$$

Similar logic can be used to rewrite (5) as

$$v(y,z)=\begin{cases} v(y,z_0)+\Big(\dfrac{v(y,z_1)-v(y,z_0)}{v(y_0,z_1)}\Big)v(y_0,z) & \text{if } z\succcurlyeq z_0,\\ v(y,z_0)+\Big(\dfrac{v(y,z_{-1})-v(y,z_0)}{v(y_0,z_{-1})}\Big)v(y_0,z) & \text{if } z\preccurlyeq z_0.\end{cases} \tag{9}$$

Evaluating (9) at y_1 for any point $z\in \mathrm{Z}$ gives

$$v(y_1,z)=\begin{cases} v(y_1,z_0)+\Big(\dfrac{v(y_1,z_1)-v(y_1,z_0)}{v(y_0,z_1)}\Big)v(y_0,z) & \text{if } z\succcurlyeq z_0,\\ v(y_1,z_0)+\Big(\dfrac{v(y_1,z_{-1})-v(y_1,z_0)}{v(y_0,z_{-1})}\Big)v(y_0,z) & \text{if } z\preccurlyeq z_0.\end{cases} \tag{10}$$

Similarly, evaluating (9) at y_{-1} for any point $z\in \mathrm{Z}$ gives

$$v(y_1,z)=\begin{cases} v(y_{-1},z_0)+\Big(\dfrac{v(y_{-1},z_1)-v(y_{-1},z_0)}{v(y_0,z_1)}\Big)v(y_0,z) & \text{if } z\succcurlyeq z_0,\\ v(y_{-1},z_0)+\Big(\dfrac{v(y_{-1},z_{-1})-v(y_{-1},z_0)}{v(y_0,z_{-1})}\Big)v(y_0,z) & \text{if } z\preccurlyeq z_0.\end{cases} \tag{11}$$

Substituting (10) and (11) into (8) and simplifying leads to the following theorem for calculating the value function for a multiattribute prospect theory problem:

Theorem 3. *For attributes* Y *and* Z *which are mutually independent, the value function required for multiattribute prospect theory for any point* (y,z) *with* $y\in \mathrm{Y}$ *and* $z\in \mathrm{Z}$ *can be evaluated as*

$$\begin{aligned} v(y,z)&=v(y_0,z)+v(y,z_0)\\ &\quad+v(y_0,z)v(y,z_0)\\ &\quad\times\begin{cases} \Big(\dfrac{v(y_1,z_1)-v(y_1,z_0)-v(y_0,z_1)}{v(y_0,z_1)v(y_1,z_0)}\Big) & \text{if } y\succcurlyeq y_0,\ z\succcurlyeq z_0,\\ \Big(\dfrac{v(y_1,z_{-1})-v(y_1,z_0)-v(y_0,z_{-1})}{v(y_0,z_{-1})v(y_1,z_0)}\Big) & \text{if } y\succcurlyeq y_0,\ z\preccurlyeq z_0,\\ \Big(\dfrac{v(y_{-1},z_1)-v(y_{-1},z_0)-v(y_0,z_1)}{v(y_0,z_1)v(y_{-1},z_0)}\Big) & \text{if } y\preccurlyeq y_0,\ z\succcurlyeq z_0,\\ \Big(\dfrac{v(y_{-1},z_{-1})-v(y_{-1},z_0)-v(y_0,z_{-1})}{v(y_0,z_{-1})v(y_{-1},z_0)}\Big) & \text{if } y\preccurlyeq y_0,\ z\preccurlyeq z_0.\end{cases} \end{aligned} \tag{12}$$

So, provided we have mutual independence, this theorem allows us to assign a value to any alternative (y, z) for any $y \in \mathrm{Y}$ and $z \in \mathrm{Z}$. The values in the piecewise function which are the coefficients of the $v(y_0, z)v(y, z_0)$ term are similar to the constant k_{YZ} in the EUT case [Keeney and Raiffa 1976, p. 234, Theorem 5.2]. In fact, if you were to choose the reference point as the worst possible values in Y and Z, you would find that you are only in the top part of the piecewise function of Theorem 3. This would give no significant differences between the utility function in EUT and the value function in PT as all outcomes would be considered as gains. As such, everyone would display risk-aversion to all options, as is the case in EUT. This shows how the significant difference in the theories results from the use and choice of a reference point.

In deriving this formula, we assigned the reference point to be (y_0, z_0). However, suppose we decide to change the reference point to another point (y, z) for any $y \in \mathrm{Y}$ and $z \in \mathrm{Z}$. This would then potentially need different points y_1, y_{-1}, z_1, z_{-1} to be chosen, which changes the constant term. So, an individual who has the same preference ordering for each attribute could change their decision using this formula based on the reference point that they choose. This shows how important the reference point is.

3.2. ***One independent attribute.*** Let us consider the case where we only have one attribute being independent of the other. Without loss of generality, we assume that attribute Z is independent of Y, meaning we can write the value function of y and z (with reference point (y_0, z_0)) in the form of (5). From here, we will use similar steps as in the proof of Theorem 5.6 in [Keeney and Raiffa 1976, pp. 244–245].

We begin with (5) and choose z_1 and z_{-1} with $z_1 \succ z_0 \succ z_{-1}$. They are chosen to satisfy the following equations which fix the origin and unit of measure of $v(y_0, z)$:

$$v(y_0, z_0) = 0, \tag{13}$$

$$v(y_0, z_1) = 1, \tag{14}$$

$$v(y_0, z_{-1}) = -1. \tag{15}$$

We can now evaluate (5) at the point $z = z_0$ for any value of $y \in \mathrm{Y}$ which leads to (using (13))

$$d_1^+(y) = d_1^-(y) = v(y, z_0). \tag{16}$$

This can now be combined with (5) to give

$$v(y, z) = \begin{cases} v(y, z_0) + d_2^+(y)v(y_0, z) & \text{if } z \succcurlyeq z_0, \\ v(y, z_0) + d_2^-(y)v(y_0, z) & \text{if } z \preccurlyeq z_0. \end{cases} \tag{17}$$

Now evaluate (17) at the points $z = z_1$ and $z = z_{-1}$. Using (14) and (15), we get

$$d_2^+(y) = v(y, z_1) - v(y, z_0), \tag{18}$$

$$d_2^-(y) = v(y, z_0) - v(y, z_{-1}). \tag{19}$$

Notice here that we have $d_2^+(y) > 0$ and $d_2^-(y) > 0$, as is required in the assumption in (3). Substituting these into (17) and rearranging leads to the following theorem.

Theorem 4. *For attributes* Y *and* Z *with* Z *independent of* Y *but without* Y *being independent of* Z*, the value function required for multiattribute prospect theory for any point* (y, z) *with* $y \in \mathrm{Y}$ *and* $z \in \mathrm{Z}$ *can be evaluated as*

$$v(y, z) = \begin{cases} v(y, z_0)[1 - v(y_0, z)] + v(y, z_1)v(y_0, z) & \text{if } z \succcurlyeq z_0, \\ v(y, z_0)[1 + v(y_0, z)] - v(y, z_{-1})v(y_0, z) & \text{if } z \preccurlyeq z_0. \end{cases} \tag{20}$$

If an individual is going to use Theorem 4 to help solve a multiattribute problem, they are required to specify a few values initially to fix the unit of measure for each of the value functions. This is something which is also done in the EUT case with one independent attribute [Keeney and Raiffa 1976, p. 244, Theorem 5.6]. We have already fixed the unit of measure for $v(y_0, z)$ for any $z \in \mathrm{Z}$. However, we still need to specify a unit of measure for the other value functions in formula (20), namely $v(y, z_0)$, $v(y, z_1)$ and $v(y, z_{-1})$.

At this point, we have already specified one value for each of these functions (each of them evaluated at y_0). Another two points are required for each value function to specify the unit of measure. This is because $v(y, z)$ is, by definition, different for gains and losses. Therefore, we must fix the unit of measure for $y \succ y_0$ and also for $y \prec y_0$. We can do this in a similar way to the EUT case.

For fixing the unit of measure of $v(y, z_0)$, we find points $y_2 \succ y_0 \succ y_{-2} \in \mathrm{Y}$ and $z_2 \succ z_0 \succ z_{-2} \in \mathrm{Z}$ such that the individual is indifferent between (y_0, z_2) and (y_2, z_0) and between (y_0, z_{-2}) and (y_{-2}, z_0). The indifference means that $v(y_0, z_2) = v(y_2, z_0)$ and $v(y_0, z_{-2}) = v(y_{-2}, z_0)$. Then, whether dealing with gains or losses in the attribute Y, we will have fixed the unit of measure for $v(y, z_0)$. The same can then be done for $v(y, z_1)$ and $v(y, z_{-1})$ to fix their unit of measure. Assuming that such points exist is a trivial assumption to make as if this is not the case, you are in a much simpler situation and do not require the theories introduced in this paper.

Clearly, if attribute Y is independent of Z instead of Z being independent of Y, (20) would have y and z swapped, with points y_1, y_{-1} chosen instead of z_1, z_{-1}. Therefore, Theorem 4 allows us to evaluate a value for all alternatives y, z with a weaker requirement of only one attribute being independent of the other. However, in exchange for this, there are more values that an individual will have to specify.

3.3. ***Calculating the coefficients for multiattribute PT.*** In this section, we are going to focus on methods that can be used to evaluate the coefficients of the value functions in (12). Within this, let us suppose that the individual has already chosen the forms that their value functions will take. This means they will have already specified $v(y, z_0)$ for all $y \in \mathrm{Y}$ and $v(y_0, z)$ for all $z \in \mathrm{Z}$. This means we have four points which remain to be specified: $v(y_1, z_1)$, $v(y_1, z_{-1})$, $v(y_{-1}, z_1)$ and $v(y_{-1}, z_{-1})$ for appropriately chosen y_1, y_{-1}, z_1, z_{-1}. But how do we make these choices of points and resulting values?

Choosing what points to use as y_1, y_{-1}, z_1 and z_{-1} is a free choice for the decision maker, provided that they satisfy $y_1 \succ y_0 \succ y_{-1}$ and $z_1 \succ z_0 \succ z_{-1}$. This is because the important consideration here is the values associated with them, not the actual points. The values then form the coefficient of the value functions in the piecewise part of (12).

Let us initially focus on working out $v(y_1, z_1)$. A simple way to do this is to attempt to find equivalences. For example, the decision maker should consider what value of $y \in \mathrm{Y}$ means that they are indifferent between (y_1, z_1) and (y, z_0). As we have already specified the value function $v(y, z_0)$, we now have a value for $v(y_1, z_1)$. The same can be done for fixing Y at y_0 and considering what value of $z \in \mathrm{Z}$ leads to indifference between (y_0, z) and (y_1, z_1). A sensible check which the decision maker can complete is to do both and ensure that the values are the same (or at least very similar). This simple method allows the decision maker to accurately fix the coefficients of their value function in (12) and ensures that the values chosen are in line with their beliefs.

If the value functions $v(y, z_0)$ and $v(y_0, z)$ have not already been specified, this leads to a slightly more complicated situation. In this case, equivalence relations would not be very useful and as such, it is likely that the decision maker will simply have to choose these values. However, logic checks can be used to ensure that the values chosen are appropriate. For example, $v(y_1, z_1)$ should be greater than $v(y_1, z_0)$ provided that $z_1 \succ z_0$. So, checking that more preferred values are given a higher value is a simple logic check. Furthermore, the decision maker could use the equivalence relations stated earlier once the value functions have been fully specified. This will then ensure that the values that they have chosen are appropriate for their beliefs.

4. Process of solving a multiattribute decision problem

We now outline a "standard" process which can be used to help an individual solve a decision problem with two attributes. The aim of this is to make it quicker and easier for the individual to solve their decision problem and to ensure the best outcome based on their beliefs. We suggest that the process of solving a multiattribute decision problem can be broken down into a few stages. They are

- formulating the problem,
- independence and choosing a theory,
- applying the theory of choice.

We will now discuss the first two stages in the following two subsections. We will not discuss applying the theory of choice as this is a very simple process once the previous two stages have been completed. However, when applying the theory of choice, logic checks should be carried out to ensure that the individual is assigning utilities/values in line with their underlying preferences and beliefs.

4.1. ***Formulating the problem.*** Formulating a multiattribute decision problem involves several stages which all need to be completed fully and carefully. In formulating the problem in this way, it will save time later in evaluating alternatives and will ensure the individual fully understands the decision problem they are faced with. At this point, we are going to assume that we know the individual who is faced with the decision problem.

First, we need to establish what the decision problem actually is. What are the alternatives that the individual is aiming to choose between? For example, the alternatives could be different treatments which a doctor is choosing between to give to a patient.

Following this, we need to be clear as to what is required as the outcome from the decision problem. This could involve choosing a single best alternative, choosing an acceptable region of alternatives or simply giving a ranking for all of the possible alternatives. This helps the individual understand the problem they are faced with and ensures they are getting the output that they desire.

It should also be specified to the decision maker whether there are any probabilities involved in the decision problem. With probabilities involved, this would then be an extra consideration for the decision maker to have when deciding between EUT and PT. This is because they will have to consider the probability distribution itself and consider whether the weighting function in PT better reflects how they would view the probabilities.

We now need to understand what the attributes are that we are using to compare the alternatives. In this paper, we have focussed on the case where there are two attributes. It is important at this point to understand what these attributes actually mean and the possible values they could take in the decision problem. Knowing the domain of values for each attribute allows us to accurately scale the utility/value function based around the best and worst possible outcomes, according to the individual. Without this, mistakes could be made in that the individual may expect a much higher value for an attribute than is actually possible. Therefore, it is vital that the individual understands the possible values each attribute can take.

The individual should also be clear as to whether the attributes are continuous or discrete. For example, a monetary attribute is continuous to two decimal places. So, if the decision problem has two of the possible outcomes as £10.00 and £15.00, we understand the values that would go between them. As such, it is conceivable to have £12.50 as the reference point if we were using PT. However, if an attribute is considered to be discrete, this would not be the case. For example, suppose the attribute is different types of fruit and that two of the possible outcomes are apple and orange. Clearly, it is not feasible to choose a reference point between these two outcomes.

It is also useful to understand a rough preference ordering for the attributes from the decision maker. For some attributes, this is simpler than others. For example, a monetary attribute is one where it is easy to suggest that an individual will simply aim to minimise expenditure or maximise income. However, other attributes such as colours of paint are much more subjective.

It could be suggested that a rough preference ordering is not needed at this stage. However, we believe there to be a couple of reasons why it would be useful to consider this now. First, it gives us a basis on which to ensure the individual is acting logically. Secondly, it is useful at this stage to allow for cancellation. Once the preference ordering has been considered, it may be that some alternatives are better than others for both attributes. If this is the case (and the decision problem is simply choosing a single best decision), then the alternative which is worse for both attributes could be ignored. So, if we preferred blue paint to red paint and blue was cheaper than red, then the red paint would be dominated and removed from the decision problem.

We have now completed the process of formulating the problem. This was done with the aim of simplifying the process later and assisting the decision maker in understanding the problem with which they are faced.

4.2. ***Independence and choosing a theory.*** Now that we have formulated the decision problem, we have to consider whether either of the attributes are independent of each other. This can be tested in the following way. Consider two particular values of Y, say $y_1, y_2 \in \text{Y}$, that you can specify a preference and strength of preference between. Now consider any value $z_1 \in \text{Z}$. What is your preference and level of preference between (y_1, z_1) and (y_2, z_1)? Now suppose that we choose a different $z_2 \in \text{Z}$ with $z_2 \neq z_1$. What is your preference and level of preference between (y_1, z_2) and (y_2, z_2)? If the preference ordering and strength of preference remains the same between the two options, irrespective of the value of Z, then we can say that Y is independent of Z. As you would expect, to show that Z is independent of Y uses a similar logic, but with the y and z values switched in the above.

However, can you be 100% sure of independence without testing the indifference for all of the possible values of Y or Z? There is no guarantee that the indifference will necessarily hold for all Y and Z. Unfortunately, attempting to test this for all possible values of Y and Z would take a significant amount of time and potentially be impossible. As such, it is easier to either test for a couple of potential Y, Z combinations or to simply make an assumption.

It is worth considering at this point how individuals actually face multiattribute decision problems — do most consider attributes as independent? Clearly the test above can be used to see whether independence is a reasonable assumption. The

theory	level of independence	number of choices
EUT	mutual independence	$2n+2$
	single independence	$3n+1$
PT	mutual independence	$2n+8$
	single independence	$3n+7$

Table 1. Comparison of the effect of independence on the number of choices for both methods.

advantage of it is that it creates a simpler model from which to evaluate the possible alternatives. Without this assumption, fitting a model to evaluate how good (or bad) each alternative is becomes difficult. One possible alternative is to assume independence and perform logic checks on the outcome to ensure rational decisions have been made.

Suppose that we are in a situation where we assume some level of independence between the attributes. A key consideration when choosing what level of independence to assume is the number of choices that will be required. Clearly, this will be different for EUT and PT and for different levels of independence. Let us consider the general case of having attributes Y and Z, with n different alternatives to choose between. We begin by considering the case of mutual independence in PT and following Theorem 3 from Section 3.1.

To begin with, we make a choice of what the individual's reference points $y_0 \in \mathrm{Y}$ and $z_0 \in \mathrm{Z}$ are. Once the individual has made these two choices, we have to choose $y_1, y_{-1} \in \mathrm{Y},\ z_1, z_{-1} \in \mathrm{Z}$ that satisfy $y_1 \succ y_0 \succ y_{-1}$ and $z_1 \succ z_0 \succ z_{-1}$. Following this, the individual will make eight choices to evaluate $v(y_i, z_j)$ for $i, j = -1, 0, 1$ using the methods in Section 3.3 (note, we already have $v(y_0, z_0) = 0$). This fixes the constant values in the piecewise function of Theorem 3.

The individual will now need to specify conditional value functions $v(y_0, z)$ for all $z \in \mathrm{Z}$ and, similarly, specify $v(y, z_0)$ for all $y \in \mathrm{Y}$. This could be considered to be four choices of value functions for each domain of Y and Z. However, they would then need to be evaluated for all appropriate $y \in \mathrm{Y}$, $z \in \mathrm{Z}$ that we have not already specified. As such, we suggest that evaluating $v(y, z_0)$ for all $y \in \mathrm{Y}$ will be $n-3$ choices and similarly for $v(y_0, z)$ for all $z \in \mathrm{Z}$.

The individual is now ready to evaluate $v(y, z)$ for any $y \in \mathrm{Y}$ and any $z \in \mathrm{Z}$ and choose the pairing (y, z) with the highest value. To get to this point, the individual has made $2+4+8+2(n-3) = 2n+8$ choices. Applying similar logic to the other three cases gives us the number of choices for each case; see Table 1.

Notice here that when n is not significantly large, there is not a huge difference between the different levels of independence. However, if n gets large, this is where having mutual independence would save significantly more time. We can also notice here that the number of choices is unlikely to have an impact when choosing between EUT and PT. These extra choices for PT can be considered to be a trade-off for the potentially more realistic modelling that PT provides.

When choosing between EUT and PT, it is worth considering whether there are probabilities involved in the decision problem. With probabilities involved, the decision can become more complicated. For example, if the probability distribution has some events occurring with certainty or with very low probabilities, the weighting function in PT treats these differently [Kahneman and Tversky 1979, pp. 280–284], making the decision more complicated. However, if the probabilities are away from 0 or 1, there is no significant difference between the probabilities and the weighting function and as such, the choice remains the same as without probabilities.

For now, we are going to consider that we do not have probabilities involved and compare the theories at a base level. The main difference comes in that PT has a reference point from which to consider gains and losses, whereas EUT does not. As such, if the alternatives contain gains and losses, PT is likely to be more useful as it was designed to reflect how individuals deal with gains and losses better than EUT. On the other hand, EUT is designed as a normative theory and as such, you would expect that the decision made will be logical and rational. If we use PT, the decision can be affected by how most people act.

Once the individual has decided the level of independence and which of the theories they prefer, we are ready to apply the theory of choice and get the outcome of the decision problem.

5. Solving multiattribute problems: breakdown cover

5.1. ***Introducing the example: purchasing breakdown cover.*** Let us consider a real-life application where an individual is aiming to purchase breakdown cover for their car. At this point, it should be emphasised that the policies that we will be comparing are for vehicle cover for a general vehicle, with the prices a basic quote. If an individual were to actually purchase breakdown cover, they would be required to publish details of the vehicle that they have. Furthermore, we are only going to consider vehicle breakdown cover as quoted by the AA[1] and the RAC[2] on the 26th February 2018.

When deciding upon which cover to purchase, there are two things the individual must consider: the cost and the extent of the cover. This is therefore an example of a problem with two attributes to consider, as we require. To simplify notation, we are going to assign the cost per month attribute to be represented by Y and the level of cover attribute to be represented by Z.

[1] https://www.theaa.com/breakdown-cover/

[2] https://www.rac.co.uk/breakdown-cover/

policy number	level of cover	cost per month	notation
P0	no cover	£0	(£0, 0)
P1	RA	£5.50	(−£5.50, 1)
P2	RA and KR	£7.50	(−£7.50, 2)
P3	RA and AH	£8.00	(−£8.00, 3)
P4	RA, AH and NR	£10.00	(−£10.00, 4)
P5	RA, NR and OT	£12.00	(−£12.00, 5)
P6	RA, AH and KR	£12.00	(−£12.00, 6)
P7	RA, NR and KR	£12.00	(−£12.00, 7)
P8	RA, AH, NR and OT	£12.50	(−£12.50, 8)
P9	RA, AH, NR and KR	£13.50	(−£13.50, 9)
P10	RA, AH, NR, OT and KR	£14.50	(−£14.50, 10)

Table 2. Summary of coverage costs for roadside assistance (RA), key replace (KR), at home (AH), national recovery (NR), and onward travel (OT).

Clearly, the attribute for cost is one which is going to be fairly simple to evaluate as it is a quantitative one and utilities/values can be based off the cost. However, the extent of the cover needs further specification. Using the AA and the RAC as the potential providers, the possible policies we are going to compare and evaluate can be seen in Table 2. To simplify the notation, we have used a numbering system to represent each level of cover. For policy PI where $\mathrm{I} \in \{0, 1, 2, \ldots, 10\}$, we represent the corresponding level of cover for that policy with the number I. Note that the cost per month is a negative value as it is an amount of money that the individual will spend on the cover. This is all summarised in Table 2.

Full details of what is included in each level of cover can be found in footnotes 1 and 2. We are going to assume that individuals prefer to minimise expenditure (and hence maximise the attribute of money). Furthermore, we will assume individuals prefer having a higher level of cover. However, differences in preferences between things such as key replace and at home cover will need to be specified by the individual in the stating of their utilities/values.

A significant difference between attributes Y and Z comes in that Y is a continuous attribute (to 2 decimal places), whereas Z is a discrete attribute. This has a significant impact on the choice of the reference point for PT, as was discussed in Section 4.1.

5.2. ***Formulating the problem.*** We are now going to apply the process discussed in Section 4 to the example of purchasing breakdown cover for a vehicle. In the introduction to the example, we have completed many of the tasks required in the formulating of the problem. We (the authors) will be the ones who will be faced with the decision problem and hence the outcome will be based on our preferences and beliefs. We have specified that we are aiming to choose a single best policy from the options P0–P10, have identified that the attributes are cost per month (Y) and level of cover (Z), and identified the relevant values they can take.

The only part of formulating the problem which we have not completed in the introduction is specifying a rough preference ordering between the values that the attributes can take. For attribute Y which represents cost per month, it is trivial to suggest that the decision maker will aim to reduce the expenditure and hence maximise the money they have. However, for attribute Z which represents level of cover, it is slightly more difficult. While we can easily say that the decision maker would prefer more cover to less cover, it is hard to distinguish between some of the options which have similar levels of cover but with slightly different things involved.

Let us suppose we have our rough preference ordering as follows:

$$0 \prec 1 \prec 2 \prec 3 \prec 6 \prec 7 \prec 4 \prec 5 \prec 9 \prec 8 \prec 10.$$

This is built on the rough idea that we prefer onward travel (OT) to national recovery (NR) to at home (AH) cover to key replace (KR). Now that we have specified this rough preference ordering, we can simplify the problem significantly. For example, we can remove P9 as it is dominated by P8 due to having a lower cost per month and a more preferable level of cover. Similarly, P6 and P7 are dominated by P4. As such, we can remove policies P6, P7 and P9 from the problem and we are left with eight different policies to choose between. We have now fully formulated the problem and can move on to the next stage.

5.3. ***Independence and choosing a theory.*** Now that we have formulated the problem, we need to consider whether we have any independence between the attributes Y and Z. At this point, we can notice we have eight alternatives and, as such, the difference in the number of choices between the different methods is not significant. Therefore, we should base the choice of independence on the tests discussed at the start of Section 4.2. For this particular example, we are going to cover both levels of independence assumption and compare the outcomes.

We are now going to move on to applying the theory. Usually at this stage, we would make a decision on whether we want to use EUT or PT. However, we are going to apply both theories for each case of independence and compare the different outcomes.

5.4. ***Applying the theories.*** In this section we will be applying the multiattribute PT formulas to the example. Within this, we will analyse the impact that each choice we make will have and will compare the outcomes to those when using multiattribute EUT. Applying multiattribute EUT to the example is presented in Appendix A.

5.4.1. *Mutually independent attributes.* We will begin by considering the attributes as mutually independent and look to apply Theorem 3 and PT to the example.

The instance of mutually independent attributes and applying EUT can be seen in Appendix A.1. An interesting consideration is how the choice of reference point impacts the decision that is made. This is something we will explore within this section. With the eight policies we are considering, we know that we will have 24 $(= 2n + 8)$ choices that we need to make.

We consider three different choices for the reference points. They are as follows:

Choice 1: $y_0 = -£10.00, \quad z_0 = 4.$

Choice 2: $y_0 = -£12.00, \quad z_0 = 3.$

Choice 3: $y_0 = -£9.00, \quad z_0 = 5.$

At this point, we can decide on what points we want to choose as y_1, y_{-1}, z_1, z_{-1} with $y_1 \succ y_0 \succ y_{-1}$ and $z_1 \succ z_0 \succ z_{-1}$. We will choose these points now and use the same points for each of the choices of reference points. We will choose $y_1 = -£7.50$, $y_{-1} = -£12.50$, $z_1 = 8$ and $z_{-1} = 2$. As in the EUT case, these choices are arbitrary.

However, unlike the EUT case, we choose what values we give to them rather than in the EUT case where it had utility 1. The values that are chosen have an impact on the coefficient of the $v(y_0, z)v(y, z_0)$ term (which could be considered as an interaction term) and so have a similar impact to the impact that the constants chosen in the EUT case have. These will therefore need to be specified for each choice of reference point. The values chosen can be seen in Appendix B.1.

Now that we have decided upon these values, we are ready to state the conditional value functions for each attribute Y and Z. In other words, we are now going to decide the values $v(y, z_0)$ for all $y \in$ Y and $v(y_0, z)$ for all $z \in$ Z. The choices made were based on the previously stated preference ordering and can be found in Appendix B.1.

After these values have been stated, we can now evaluate the values of each of the policies using (12) from Theorem 3. This then gives us the following:

$v(\text{policy}) = v(y, z)$	choice 1	choice 2	choice 3
P0 = (£0, 0)	0.645	−3.501	−0.663
P1 = (−£5.50, 1)	0.401	−0.973	−0.370
P2 = (−£7.50, 2)	0.200	.750	−0.350
P3 = (−£8.00, 3)	0.252	**1.350**	−0.276
P4 = (−£10.00, 4)	0.000	1.080	−0.632
P5 = (−£12.00, 5)	−0.038	0.800	−0.750
P8 = (−£12.50, 8)	0.400	0.700	−0.150
P10 = (−£14.50, 10)	**0.972**	−0.238	**1.210**

We have highlighted the best policy under each choice in bold. We can see that with choices 1 and 3 for the reference point, we choose policy P10 to be the best option. However, when we have choice 2 for the reference point, we choose policy P3. In fact, for choice 2, policy P10 was one of the worst options. This is clearly significantly different and shows how changing the reference point can impact what decision is made regarding an "acceptable" level of cover for the appropriate cost.

5.4.2. *One independent attribute.* We are now going to focus on the instance where we have independence in one direction. In this paper, we will focus on the case where Z is independent of Y. This means we will be applying Theorem 4 from Section 3.2 to the breakdown cover example. As such, it will require 31 ($= 3n + 7$) choices to be made by the decision maker, more than has been required in any other circumstance. We begin by deciding on the reference point (y_0, z_0). When making this decision in the mutual independence case, we considered several different choices of reference point and considered the impact they would have on the final decision. Therefore, we are once again going to consider the same three pairs of reference points and compare the outcomes.

Following the choice of reference point, we must choose $z_1, z_{-1} \in \mathrm{Z}$ such that $z_1 \succ z_0 \succ z_{-1}$. These choices need to be carefully made as we need z_1 such that $v(y_0, z_1) = 1$ and z_{-1} such that $v(y_0, z_{-1}) = -1$. Let our choices be the following:

Choice 1: $y_0 = -£10.00, \quad z_0 = 4, \quad z_1 = 8, \quad z_{-1} = 2.$

Choice 2: $y_0 = -£12.00, \quad z_0 = 3, \quad z_1 = 5, \quad z_{-1} = 1.$

Choice 3: $y_0 = -£9.00, \quad z_0 = 5, \quad z_1 = 8, \quad z_{-1} = 3.$

Once these choices have been made, we need to fix the units of measure of $v(y, z_0)$, $v(y, z_1)$ and $v(y, z_{-1})$. As with the other theories, this is done by finding equivalences. So, for example, to fix the unit of measure for $v(y, z_0)$, we need to choose $y_2, y_{-2} \in \mathrm{Y}$ and $z_2, z_{-2} \in \mathrm{Z}$ such that $(y_0, z_2) \sim (y_2, z_0)$ and $(y_0, z_{-2}) \sim (y_{-2}, z_0)$. Similarly, we need $y_3, y_{-3}, y_4, y_{-4} \in \mathrm{Y}$ and $z_3, z_{-3}, z_4, z_{-4} \in \mathrm{Z}$ such that we have $(y_3, z_1) \sim (y_0, z_3)$, $(y_{-3}, z_1) \sim (y_0, z_{-3})$, $(y_4, z_{-1}) \sim (y_0, z_4)$ and $(y_{-4}, z_{-1}) \sim (y_0, z_{-4})$. The choices we will make for each choice of reference point are:

Choice 1:
$$z_2 = z_3 = z_4 = 5, \quad z_{-2} = z_{-3} = z_{-4} = 3,$$
$$y_2 = -£8.00, \quad y_3 = -£12.00, \quad y_4 = -£6.00,$$
$$y_{-2} = -£12.50, \quad y_{-3} = -£14.50, \quad y_{-4} = -£9.00.$$

Choice 2:
$$z_2 = z_3 = z_4 = 4, \quad z_{-2} = z_{-3} = z_{-4} = 2,$$
$$y_2 = -£10.00, \quad y_3 = -£13.50, \quad y_4 = -£8.00,$$
$$y_{-2} = -£13.50, \quad y_{-3} = -£16.50, \quad y_{-4} = -£10.00.$$

Choice 3: $z_2 = z_3 = z_4 = 10, \quad z_{-2} = z_{-3} = z_{-4} = 4,$

$$y_2 = -£7.50, \quad y_3 = -£8.00, \quad y_4 = -£2.50$$
$$y_{-2} = -£11.00, \quad y_{-3} = -£13.00, \quad y_{-4} = -£8.00.$$

Notice here that similar values of attribute Z are chosen to formulate equivalences with. This is because Z is a discrete attribute with only eight conceivable outcomes so equivalences would be difficult to work with if changing Z at all times. Furthermore, as Y is continuous to two decimal places, we can fix Z at certain levels and change the attribute Y to allow for the required equivalence relations.

Now that these choices have been made, we can state the conditional value functions $v(y_0, z)$ for all $z \in Z$ and $v(y, z_{-1})$, $v(y, z_0)$, $v(y, z_1)$ for all $y \in Y$. These can be found in Appendix B.2.

Now that all these have been specified, we are in a position to be able to evaluate the value of each of the policies. The results are as follows:

v(policy) $= v(y, z)$	choice 1	choice 2	choice 3
P0 = (£0, 0)	**0.735**	1.100	**0.715**
P1 = (−£5.50, 1)	0.413	0.900	0.100
P2 = (−£7.50, 2)	−0.100	0.685	−0.725
P3 = (−£8.00, 3)	−0.020	0.750	−0.450
P4 = (−£10.00, 4)	0.000	0.780	−0.695
P5 = (−£12.00, 5)	−0.140	1.000	−0.600
P8 = (−£12.50, 8)	0.200	**1.110**	−0.100
P10 = (−£14.50, 10)	−0.375	0.475	−0.380

As we can see, in this situation, the best policy is P0 for choices 1 and 3 and P8 for choice 2. It is interesting to see here that the best policy for choices 1 and 3 is to have no breakdown cover at all. Furthermore, for choice 2, policy P0 is very close to being chosen as the best option. This clearly suggests that for this context, there is an underlying attitude that losing money is worse than having a lower level of cover.

While choosing cheaper policies has an obvious financial benefit in the short term, it could lead to financial problems in the long term if the vehicle encountered problems. This is something we could potentially account for if we included probabilities and the corresponding financial outlay if problems occurred. However, it would reduce it to a single attribute problem in which the costs and corresponding probabilities are difficult to construct.

6. Concluding remarks

In this paper, we have applied PT to multiattribute problems with different independence assumptions. We have also introduced a process which can be followed to assist in effectively and efficiently solving a multiattribute decision problem. Using these two developments allows an individual to apply PT to a decision problem with two attributes. This is especially useful for problems where there are gains and losses to consider with respect to certain attributes.

An example of where this can be useful is seen in Section 5 where we apply the results of this paper to a real-life example of purchasing breakdown cover. If you already had a certain level of breakdown cover, it is easy to see how you could consider the different costs and levels of cover as gains and losses. This shows how multiattribute PT can be useful in solving real-life problems. However, there is a lot more further research which can be done into applying PT to multiattribute problems.

First, extending PT so that it can be applied to problems with more than two attributes is interesting. This may require different methods to be used, although independence assumptions could still be useful. Secondly, research into the reference point in PT will be interesting. Can the reference point be adapted so that you can choose a set of values as a reference point? This would allow the individual to select an acceptable region as their reference point and then consider gains and losses from that set. Finally, this paper focusses on the instance where we have at least a certain level of independence assumption between the attributes. Could PT be adapted so that we do not require any independence between the attributes?

Appendix A: Breakdown cover EUT calculations

A.1. ***Mutually independent attributes — multiattribute EUT.*** Using multiattribute EUT for this context of mutually independent attributes, we will be applying Theorem 5.2 from [Keeney and Raiffa 1976, p. 234]. For this instance, we choose $y_0 = -£14.50$, $y_1 = -£12.00$, $y_* = 0$, $z_0 = 0$, $z_1 = 4$ and $z_* = 10$.

We are now required to choose the values for the constants k_Y and k_Z which subsequently decide k_{YZ} and k. These choices have a significant impact on the overall utilities we assign to each of the policies. As such, we are going to consider three possible pairs of values and see how changing these values affects the overall utilities and decisions. The pairs of values we are going to consider are as follows:

Choice 1: $k_Y = 0.2, \quad k_Z = 0.35.$

Choice 2: $k_Y = k_Z = 0.5.$

Choice 3: $k_Y = 0.6, \quad k_Z = 0.8.$

So, with choice 1, we have that $k_{YZ} = 0.45$ and $k = \frac{45}{7} > 0$, which implies that the attributes Y and Z are complimentary. Choice 2 would give the formulas from additive independence and would also imply there is no interaction between the attributes. Choice 3 would make $k_{\mathrm{YZ}} = -0.4$ and $k = -\frac{5}{6} < 0$, which implies the attributes Y and Z are substitutes.

After specifying the values for the constants, we have now fully defined what we need for the theorem and hence can begin to evaluate the utilities for all of the policies. To do this, we are going to consider the conditional utility functions for each of the attributes Y and Z and evaluate a utility for each alternative. We choose the utility values as follows:

$$\begin{aligned} &u_{\mathrm{Y}}(-£14.50) = 0, && u_{\mathrm{Y}}(-£8.00) = 1.75,\\ &u_{\mathrm{Y}}(-£12.50) = 0.8, && u_{\mathrm{Y}}(-£7.50) = 1.9,\\ &u_{\mathrm{Y}}(-£12.00) = 1, && u_{\mathrm{Y}}(-£5.50) = 2.2,\\ &u_{\mathrm{Y}}(-£10.00) = 1.35, && u_{\mathrm{Y}}(£0) = 2.9,\end{aligned}$$

$$\begin{aligned} &u_{\mathrm{Z}}(0) = 0, && u_{\mathrm{Z}}(4) = 1,\\ &u_{\mathrm{Z}}(1) = 0.25, && u_{\mathrm{Z}}(5) = 1.25,\\ &u_{\mathrm{Z}}(2) = 0.4, && u_{\mathrm{Z}}(8) = 1.6,\\ &u_{\mathrm{Z}}(3) = 0.65, && u_{\mathrm{Z}}(10) = 2.2.\end{aligned}$$

The result of choosing these utility values and applying Theorem 5.2 is the following:

$u(\text{policy}) = u(y, z)$	choice 1	choice 2	choice 3
P0 = (£0, 0)	0.580	**1.450**	1.740
P1 = (−£5.50, 1)	0.775	1.225	1.300
P2 = (−£7.50, 2)	0.862	1.150	1.156
P3 = (−£8.00, 3)	1.089	1.200	1.115
P4 = (−£10.00, 4)	1.228	1.175	1.070
P5 = (−£12.00, 5)	1.200	1.125	1.100
P8 = (−£12.50, 8)	**1.296**	1.200	1.248
P10 = (−£14.50, 10)	0.770	1.100	**1.760**

We have highlighted the best option for each choice in bold. We can see that for choice 1, the best option is policy P8; for choice 2, the best option is policy P0 and for choice 3, the best option is P10. It is not a surprise to see one of the more extreme options (i.e., P0 or P10) being the chosen option for choice 3 as this was the instance where the attributes are substitutes.

A.2. ***One independent attribute — multiattribute EUT.*** We are now going to focus on the instance where we only have one attribute which is independent of the other. For this paper, we are going to focus on the instance where we have Z independent of Y. This involves applying Theorem 5.6 from [Keeney and Raiffa 1976, p. 244].

We begin by choosing $y_0 \in \mathrm{Y}$ and $z_0, z_1 \in \mathrm{Z}$ such that $u(y_0, z_0) = 0$ and $u(y_0, z_1) = 1$. In this instance, we are going to choose $y_0 = -£14.50$, $z_0 = 0$ (as in the mutual independence case) and $z_1 = 4$. Following this, we are required to fix the unit of measure of $u(y, z_0)$ and $u(y, z_1)$. This comes from choosing $y_2 \in \mathrm{Y}$, $z_2 \in \mathrm{Z}$ such that $(y_0, z_2) \sim (y_2, z_0)$ and, similarly, $y_3 \in \mathrm{Y}$, $z_3 \in \mathrm{Z}$ such that $(y_0, z_3) \sim (y_3, z_1)$. Let us choose $y_2 = -£5.50$, $z_2 = 3$, $y_3 = -£12.00$ and $z_3 = 8$.

Now that these decisions have been made, we are required to evaluate the conditional utility functions $u(y_0, z)$ for all $z \in \mathrm{Z}$ and $u(y, z_0), u(y, z_1)$ for all $y \in \mathrm{Y}$. To begin with, we assign the utilities $u(y_0, z)$ for all $z \in \mathrm{Z}$ and using these, then assign the remaining $u(y, z_0)$ and $u(y, z_1)$ for all $y \in \mathrm{Y}$. The choices we make are as follows:

$$\begin{aligned} u(y_0, 0) &= 0, & u(y_0, 4) &= 1, \\ u(y_0, 1) &= 0.2, & u(y_0, 5) &= 1.25, \\ u(y_0, 2) &= 0.45, & u(y_0, 8) &= 1.6, \\ u(y_0, 3) &= 0.8, & u(y_0, 10) &= 2.1, \end{aligned}$$

$$\begin{aligned} u(-£14.50, z_0) &= 0, & u(-£8.00, z_0) &= 0.45, \\ u(-£12.50, z_0) &= 0.1, & u(-£7.50, z_0) &= 0.55, \\ u(-£12.00, z_0) &= 0.15, & u(-£5.50, z_0) &= 0.8, \\ u(-£10.00, z_0) &= 0.25, & u(£0.00, z_0) &= 1.25, \end{aligned}$$

$$\begin{aligned} u(-£14.50, z_1) &= 1, & u(-£8.00, z_1) &= 2.6, \\ u(-£12.50, z_1) &= 1.6, & u(-£7.50, z_1) &= 2.7, \\ u(-£12.00, z_1) &= 1.8, & u(-£5.50, z_1) &= 3.1, \\ u(-£10.00, z_1) &= 2.25, & u(£0.00, z_1) &= 3.7. \end{aligned}$$

Now that these choices have been made, we can calculate the utilities of each of the policies and choose the one with the highest utility value. The corresponding utilities are

$$\begin{aligned} u(\mathrm{P0}) &= 1.25, & u(\mathrm{P4}) &= 2.25, \\ u(\mathrm{P1}) &= 1.26, & u(\mathrm{P5}) &= 2.2125, \\ u(\mathrm{P2}) &= 1.5175, & \boldsymbol{u}(\mathbf{P8}) &= \mathbf{2.5}, \\ u(\mathrm{P3}) &= 2.17, & u(\mathrm{P10}) &= 2.1. \end{aligned}$$

We can see from this that when we have Z independent of Y, we choose policy P8 to be the best policy. P8 was the same policy that we chose for choice 1 of the mutually independent EUT case but different from all other choices we have made so far. This shows how different choices can come from the same decision maker dependent on how they formulate the problem and choose independence.

Appendix B: Breakdown cover PT choices

B.1. *Mutually independent attributes.*

Choice 1: $v(y_1, z_1) = 1.7$, $v(y_1, z_0) = 0.8$, $v(y_1, z_{-1}) = 0.2$,
$v(y_0, z_1) = 0.85$, $v(y_0, z_0) = 0$, $v(y_0, z_{-1}) = -0.7$,
$v(y_{-1}, z_1) = 0.4$, $v(y_{-1}, z_0) = -0.5$, $v(y_{-1}, z_{-1}) = -1.3$.

Choice 2: $v(y_1, z_1) = 2.5$, $v(y_1, z_0) = 1.6$, $v(y_1, z_{-1}) = 0.75$,
$v(y_0, z_1) = 1.4$, $v(y_0, z_0) = 0$, $v(y_0, z_{-1}) = -0.3$,
$v(y_{-1}, z_1) = 0.7$, $v(y_{-1}, z_0) = -0.4$, $v(y_{-1}, z_{-1}) = -0.7$.

Choice 3: $v(y_1, z_1) = 1.1$, $v(y_1, z_0) = 0.65$, $v(y_1, z_{-1}) = -0.35$,
$v(y_0, z_1) = 0.4$, $v(y_0, z_0) = 0$, $v(y_0, z_{-1}) = -0.95$,
$v(y_{-1}, z_1) = -0.15$, $v(y_{-1}, z_0) = -0.9$, $v(y_{-1}, z_{-1}) = -1.7$.

Our choices for $v(y, z_0)$ for all possible $y \in \mathrm{Y}$ are as follows:

Choice 1: $v(£0, z_0) = 1.9$, $v(-£10.00, z_0) = 0$,
$v(-£5.50, z_0) = 1.35$, $v(-£12.00, z_0) = -0.35$,
$v(-£7.50, z_0) = 0.8$, $v(-£12.50, z_0) = -0.5$,
$v(-£8.00, z_0) = 0.65$, $v(-£14.50, z_0) = -1.05$.

Choice 2: $v(£0, z_0) = 3.1$, $v(-£10.00, z_0) = 0.7$,
$v(-£5.50, z_0) = 2.35$, $v(-£12.00, z_0) = 0$,
$v(-£7.50, z_0) = 1.6$, $v(-£12.50, z_0) = -0.4$,
$v(-£8.00, z_0) = 1.35$, $v(-£14.50, z_0) = -1.1$.

Choice 3: $v(£0, z_0) = 1.8$, $v(-£10.00, z_0) = -0.25$,
$v(-£5.50, z_0) = 1.1$, $v(-£12.00, z_0) = -0.75$,
$v(-£7.50, z_0) = 0.65$, $v(-£12.50, z_0) = -0.9$,
$v(-£8.00, z_0) = 0.45$, $v(-£14.50, z_0) = -1.6$.

Similarly, we can state our values $v(y_0, z)$ for all $z \in \mathrm{Z}$ as:

Choice 1: $v(y_0, 0) = -1.9$, $v(y_0, 3) = -0.45$, $v(y_0, 8) = 0.85$,
$v(y_0, 1) = -1.25$, $v(y_0, 4) = 0$, $v(y_0, 10) = 1.8$.
$v(y_0, 2) = -0.7$, $v(y_0, 5) = 0.3$,

Choice 2: $v(y_0, 0) = -1.45$, $v(y_0, 3) = 0$, $v(y_0, 8) = 1.4$,
$v(y_0, 1) = -0.9$, $v(y_0, 4) = 0.45$, $v(y_0, 10) = 2.1$.
$v(y_0, 2) = -0.3$, $v(y_0, 5) = 0.8$,

Choice 3: $v(y_0, 0) = -2.15$, $v(y_0, 3) = -0.7$, $v(y_0, 8) = 0.4$,
$v(y_0, 1) = -1.35$, $v(y_0, 4) = -0.4$, $v(y_0, 10) = 1.1$.
$v(y_0, 2) = -0.95$, $v(y_0, 5) = 0$,

B.2. ***Single independence.*** We evaluate $v(y_0, z)$ for all $z \in \mathrm{Z}$ as:

Choice 1: $v(y_0, 0) = -1.7$, $v(y_0, 3) = -0.6$, $v(y_0, 8) = 1$,
$v(y_0, 1) = -1.25$, $v(y_0, 4) = 0$, $v(y_0, 10) = 1.45$.
$v(y_0, 2) = -1$, $v(y_0, 5) = 0.4$,

Choice 2: $v(y_0, 0) = -1.5$, $v(y_0, 3) = 0$, $v(y_0, 8) = 1.4$,
$v(y_0, 1) = -1$, $v(y_0, 4) = 0.4$, $v(y_0, 10) = 1.75$.
$v(y_0, 2) = -0.55$, $v(y_0, 5) = 1$,

Choice 3: $v(y_0, 0) = -2.1$, $v(y_0, 3) = -1$, $v(y_0, 8) = 1$,
$v(y_0, 1) = -1.7$, $v(y_0, 4) = -0.45$, $v(y_0, 10) = 1.45$.
$v(y_0, 2) = -1.45$, $v(y_0, 5) = 0$,

Following this, we evaluate $v(y, z_0)$ for all $y \in \mathrm{Y}$ as:

Choice 1: $v(-£14.50, z_0) = -1.1$, $v(-£8.00, z_0) = 0.4$,
$v(-£12.50, z_0) = -0.6$, $v(-£7.50, z_0) = 0.45$,
$v(-£12.00, z_0) = -0.5$, $v(-£5.50, z_0) = 0.85$,
$v(-£10.00, z_0) = 0$, $v(£0.00, z_0) = 1.5$.

Choice 2: $v(-£14.50, z_0) = -0.75$, $v(-£8.00, z_0) = 0.75$,
$v(-£12.50, z_0) = -0.15$, $v(-£7.50, z_0) = 0.85$,
$v(-£12.00, z_0) = 0$, $v(-£5.50, z_0) = 1.2$,
$v(-£10.00, z_0) = 0.4$, $v(£0.00, z_0) = 1.85$.

Choice 3: $v(-£14.50, z_0) = -1.25$, $v(-£8.00, z_0) = 1.05$,
$v(-£12.50, z_0) = -0.8$, $v(-£7.50, z_0) = 1.45$,
$v(-£12.00, z_0) = -0.6$, $v(-£5.50, z_0) = 1.8$,
$v(-£10.00, z_0) = -0.2$, $v(£0.00, z_0) = 2.5$.

Similarly, we evaluate $v(y, z_1)$ for all $y \in \mathrm{Y}$ as:

Choice 1:
$$v(-£14.50, z_1) = -0.6, \quad v(-£8.00, z_1) = 1.4,$$
$$v(-£12.50, z_1) = 0.2, \quad v(-£7.50, z_1) = 1.55,$$
$$v(-£12.00, z_1) = 0.4, \quad v(-£5.50, z_1) = 1.8,$$
$$v(-£10.00, z_1) = 1, \quad v(£0.00, z_1) = 2.4.$$

Choice 2:
$$v(-£14.50, z_1) = -0.05, \quad v(-£8.00, z_1) = 1.8,$$
$$v(-£12.50, z_1) = 0.75, \quad v(-£7.50) = 1.9,$$
$$v(-£12.00, z_1) = 1, \quad v(-£5.50, z_1) = 2.35,$$
$$v(-£10.00, z_1) = 1.35, \quad v(£0.00, z_1) = 2.9.$$

Choice 3:
$$v(-£14.50, z_1) = -0.65, \quad v(-£8.00, z_1) = 1.45,$$
$$v(-£12.50, z_1) = -0.1, \quad v(-£7.50, z_1) = 1.6,$$
$$v(-£12.00, z_1) = 0.3, \quad v(-£5.50, z_1) = 2.05,$$
$$v(-£10.00, z_1) = 0.85, \quad v(£0.00, z_1) = 2.75.$$

Finally, we evaluate $v(y, z_{-1})$ for all $y \in \mathrm{Y}$ as:

Choice 1:
$$v(-£14.50, z_{-1}) = -2.05, \quad v(-£8.00, z_{-1}) = -0.3,$$
$$v(-£12.50, z_{-1}) = -1.65, \quad v(-£7.50, z_{-1}) = -0.1,$$
$$v(-£12.00, z_{-1}) = -1.5, \quad v(-£5.50, z_{-1}) = 0.5,$$
$$v(-£10.00, z_{-1}) = -1, \quad v(£0, z_{-1}) = 1.05.$$

Choice 2:
$$v(-£14.50, z_{-1}) = -1.6, \quad v(-£8.00, z_{-1}) = 0.4,$$
$$v(-£12.50, z_{-1}) = -1.15, \quad v(-£7.50, z_{-1}) = 0.55,$$
$$v(-£12.00, z_{-1}) = -1, \quad v(-£5.50, z_{-1}) = 0.9,$$
$$v(-£10.00, z_{-1}) = -0.55, \quad v(£0.00, z_{-1}) = 1.35.$$

Choice 3:
$$v(-£14.50, z_{-1}) = -2.55, \quad v(-£8.00, z_{-1}) = -0.45,$$
$$v(-£12.50, z_{-1}) = -2, \quad v(-£7.50, z_{-1}) = -0.05,$$
$$v(-£12.00, z_{-1}) = -1.85, \quad v(-£5.50, z_{-1}) = 0.8,$$
$$v(-£10.00, z_{-1}) = -1.3, \quad v(£0, z_{-1}) = 1.65.$$

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