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Shabat polynomials and monodromy groups of trees uniquely determined by ramification type

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A dessin d'enfant or dessin is a bicolored graph embedded into a Riemann surface. Acyclic dessins can be described analytically by preimages of Shabat polynomials and algebraically by their monodromy groups. We determine the Shabat polynomials and monodromy groups of planar acyclic dessins that are uniquely determined by their ramification types.

1. Introduction

Popularized by Grothendieck in his "Esquisse d'un programme", the theory of *dessins* reaches across and connects multiple disciplines, including graph theory, topology, geometry, algebra and complex analysis. Our motivation for this paper is rooted in one of the fundamental questions in the theory of dessins — that is, how to distinguish classes of dessins by means of topological, algebraic or combinatorial invariants. In this paper, we focus our attention on this question by studying dessins which are also trees. Since such dessins by any measure might be considered among the simplest, it is worthwhile to have a complete catalog of the Belyi maps and monodromy groups to which they correspond.

Our main objective in this paper is to determine the Shabat polynomials (up to isomorphism) and monodromy groups corresponding to every known planar connected acyclic dessin uniquely determined by its ramification type, the complete list of which was given in [Shabat and Zvonkin 1994]. We begin in Section 1 by providing the main result of the paper, followed by definitions and notation needed to describe the class of dessins with which we are concerned, as well as some necessary background about Shabat polynomials and wreath products. Readers already acquainted with these subjects may wish to read Section 1A and skip Section 1B. In Section 2 we provide a unique (up to isomorphism) Shabat polynomial

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for each ramification type corresponding to exactly one (planar) bicolored tree; in Section 3 we provide the monodromy groups for each such ramification type. In Section 4, we suggest future directions that may be taken from the results presented here.

1A. ***Main results.*** Here, we state the main result of the paper in the following theorem. The remainder of this section provides the background and preliminaries for the rest of the paper. Theorem 1.1 lists the ramification types which correspond to exactly one dessin which is a tree, along with the associated monodromy groups and Shabat polynomials. Theorem 1.1 contains every such ramification type, as asserted in [Shabat and Zvonkin 1994]. In Sections 2 and 3, we argue that Theorem 1.1 lists the correct Shabat polynomials and monodromy groups.

Theorem 1.1. *The following list includes all seven ramification types* (*degrees of black vertices followed by degrees of white vertices*) *that produce exactly one dessin which is a tree* (see [Shabat and Zvonkin 1994]). *Each ramification type given on the list is followed by* (*a*) *the Shabat polynomial* (*unique up to isomorphism*) *and* (*b*) *the monodromy group for the dessin.*

(1) $[r; 1^r]$

  (a) z^r

  (b) C_r

(2) $[2^r, 1; 2^r, 1]$

  (a) $\frac{1}{2}(1+\cos((2r+1)\arccos(z)))$

  (b) $D_{2(2r+1)}$, *where D_m denotes the dihedral group of order m*

(3) $[2^r; 2^{r-1}, 1^2]$

  (a) $\frac{1}{2}(1+\cos(2r\arccos(z)))$

  (b) $D_{2(2r)}$

(4) $[s^{r-1}, t; r, 1^{(r-1)(s-1)+(t-1)}]$ *for* $r > 1,\ t > 0$

  (a) $(1-z)^t\big(\sum_{k=0}^{r-1}\left(\frac{t}{s}\right)_k\frac{z^k}{k!}\big)^s$

  (b) $\begin{cases} C_r \wr C_s, & s = t, \\ S_{n/d} \wr C_d, & s \neq t,\ r \text{ even}, \\ A_{n/d} \wr C_d, & s \neq t,\ r \text{ odd and } \frac{t}{d} \text{ is odd}, \\ (A_{n/d})^d \rtimes C_{2d}, & s \neq t,\ r \text{ odd},\ \frac{t}{d} \text{ even}, \end{cases}$

  where $n = s(r-1)+t,\ d = \gcd(s,t)$.

(5) $[r, t, 1^{r+t-2}; 2^{r+t-1}],\ r, t > 1$

  (a) $4z^r(1-z)^t\big(\sum_{j=0}^{r-1}\binom{t-1+j}{t-1}z^j\big)\big(\sum_{j=0}^{t-1}\binom{r-1+j}{r-1}\binom{r+t-1}{r+j}(-1)^j z^j\big)$

(b) $\begin{cases} A_{2r-1} \times C_2, & r = t,\ r \text{ odd}, \\ S_{2r-1} \times C_2, & r = t,\ r \text{ even}, \\ A_{r+t-1} \wr C_2, & r \neq t,\ \text{both odd}, \\ R_2, & r \neq t,\ \text{both even}, \\ S_{r+t-1} \wr C_2, & r \neq t,\ \text{else}, \end{cases}$

where R_m denotes the index-2 subgroup of $S_{n/m} \wr C_m$ such that, for all $(\tau_1, \dots, \tau_m, g) \in R_m$, the permutation $\tau_1 \tau_2 \cdots \tau_m$ is even.

(6) $[r^2, 1^{4r-3}; 3^{2r-1}]$

(a) $-3\sqrt{3}\, i\ S_r(z)(1 - S_r(z))\big(S_r(z) - \frac{1}{2}(1 - i\sqrt{3})\big)$

(b) $\begin{cases} A_{2r-1} \wr C_3, & r \text{ odd}, \\ R_3, & r \text{ even} \end{cases}$

(7) $[3^3, 1^5; 2^7]$

(a) $-\frac{4}{531441}(z-1)z^3(2z^2+3z+9)^3(8z^4+28z^3+126z^2+189z+378)$

(b) $A_7 \wr C_2$

1B. ***Background and preliminaries.*** We begin by providing a terse exploration of the object known as a *dessin*. For more detailed and comprehensive literature on the subject, see [Shabat and Zvonkin 1994; Wood 2006]. For the purposes of this paper, we begin with the observation that dessins may be realized by meromorphic functions known as *Belyi maps*. The arithmetic dynamics of these Belyi maps have been studied in some cases [Anderson et al. 2018].

Definition 1.2. Let X be a compact Riemann surface. A *Belyi map* is a meromorphic function $F : X \to \mathbb{P}^1(\mathbb{C})$ that is unramified outside of $\{0, 1, \infty\}$. That is, all critical values of F are contained in $\{0, 1, \infty\}$. Here we may consider $\mathbb{P}^1(\mathbb{C})$ as just $\mathbb{C} \cup \{\infty\}$.

Grothendieck's notion of a *dessin d'enfant* or *dessin* for short is a way to combinatorially characterize Belyi maps. If F is a Belyi map, then $F^{-1}([0, 1])$, that is, the preimage of the interval $[0, 1]$, has the structure of a bicolored connected graph embedded in X. The basic structure of the bicolored graph Δ_F associated with a Belyi map F is given when we identify $F^{-1}(0)$ as the set of black vertices, $F^{-1}(1)$ as the set of white vertices, $F^{-1}((0, 1))$ as the set of edges and $F^{-1}(\mathbb{P}^1(\mathbb{C}) - [0, 1])$ as the set of faces. Note that the degrees of the black and white vertices of Δ_F correspond to the multiplicities of the roots of F and $F - 1$, respectively. Furthermore, the dessin Δ_F recovered from a Belyi map F is planar if and only if F is defined on $\mathbb{P}^1(\mathbb{C})$, while Δ_F is a tree if and only if F is a polynomial. Throughout this paper, we assume $X = \mathbb{P}^1(\mathbb{C})$.

These structure of Δ_F can be captured by the notion of a *dessin*, the relatively simple combinatorial characterization given by Grothendieck.

Definition 1.3. A *dessin d'enfant* or *dessin* is a connected bicolored graph equipped with a cyclic ordering of the edges (oriented counterclockwise) around each vertex.

Given a Belyi map F, it is not difficult to use the procedure described above to visualize the dessin Δ_F to which F corresponds. However, recovering a Belyi map from a given dessin is a much more difficult proposition. Given a dessin Δ_F, a corresponding Belyi map F can be determined (uniquely up to isomorphism over $\mathbb{C} \cup \{\infty\}$) by considering the degrees of the vertices of Δ_F and the resulting system of polynomial equations involving roots and poles of F. Various methods of calculating Belyi maps may be found in [Couveignes 1994; Matiyasevich 1996; Schneps 1994; Sijsling and Voight 2014].

Definition 1.4. A *Shabat polynomial* is a polynomial $F : \mathbb{C} \to \mathbb{C}$ whose critical values are contained in $\{0, 1\}$.

That is, a Shabat polynomial is a Belyi map which has only one pole (which is at infinity); hence, its corresponding dessin will be a tree. (Shabat polynomials can be defined more broadly as in [Shabat and Zvonkin 1994] as generalized Chebyshev polynomials which have at most two critical values. Without loss of generality, we choose in this paper to identify the two critical values 0 and 1.)

Definition 1.5. We say that two Shabat polynomials F, G are isomorphic if there exist $\alpha \in \mathbb{C}^\times$ and $\beta \in \mathbb{C}$ such that $F(z) = G(\alpha z + \beta)$.

Assume we have a dessin which is a tree and we label the edges with the numbers $1, 2, \ldots, n$. We can associate the dessin with a pair of permutations σ_0, $\sigma_1 \in S_n$, where n is number of edges, such that the cycles of σ_0 correspond to the cyclic ordering (read counterclockwise) of the edges around the black vertices and the cycles of σ_1 correspond to the ordering (read counterclockwise) of the edges around the white vertices. For example, see Figure 1, where we have a bicolored tree, whose edges are labeled $1, 2, \ldots, 7$ inducing a pair of permutations $\sigma_0, \sigma_1 \in S_7$ associated with the black and white vertices, respectively. In general, by σ_0 (respectively, σ_1), we mean the product of the cycle permutations associated with the edges about all of the black (respectively, white) vertices. The group that σ_0 and σ_1 generate is a central focus of this paper.

Definition 1.6. The *monodromy group* of a dessin with n edges is $\langle \sigma_0, \sigma_1, \sigma_\infty \rangle$, the group generated by $\sigma_0, \sigma_1, \sigma_\infty \in S_n$, where σ_0, σ_1 are as described in the preceding paragraph and σ_∞ is such that $\sigma_0 \sigma_1 \sigma_\infty = 1$.

We remark that since $\sigma_\infty = (\sigma_0 \sigma_1)^{-1}$, we may remove it from the generating set for the monodromy group, but we keep it in the definition to be consistent with the wider literature on this subject, which goes well beyond the consideration of Shabat polynomials. For the remainder of the paper, when we refer to the generators of the monodromy group, we are talking about σ_0 and σ_1. When a dessin is connected,

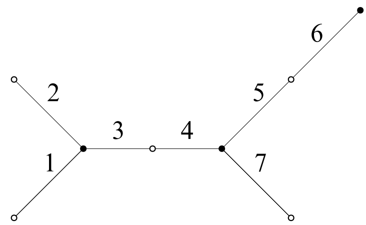

Figure 1. A dessin determined by the pair of permutations $\sigma_0 = (1, 3, 2)(4, 7, 5)$ and $\sigma_1 = (3, 4)(5, 6)$ whose monodromy group $\langle \sigma_0, \sigma_1 \rangle$ is isomorphic to $GL_3(\mathbb{F}_2)$, a transitive subgroup of S_7.

its monodromy group will be a transitive subgroup of S_n, where n is the number of edges in the dessin.

To every dessin, we may associate an invariant known as its *ramification type*. The ramification type of a dessin with n edges is given by the three partitions of n corresponding to the degrees of the black vertices, the degrees of the white vertices and the degrees of the faces. In the case of a dessin having one face, the latter partition is simply $n = n$. Since we focus exclusively on dessins with one face in this paper, we will omit from the notation for ramification type the last partition corresponding to the degrees of the faces.

Definition 1.7. The *ramification type* of a dessin with n edges (and exactly one face) consists of the two partitions of n

$$[b_1^{\beta_1} b_2^{\beta_2} \cdots b_k^{\beta_k}; w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_\ell^{\alpha_\ell}]$$

written in exponential notation, where b_1, b_2, ..., b_k are the distinct degrees of the black vertices, w_1, w_2, ..., w_ℓ are the distinct degrees of the white vertices, β_i is the number black vertices of degree b_i and α_i is the number white vertices of degree w_i.

Note that $b_1^{\beta_1} b_2^{\beta_2} \cdots b_k^{\beta_k}$ and $w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_\ell^{\alpha_\ell}$ are both partitions of n, where n is the number of edges, and these two partitions correspond to the cycle type of σ_0 and σ_1, respectively.

While each dessin has a unique ramification type, one may ask how many distinct dessins (or equivalently nonisomorphic Shabat polynomials) are associated with a given ramification type. Our focus in this paper will be narrowed to ramification types which admit unique dessins.

We sometimes use the concept of tree composition to decompose a dessin into smaller dessins. Composition will also help us compute new Shabat polynomials as it corresponds with the usual polynomial composition. It is an easy exercise in calculus to show that the composition of two Shabat polynomials is again a Shabat polynomial.

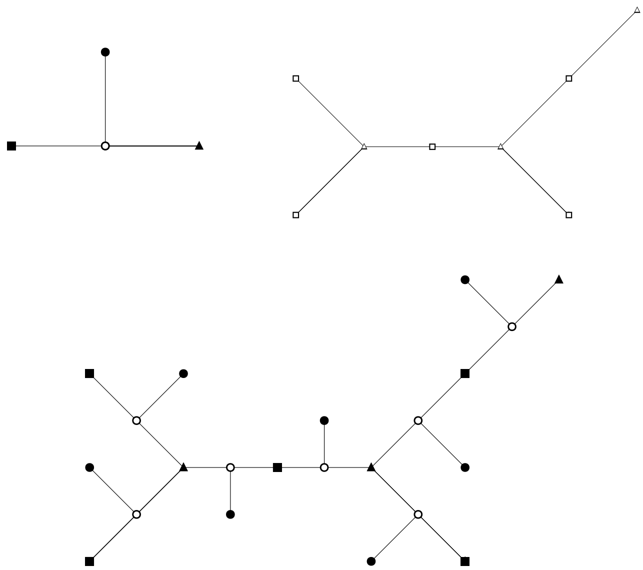

Figure 2. Top, left: P, with two vertices marked square □ and triangle △. Top, right: Q, with black vertices marked □, white vertices marked △. Bottom: The composition $P \circ Q$ of two dessins P, Q.

Many of the dessins that we study can be constructed by a composition process given by Adrianov and Zvonkin [1998]. Given two dessins, P and Q, we begin the composition $P \circ Q$ by first distinguishing two vertices of P — label them with a square and a triangle. The vertices of Q will be preimages of the square and triangle, so we mark every black vertex of Q with a square and similarly every white vertex of Q with a triangle. The process of composition is as follows:

(1) Replace each edge of Q with the union of the path from the square to the triangle in P along with every branch connected to that path.

(2) Adjoin to each square (resp., triangle) vertex of Q the union of every branch connected to the square (triangle) in P except for the one in the path to the triangle (square). Do this as many times as the degree of the vertex.

The resulting graph should resemble n copies of P arranged in the shape of Q, where n is the number of edges of Q. We demonstrate this process in Figure 2.

Remark 1.8. Let G_P, G_Q denote the respective monodromy groups of P and Q. According to a theorem of Adrianov and Zvonkin [1998], the monodromy group of $P \circ Q$ is a subgroup of $G_Q \wr G_P$, where $\wr$ denotes the wreath product.

This process also gives a way to compute Shabat polynomials. If p, q are the respective Shabat polynomials of P, Q such that $p(0), p(1) \in \{0, 1\}$ then the Shabat polynomial of $P \circ Q$ is $p \circ q$ (where $\circ$ denotes the conventional composition of functions, i.e., $(f \circ g)(x) = f(g(x))$. Later on, when we compute Shabat polynomials of more complicated dessins, we will make extensive use of this fact.

We will often call upon the idea of the wreath product of groups to describe our monodromy groups. The composition process produces dessins whose monodromy groups are subgroups of wreath products. While there are numerous examples for which the containment is proper, often equality of the groups is achieved. As far as the present authors can tell, the exact conditions that ensure equality are not known.

Definition 1.9. Let d be a positive integer. Let $G \leq S_d$ and H be groups. Let K be the direct product of d copies of H. If $h = (h_1, \dots, h_d) \in K$, then we define the action of $\sigma \in G$ on K by $\sigma \cdot h = (h_{\sigma(1)}, \dots, h_{\sigma(d)})$. The *wreath product* of H by G is the semidirect product $K \rtimes G$ with respect to the action above, and we denote this group by $H \wr G$.

In this paper, G is typically C_d, the cyclic group of order d.

2. Shabat polynomials for trees uniquely determined by ramification type

In this section, we summarize the list of Shabat polynomials (up to isomorphism) corresponding to dessins which are trees and are uniquely determined by ramification type. The complete list of ramification types for such dessins was given in [Shabat and Zvonkin 1994]. For the Shabat polynomials corresponding to these ramification types, we adopt the convention described in Definition 1.7.

Proposition 2.1. *The ramification types* $[r; 1^r]$, $[2^r, 1; 2^r, 1]$, $[2^r; 2^{r-1}, 1^2]$ *have respective Shabat polynomials*

$$z^r, \quad \tfrac{1}{2}(1 + \cos((2r+1)\arccos(z))), \quad \tfrac{1}{2}(1 + \cos((2r)\arccos(z))),$$

all unique up to isomorphism.

This result is already well known in the literature and can be found on pages 3–4 of [Shabat and Zvonkin 1994]. See Figure 3.

Proposition 2.2 [Adrianov 2007]. *Up to isomorphism, the unique Shabat polynomial for the ramification type* $[s^{r-1}, t; r, 1^{(r-1)(s-1)+(t-1)}]$ *is*

$$F(z) = (1-z)^t \left(\sum_{k=0}^{r-1} \left(\frac{t}{s}\right)_k \frac{z^k}{k!} \right)^s,$$

where

$$(a)_k = a(a+1)(a+2)\cdots(a+k-1)$$

denotes the Pochhammer symbol.

The proof for this proposition can be found in [Adrianov 2007].

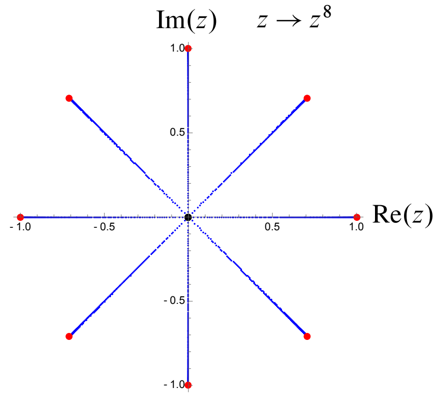

Figure 3. The dessin with ramification type $[8; 1^8]$.

Proposition 2.3. *Let $r > 1$. Up to isomorphism, the Shabat polynomial for the tree having ramification type*

$$[r, t, 1^{r+t-2}; 2^{r+t-1}]$$

with a black vertex of degree r located at $z = 0$ and a black vertex of degree t located at $z = 1$ is given by

$$F(z) = 4z^r \binom{r+t-1}{r} {}_2F_1(t-1, r; r+1; z) \times \left(1 - (1-z)^t z^r \binom{r+t-1}{t-1} {}_2F_1(1, r+t; r+1; z)\right),$$

where ${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

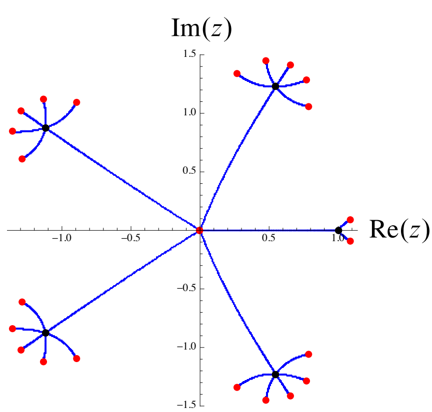

Figure 4. The dessin obtained by the Shabat polynomial given in Proposition 2.2 when $s = 6$, $r = 5$, $t = 3$.

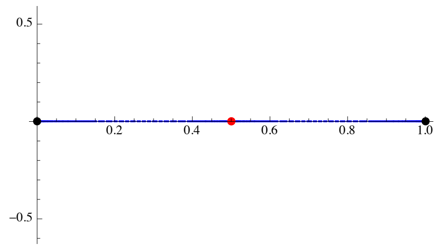

Figure 5. The dessin (path graph) obtained by the Shabat polynomial $\beta(z) = 4z(1-z)$.

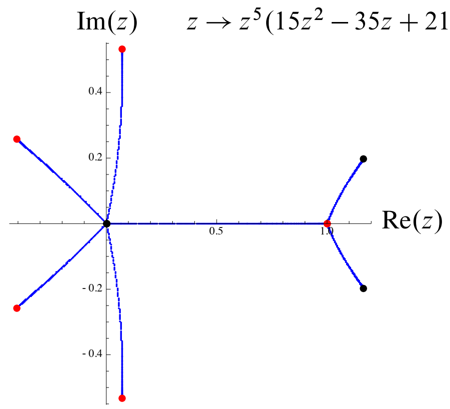

Figure 6. The tree obtained by the Shabat polynomial in Proposition 2.2 where $s = 1, r = 3, t = 5$.

Proof. Let $S_{r,t}(z)$ be the Shabat polynomial for the ramification type $[t, 1^{r-1}; r, 1^{t-1}]$. By Proposition 2.2, with $s = 1$,

$$S_{r,t}(z) = (1-z)^t \sum_{j=0}^{r-1} \binom{t-1+j}{t-1} z^j.$$

Consider the map $\beta(z) = 4z(1-z)$ with the dessin Δ_β (see Figure 5) and $S_{r,t}(z)$ with the dessin Δ_S (see Figure 6). The composition $\beta(z) \circ S_{r,t}(z)$ is a Shabat polynomial that produces the dessin obtained by coloring the vertices of Δ_S to black and adding a white vertex of degree 2 inside every edge (in other words, replacing every edge of Δ_S with Δ_β). Note the number of edges in $S_{r,t}(z)$ is $r+t-1$. The composition produces the new dessin Δ_F (see Figure 7) and Shabat polynomial $F(z) = \beta(z) \circ S_{r,t}(z)$ with ramification type $[r, t, 1^{r+t-2}; 2^{r+t-1}]$, and therefore $F(z)$ equals

$$4z^r(1-z)^t \left(\sum_{j=0}^{r-1} \binom{t-1+j}{t-1} z^j\right) \left(\sum_{j=0}^{t-1} \binom{r-1+j}{r-1} \binom{r+t-1}{r+j} (-1)^j z^j\right),$$

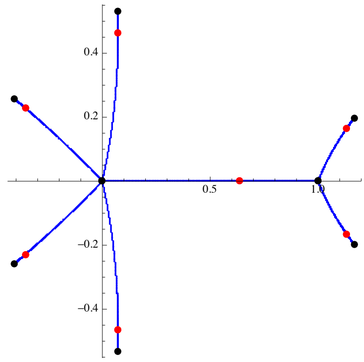

Figure 7. The tree obtained by the Shabat polynomial in Proposition 2.3 with $r = 5, t = 3$.

which can be rewritten in terms of hypergeometric functions, as in the statement of the present proposition. □

Proposition 2.4. *The Shabat polynomial for the unique tree having ramification type*

$$[r^2, 1^{4r-3}; 3^{2r-1}]$$

with two black vertices of degree r located at $z = 0$ and $z = 1$ is given by

$$F(z) = (T \circ S_r)(z),$$

where

$$T(z) = -3i\sqrt{3}z(1-z)(z+\rho), \quad \rho = \tfrac{1}{2}(-1+i\sqrt{3}),$$

and

$$S_r(z) = (1-z)^r \sum_{j=0}^{r-1} \binom{r-1+j}{r-1} z^j.$$

$F(z)$ is unique up to isomorphism.

Proof. First we will show that $T(z) := -3i\sqrt{3}z(1-z)(z+\rho)$ corresponds to a 3-star with a white center and black leaves at $z = 0$ and $z = 1$. Considering $T(z)$, we see immediately three distinct roots of multiplicity 1 at $z = 0, 1, \frac{1}{2}(1-i\sqrt{3})$ representing three black leaves in Δ_F. Next we consider the derivative of $T(z)$,

$$T'(z) = -3i\sqrt{3}(\rho + 2(1-\rho)z - 3z^2),$$

which has a single root of multiplicity 2 (note that the discriminant of $T'(z)$ is zero). Since the multiplicity of the black vertices is 1, we may assume that the multiple root in $T'(s)$ must refer to a root of multiplicity 3 in $F(z) - 1$, representing the white vertex of degree 3. Therefore, $T(z)$ must be a 3-star with black

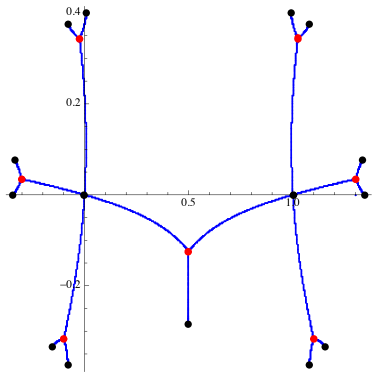

Figure 8. An illustration of the tree derived from the Shabat polynomial in Proposition 2.4 where $r = 4$.

leaves at $z = 1$ and $z = 0$. We can now use the idea of composition to replace every edge of the tree having Shabat polynomial $S_r(z) := S_{r,r}(z)$, where $S_{r,t}(z)$ is the polynomial as defined in the proof of Proposition 2.3, with the 3-star by computing the composition $(T \circ S_r)(z)$. This will add a white vertex of degree 3 and an additional black leaf for every edge. Note that $S_r(z)$ corresponds to a tree with $2r-1$ edges and $4r-2$ vertices. Therefore Δ_F will have $2r-1$ white vertices of degree 3 and $4r-3$ black leaves, in addition to the two black vertices of degree r.

Note: An anonymous referee pointed out that we may go one step further here by letting $z' := i\sqrt{3}z - \rho^2$. A quick computation shows that $\overline{S_r(z')} = S_r(1-z')$. One can also show that $S_r(z) = 1 - S_r(1-z)$ using the following argument. Observe that 0 is a root of order r of $S_r(z)$ and $1 - S_r(1-z)$. Further observe that 1 is a root of order r of $S_r(z) - 1$ and $1 - S_r(1-z) - 1$. Thus we deduce that $S_r(z) = 1 - S_r(1-z)$ using the uniqueness of the Shabat polynomial from Proposition 2.2. Hence, $\overline{S_r(z')} = 1 - S_r(z')$. A few simple calculations yield the equality $\overline{T(S_r(z'))} = T(S_r(z'))$, which implies $T(S_r(z')) \in \mathbb{Q}[z]$. □

Proposition 2.5. *For the tree with ramification type* $[3^3, 1^5; 2^7]$, *a black vertex of degree* 3 *at* $z = 0$ *and a black vertex of degree* 1 *at* $z = 1$, *the Shabat polynomial is*

$$F(z) = -\tfrac{4}{531441}(z-1)z^3(2z^2+3z+9)^3(8z^4+28z^3+126z^2+189z+378).$$

Proof. We can write $F(z) = (\beta \circ f)(z)$, where

$$\beta(z) = 4z(1-z) \quad \text{and} \quad f(z) = -\tfrac{1}{729}(z-1)(9+3z+2z^2)^3,$$

which is the Shabat polynomial for ramification type $[3^2, 1; 3, 1^4]$ obtained by letting $r = 3$, $s = 3$, $t = 1$ in Proposition 2.2. □

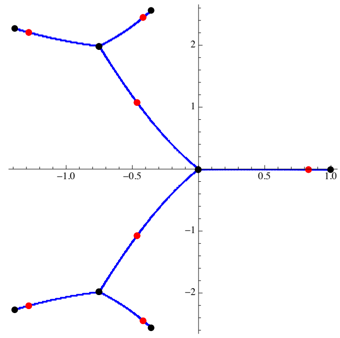

Figure 9. An illustration of the tree described in Proposition 2.5.

3. Monodromy groups for trees uniquely determined by ramification type

In this section, we provide proofs for the monodromy groups associated with each ramification type listed in Theorem 1.1. In all of our proofs, we proceed by choosing a particular labeling of the edges of the dessin. Though the monodromy group does not depend on the choice of labels, some choices better illustrate how σ_0 and σ_1 generate the monodromy group.

Proposition 3.1. *The ramification types* $[r; 1^r]$, $[2^r, 1; 2^r, 1]$, *and* $[2^r; 2^{r-1}, 1^2]$ *have respective monodromy groups* C_r, $D_{2(2r+1)}$, *and* $D_{2(2r)}$, *where* D_m *denotes the dihedral group of order* m.

Proof. The first ramification type gives the r-star dessin with monodromy group generated by an r-cycle and the identity permutation. It follows that the monodromy group is the cyclic group C_r. The second and third ramification types yield the path dessins with $2r+1$ and $2r$ edges respectively. We handle these two cases simultaneously, since the argument is essentially the same. The dessins in Figure 10 are examples of path dessins.

In both cases, the generators of the groups σ_0 and σ_1 have order 2, and the respective σ_∞'s have order $2r+1$ and $2r$. Since in this case $\sigma_\infty = (\sigma_0\sigma_1)^{-1} = \sigma_1\sigma_0$, we may view the monodromy group as $\langle\sigma_0, \sigma_\infty\rangle$. We let n denote the order of σ_∞; note that n is either $2r+1$ or $2r$ depending on the ramification type. The relations $\sigma_0^2 = \sigma_\infty^r = 1$ and $\sigma_0\sigma_\infty = (\sigma_0\sigma_1)\sigma_0 = (\sigma_1\sigma_0)^{-1}\sigma_0 = (\sigma_\infty)^{-1}\sigma_0$ hold. The conclusion is that the monodromy groups of these dessins are isomorphic to the dihedral groups of order $2n$. □

Figure 10. The path dessins of 3 and 4 edges, respectively.

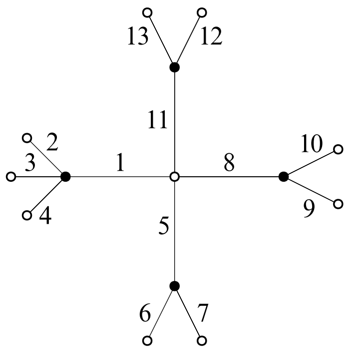

Figure 11. An example of a dessin from Proposition 3.2, where $r = 4,\ s = 3,\ t = 4$.

Proposition 3.2. *Assume $r > 1$. The ramification type $[s^{r-1}, t; r, 1^{(r-1)(s-1)+(t-1)}]$ has $n = (r-1)s + t$ edges and a unique tree with monodromy group G, with*

$$G \cong \begin{cases} C_r \wr C_s, & s = t, \\ S_{n/d} \wr C_d, & s \neq t,\ r \text{ even}, \\ A_{n/d} \wr C_d, & s \neq t,\ r \text{ is odd and } \frac{t}{d} \text{ is odd}, \\ (A_{n/d})^d \rtimes C_{2d}, & s \neq t,\ r \text{ odd},\ \frac{t}{d} \text{ even}, \end{cases}$$

where $d = \gcd(s, t)$.

Proof. The ramification type $[s^{r-1}, t; r, 1^{(r-1)(s-1)+(t-1)}]$ produces a tree of diameter 4 with $n = (r-1)s + t$ edges in the nondegenerate cases. See Figure 11.

In general, σ_0 is the product of one t-cycle and $(r-1)$-many s-cycles and σ_1 is an r-cycle. We label our edges so that we compute the permutations $\sigma_0, \sigma_1, \sigma_\infty$ as

$$\begin{aligned} \sigma_0 &= (1, \dots, t)(t+1, \dots, t+s) \\ &\qquad \cdot (t+s+1, \dots, t+2s) \cdots (t+(r-2)s+1, \dots, t+(r-1)s), \\ \sigma_1 &= (1, t+1, t+s+1, t+2s+1, \dots, t+(r-2)s+1), \\ \sigma_\infty^{-1} &= \sigma_0\sigma_1 = (1, 2, \dots, n). \end{aligned}$$

(Note that we go left to right when computing permutation products.)

Case 1: $s = t \implies G = C_r \wr C_s$. Assume $s = t$. Then our dessin is the composition of an s-star with an r-star, which means G is a subgroup of $C_r \wr C_t$ by Remark 1.8. Define $\tau_i := \sigma_0^{-i}\sigma_1\sigma_0^i$. Referring to the above where we already computed σ_0 and σ_1, we see

$$\begin{aligned} \tau_0 &= (1, t+1, 2t+1, \dots, (r-1)t+1) = \sigma_1, \\ \tau_1 &= (2, t+2, 2t+2, \dots, (r-1)t+2), \\ &\ \ \vdots \\ \tau_{t-1} &= (t, 2t, 3t, \dots, rt). \end{aligned}$$

Each τ_i is an r-cycle and generates C_r. Since the τ_i's partition $\{1, 2, \ldots, rt\}$, they must commute with each other and we see that together they generate C_r^t. Also, σ_0 is a product of t-cycles satisfying $\sigma_0^{-1}\tau_i\sigma_0 = \tau_{i+1}$, where the subscripts are reduced modulo t. These relations are sufficient to recognize that G contains $\langle\sigma_0, \tau_1, \tau_2, \ldots, \tau_{t-1}\rangle \cong C_r \wr C_t$.

Case 2: $s \neq t$, $\gcd(s,t) = 1 \implies G = A_n$ for r, t odd and $G = S_n$ otherwise. Assume that $\gcd(s,t) = 1$, with s or $t > 1$. It is known that a permutation group containing $(1,2,3)$ and $(1,2,\ldots,n)$ contains A_n; see Lemma A.1. Our goal is to show that $A_n \leq G \leq S_n$ and then use a parity argument to determine which containment is improper. Given that $\sigma_0\sigma_1 = (1,2,\ldots,n) \in G$, we proceed to show $(1,2,3) \in G$.

Assume $t = 1$ and $s > 1$. We claim $\rho := (\sigma_0^{-1}\sigma_1^{-1}\sigma_0)(\sigma_\infty\sigma_1\sigma_\infty^{-1}) = (1,2,3)$. Since $t = 1$, we know σ_0 is a product of $(r-1)$ s-cycles, while σ_1, σ_1^{-1} remain r-cycles. We see that

$$\begin{aligned}\rho &= (\sigma_0^{-1}\sigma_1^{-1}\sigma_0)(\sigma_\infty\sigma_1\sigma_\infty^{-1})\\ &= (1, (r-2)s+3, \ldots, 2s+3, s+3, 3)(2, 3, s+3, 2s+3, \ldots, (r-2)s+3).\end{aligned}$$

One may verify that $\rho(1) = 2$, $\rho(2) = 3$, $\rho(3) = 1$ and, for $k > 3$, $\rho(k) = k$. It follows that $A_n \leq G$.

If $t = 2$, we have $\sigma_0^s = (1,2) \in G$. Since G contains the transposition $(1,2)$ and the cycle $(1,2,\ldots,n)$, we have $S_n \leq G$.

Now suppose $t \geq 3$, we first set k to be the smallest positive integer such that k satisfies $k \equiv 0 \pmod s$ and $k \equiv -1 \pmod t$. The existence of such a number is guaranteed by the Chinese remainder theorem. We claim $\rho := (\sigma_1^{-1}\sigma_0^k\sigma_1)\sigma_0^k(\sigma_1^{-1}\sigma_0^{-2k}\sigma_1) = (1,2,3)$. Notice that

$$(\sigma_1^{-1}\sigma_0^k\sigma_1)\sigma_0^k(\sigma_1^{-1}\sigma_0^{-2k}\sigma_1) = (t+1, t, \ldots, 3, 2)(1, t, \ldots, 3, 2)(t+1, 2, 3, \ldots, t)^2.$$

One may verify that $\rho(1) = 2$, $\rho(2) = 3$, $\rho(3) = 1$ and $\rho(k) = k$ for $k > 3$. Thus $\rho = (1,2,3) \in G$ and therefore $A_n \subseteq G$.

For every triple s, t such that $\gcd(s,t) = 1$ and s or $t > 1$, we have shown that $A_n \subseteq G$. Since we also have $G \leq S_n$, by index considerations G is either the symmetric or alternating group of appropriate order. Otherwise if r or t is even, σ_0, being the product of a t-cycle and $(r-1)$ s-cycles, is an odd permutation (note s must be odd if t is even), so $G \cong S_n$. Since both σ_0 and σ_1 are even permutations when r and t are odd, we deduce that $G \leq A_n$ and thus the double inclusion gives us $G \cong A_n$.

Case 3: In this final case, we assume $\gcd(s,t) = d > 1$. This tree is the composition $P \circ Q$, where P is the d-star and Q is the dessin corresponding to the passport

$$\left[\left(\frac{s}{d}\right)^{r-1}, \frac{t}{d};\ r,\ 1^{(r-1)(s/d-1)+(t/d-1)}\right].$$

Hence, the monodromy group G is a subgroup of the wreath product $G_Q \wr C_d$, where G_Q is the monodromy group for Q.

Consider the partition of $\{1,\dots,n\}$ into the d sets

$$\{1, d+1, \dots, n-d+1\}, \quad \{2, d+2, \dots, n-d+2\}, \quad \dots, \quad \{d, 2d, \dots, n\},$$

each of size $\frac{n}{d}$, and denote them by $P_1, \dots, P_d$ respectively. Recall that σ_0 is the disjoint product of a t-cycle and $(r-1)$ s-cycles, and moreover every element in $\{1,2,\dots,n\}$ is moved by exactly one of these cycles under the canonical group action. Because d divides both s and t, we know $\tau := \sigma_0^d$ is the disjoint product of d $\frac{t}{d}$-cycles and $d(r-1)$ $\frac{s}{d}$-cycles. Moreover, each disjoint cycle of τ permutes elements in exactly one of the P_i while fixing the rest. Similarly, because d divides n, we know σ_∞^d is the disjoint product of d $\frac{n}{d}$-cycles, and each disjoint cycle of σ_∞^d likewise permutes elements in exactly one of the P_i. Note that σ_1 permutes only the elements of P_1.

Let k be the smallest positive integer such that k satisfies $k \equiv 0 \left(\bmod \frac{s}{d}\right)$ and $k \equiv -1 \left(\bmod \frac{t}{d}\right)$. One may verify that $\rho := \sigma_1^{-1}\tau^k\sigma_1\tau^k\sigma_1^{-1}\tau^{-2k}\sigma_1 = (1, d+1, 2d+1)$. (Note that in the case where $t = d$, we let $\rho := (\tau^{-1}\sigma_1^{-1}\tau)(\sigma_\infty^d\sigma_1\sigma_\infty^{-d})$ and proceed with the same argument.)

We can conclude that the subgroup

$$N = \langle \rho,\ \sigma_\infty^{-d}\rho\sigma_\infty^{d},\ \sigma_\infty^{-2d}\rho\sigma_\infty^{2d},\ \dots,\ \sigma_\infty^{-(n-d)}\rho\sigma_\infty^{n-d},\ \sigma_1 \rangle$$

is isomorphic to $S_{n/d}$ when r is even and isomorphic to $A_{n/d}$ when r is odd (see Lemma A.4). Furthermore, we observe that N, $\sigma_\infty^{-1}N\sigma_\infty$, $\sigma_\infty^{-2}N\sigma_\infty^2, \dots,$ $\sigma_\infty^{-d+1}N\sigma_\infty^{d-1}$ are all isomorphic to $S_{n/d}$ or $A_{n/d}$ (depending on whether r is even or odd) and $\sigma_\infty^{-i+1}N\sigma_\infty^{i+1}$ permutes elements of P_i. Hence

$$H := \langle N,\ \sigma_\infty^{-1}N\sigma_\infty,\ \sigma_\infty^{-2}N\sigma_\infty^2,\ \dots,\ \sigma_\infty^{-d+1}N\sigma_\infty^{d-1}\rangle \cong \begin{cases} (S_{n/d})^d & \text{if } r \text{ even,} \\ (A_{n/d})^d & \text{if } r \text{ odd.} \end{cases}$$

One can check that $\sigma_0^{-1}H\sigma_0 = H$ and $\sigma_1^{-1}H\sigma_1 = H$. Hence, $H \triangleleft G$. Observe that $\sigma_1 \in H$. Therefore, $H\sigma_\infty$ generates the quotient group $G\backslash H$. When r is even, the smallest power of σ_∞ in H is d, when r is odd and $\frac{n}{d}$ is odd, the smallest power of σ_∞ in H is d, and when r is odd and $\frac{n}{d}$ is even, the smallest power of σ_∞ in H is actually $2d$. (Note that when r is odd, the parities of $\frac{t}{d}$ and $\frac{n}{d}$ are the same.) In order to show that G is isomorphic to a semidirect product, we will use the splitting lemma. In our case, if we can find an element of $H\sigma_\infty$ of order d or $2d$ (depending on the case), we have shown G is a semidirect product.

First we consider the case where r is even. In this case, observe that

$$\begin{aligned}(d, 2d, 3d, \dots, n)^{-1}&(1,2,3,4,\dots,n) \\ &= (1,2,\dots,d)(d+1, d+2, \dots, 2d)\cdots(n-d+1, n-d+2, \dots, n).\end{aligned}$$

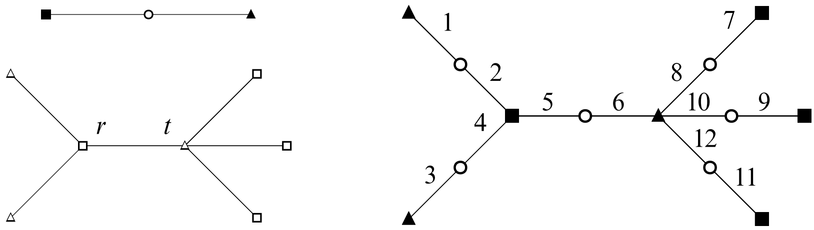

Figure 12. P and Q on the left; $P \circ Q$ on the right.

Hence, there is an element of order d in $G \backslash H$ and $G \cong (S_{n/d})^d \rtimes C_d$ by the splitting lemma for semidirect products, and in fact $G \cong S_{n/d} \wr C_d$. This also shows that $H\sigma_\infty$ contains an element of order d in the case where r and $\frac{n}{d}$ are odd since the cycle $(d, 2d, 3d, \ldots, n)$ is an element of A_n in this case. Therefore, when $\frac{n}{d}$ and r are odd, $G \cong (A_{n/d})^d \rtimes C_d$ and in fact $G \cong A_{n/d} \wr C_d$.

Now we consider the case where r is odd and $\frac{n}{d}$ is even. Observe that

$$(2d, 3d, \ldots, n)^{-1}(1, 2, 3, \ldots, n)$$
$$= (1, 2, 3, \ldots, 2d)(2d+1, 2d+2, \ldots, 3d) \cdots (n-d+1, n-d+2, \ldots, n).$$

Hence, there is an element of order $2d$ in $G \backslash H$ and thus $G \cong (A_{n/d})^d \rtimes C_{2d}$. □

Proposition 3.3. *Let $r, t > 1$. The ramification type $[r, t, 1^{r+t-2}; 2^{r+t-1}]$ produces a unique tree with monodromy group G, where*

$$G \cong \begin{cases} A_{2r-1} \times C_2, & r = t,\ r \text{ odd}, \\ S_{2r-1} \times C_2, & r = t,\ r \text{ even}, \\ A_{r+t-1} \wr C_2, & r \neq t, \text{ both odd}, \\ R_2, & r \neq t, \text{ both even}, \\ S_{r+t-1} \wr C_2, & r \neq t, \text{ else}, \end{cases}$$

where R_2 denotes the index-2 subgroup of $S_{r+t-1} \wr C_2$ such that $\tau_1\tau_2$ is an even permutation for all $(\tau_1, \tau_2, g) \in R_2$.

Proof. First, we note that this dessin is the composition $P \circ Q$, where P is the 2-star and Q is the dessin of Proposition 3.2 with $s = 1$. See Figure 12.

Let $G_Q = \langle (1, 2, \ldots, r), (r, r+1, \ldots, r+t-1) \rangle$ be the monodromy group of Q. By Proposition 3.2, we know that

$$G_Q \cong \begin{cases} A_{r+t-1}, & r, t \text{ both odd}, \\ S_{r+t-1}, & \text{otherwise}. \end{cases}$$

The dessin with ramification type $[r, t, 1^{r+t-2}; 2^{2r-1}]$ is the composition of P and Q, and so its monodromy group G satisfies $G \leq G_Q \wr C_2$ by Remark 1.8. We consider G in two cases: $r = t$ and $r \neq t$.

Case 1: $r \neq t$. In the first case, we have $r \neq t$. We label our edges in such a way that

$$\sigma_0 = (1, 2, \ldots, r)(\bar{r}, \overline{r+1}, \ldots, \overline{r+t-1}),$$
$$\sigma_1 = (1, \bar{1})(2, \bar{2}) \cdots (r+t-1, \overline{r+t-1}).$$

Note that σ_0 is the disjoint product of an r-cycle with a t-cycle; call these cycles π_1 and π_2 respectively. Consider the embedding $\phi : G \to S_{r+t-1} \wr C_2$ given by

$$\sigma_0 \mapsto (\pi_1, \pi_2, 0),$$
$$\sigma_1 \mapsto (\mathrm{id}, \mathrm{id}, 1).$$

Note that $\sigma_1^{-1}\sigma_0\sigma_1$ is mapped to $(\pi_2, \pi_1, 0)$. Apply Lemma A.5 to $n = r + t - 1$ (assume $n \geq 5$ for now), $\pi_1, \pi_2 \in S_{r+t-1}$. We have $G_Q = \langle \pi_1, \pi_2 \rangle \geq A_n$ as noted above. Lemma A.5 implies

$$A_{r+t-1} \wr C_2 \leq \phi(G) \leq S_{r+t-1} \wr C_2.$$

When r, t are odd, both π_1 and π_2 are even permutations, and we see that $\phi(G) \cong A_{r+t-1} \wr C_2$. When r and t have different parity, we know $\langle \pi_1, \pi_2 \rangle \cong S_{r+t-1}$, so $\phi(G) \cong S_{r+t-1} \wr C_2$. When r, t are both even, for any $(\rho_1, \rho_2, g) \in \phi(G)$, ρ_1 and ρ_2 will share the same parity. Since we can take $\rho_1 = \pi_1$, an odd permutation, we see that $\phi(G)$ is properly contained in between $A_{r+t-1} \wr C_2$ and $S_{r+t-1} \wr C_2$. It is in fact the group R_2 described earlier after Theorem 1.1. In the finite number of cases where $r + t - 1 < 5$, one can verify the result by hand.

Case 2: $r = t$. In the second case, we consider $r = t$. We can label our dessin in such a way that

$$\sigma_0 = (1, 2, \ldots, r)(\bar{1}, \bar{2}, \ldots, \bar{r}),$$
$$\sigma_1 = (1, \overline{r+1})(2, \overline{r+2}) \cdots (r-1, \overline{2r-1})(r, \bar{r})(r+1, \bar{1}) \cdots (2r-1, \overline{r-1}).$$

Observe that

$$\sigma_\infty^{(2r-1)} = (1, \bar{1}) \cdots (2r-1, \overline{2r-1}),$$
$$\tau_1 = \sigma_1\sigma_0\sigma_1^{-1} = (r, r+1, \ldots, 2r-1)(\bar{r}, \overline{r+1}, \ldots, \overline{2r-1}),$$
$$\tau_2 = \sigma_\infty^{(2r-1)}\sigma_1 = (\bar{1}, \overline{r+1})(1, r+1) \cdot (\bar{2}, \overline{r+2})(2, r+2) \cdots (\overline{r-1}, \overline{2r-1})(r-1, 2r-1)(r)(\bar{r}),$$

and $G_Q = \langle \sigma_\infty^{(2r-1)}, \tau_1, \tau_2 \rangle$ is a subgroup of $S_{2r-1} \times \mathbb{Z}_2$. Furthermore,

$$\tau_3 = \tau_2\sigma_1\sigma_0\sigma_1^{-1}\tau_2^{-1} = (1, 2, \ldots, r)(\bar{1}, \bar{2}, \ldots, \bar{r}).$$

By Proposition 3.2, we see that $\langle \tau_1, \tau_3 \rangle$ is S_{2r-1} if r even and A_{2r-1} if r odd, and thus we have our result. □

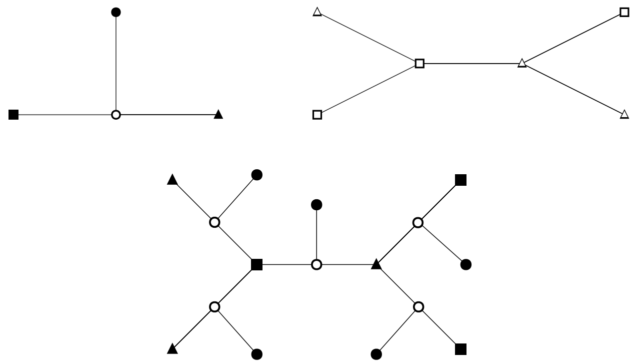

Figure 13. Top, left: P, with vertices marked. Top, right: Q, with vertices marked. Bottom: An example of the composition for $r = 3$.

Proposition 3.4. *The ramification type* $[r^2, 1^{4r-3}; 3^{2r-1}]$ *produces a unique tree with monodromy group* G, *where*

$$G \cong \begin{cases} A_{2r-1} \wr C_3, & r \text{ odd}, \\ R_3, & r \text{ even}, \end{cases}$$

where R_3 *denotes the index-2 subgroup of* $S_{2r-1} \wr C_3$ *such that* $\tau_1\tau_2\tau_3$ *is an even permutation for all* $(\tau_1, \tau_2, \tau_3, g) \in R_3$.

Proof. The procedure here is similar to the proof of the previous proposition. We observe that this dessin is the composition $P \circ Q$, where P is the 3-star with ramification type $[1^3; 3]$ and Q is the dessin from Proposition 3.2 where $s = 1$, $r = t$. See Figure 13.

We can label the dessin so that

$$\sigma_0 = (1, 2, \dots, r)(\bar{r}, \overline{r+1}, \dots, \overline{2r-1}),$$

$$\sigma_1 = (1, \bar{1}, \hat{1})(2, \bar{2}, \hat{2}) \cdots (2r-1, \overline{2r-1}, \widehat{2r-1}).$$

Note that σ_0 is the product of two r-cycles (call them π_1 and π_2 respectively) and that σ_1 is the product of $(2r-1)$ 3-cycles. Consider the embedding $\phi : G \to S_{2r-1} \wr C_3$ defined by

$$\sigma_0 \mapsto (\pi_1, \pi_2, \mathrm{id}, 0),$$

$$\sigma_1 \mapsto (\mathrm{id}, \mathrm{id}, \mathrm{id}, 1).$$

Under this homomorphism, successive conjugations of σ_0 by σ_1 are mapped to $(\mathrm{id}, \pi_1, \pi_2, 0)$ and $(\pi_2, \mathrm{id}, \pi_1, 0)$. Applying Lemma A.5 to π_1, π_2, and $\phi(G)$, we have $A_{2r-1} \wr C_3 \le \phi(G)$. When r is odd, both π_1 and π_2 are even permutations, so $A_{2r-1} \wr C_3 \ge \phi(G)$, giving a double inclusion. When r is even, we consider the quotient group

$$(S_{2r-1} \wr C_3)/(A_{2r-1} \wr C_3) \cong C_2 \times C_2 \times C_2.$$

Observe that when r is even, $\phi(G) \leq R_3$ and $(\pi_1, \pi_2, \mathrm{id}, 0)$ is equal to $(1, 1, 0)$ in the quotient group $\phi(G)/A_{2r-1} \wr C_3$. We similarly have $(0, 1, 1)$ and $(1, 0, 1)$ in the quotient group. Hence, we see that $\phi(G)$ is an index-2 subgroup of $S_{2r-1} \wr C_3$ and thus $\phi(G) \geq R_3$. □

Proposition 3.5. *The ramification type* $[3^3, 1^5; 2^7]$ *produces a unique tree with monodromy group* $G \cong A_7 \wr C_2$.

Proof. This is a sporadic case that may be verified by hand. □

4. Future directions

The reader will notice that there are some obvious pathways left open by this paper. In Theorem 1.1 each entry refers to a tree uniquely determined by ramification type. For each entry there exists a Shabat polynomial with rational coefficients. However, we were not able to find a closed form expression for the coefficients of the rational Shabat polynomial given for the tree with ramification type $[r^2, 1^{4r-3}; 3^{2r-1}]$.

As for another direction of further inquiry, we note that the present paper focuses exclusively on (planar) trees uniquely determined by ramification type. However, we know that there exists an exhaustive list of ramification types that produce exactly two distinct trees, and perhaps there are other such lists for ramification types that produce larger numbers of trees [Shabat and Zvonkin 1994]. At the very least, it would be interesting to see the complete list of monodromy groups for ramification types that produce two trees in comparison with the completion of Theorem 1.1. Finally, it would also be of interest to see similar results for classes of dessins having at least one cycle or for dessins with genus greater than 1.

Appendix

In this section we prove a few technical results used in the paper. We learned of the following results (Lemmas A.1, A.2, A.3, A.4) and their proofs from Keith Conrad. Recall that we multiply permutations left to right.

Lemma A.1. *For* $n \geq 5$, *the subgroup generated by* $(1, 2, 3)$ *and* $(1, 2, \ldots, n)$ *contains* A_n.

We prove this lemma through a sequence of lemmas.

Lemma A.2. *For* $n \geq 5$, *every element of* A_n *is a product of* 3*-cycles.*

Proof. The set of 3-cycles is a conjugacy class that is a subset of A_n. Therefore, the subgroup generated by the set of 3-cycles is a normal subgroup of A_n. Since A_n is simple for $n \geq 5$, we conclude that the set of 3-cycles generates A_n and every element of A_n is a product of 3-cycles. □

Lemma A.3. *For* $n \geq 5$, *the group* A_n *is generated by elements of the form* $(1, 2, k)$.

Proof. First observe that A_n is generated by 3-cycles of the form $(1,i,j)$. This is easily seen by observing that for any 3-cycle (a,b,c) not containing 1, we have $(a,b,c)=(1,b,c)(1,a,b)$. By Lemma A.2 we see that A_n is generated by 3-cycles of the form $(1,i,j)$.

Now we consider the 3-cycles of the form $(1,2,k)$. Since $(1,2,k)^{-1}=(1,k,2)$, any 3-cycle with 1 and 2 is generated by 3-cycles of the form $(1,2,k)$. For a 3-cycle $(1,i,j)$ not containing 2, we have $(1,i,j)=(1,2,j)(1,2,i)(1,2,j)(1,2,j)$. Hence, every element of A_n is generated by elements of the form $(1,2,k)$. □

Lemma A.4. *For $n \geq 5$, the consecutive 3-cycles $(i,i+1,i+2)$ with $1 \leq i \leq n-2$ generate A_n.*

Proof. This can be shown to be true for A_5 by computation. We proceed to prove this for $n>5$ by induction.

Assume this is true for A_n. Consider A_{n+1}. By induction, we know that cycles of the form $(i,i+1,i+2)$ generate the elements $(1,2,k)$ for $3 \leq k \leq n$. Therefore, by Lemma A.3, we need only show that we can generate $(1,2,n+1)$ in order to show that cycles of the form $(i,i+1,i+2)$ generate A_{n+1}. Observe that $(1,2,n+1)=(1,2,n)(1,2,n-1)(n-1,n,n+1)(1,2,n)(1,2,n-1)$ and thus we have proven our result. □

Now we proceed with the proof of Lemma A.1

Proof of Lemma A.1. Let $\sigma=(1,2,\dots,n)$. Observe that

$$\sigma^{-k}(1,2,3)\sigma^k=(\sigma^k(1),\sigma^k(2),\sigma^k(3))=(k+1,k+2,k+3)$$

if $0 \leq k \leq n-3$. Thus by Lemma A.4, $(1,2,3)$ and $(1,2,\dots,n)$ generate a subgroup that contains A_n. □

Lemma A.5. *Suppose that $\pi_0,\pi_1 \in S_n$ with $\langle \pi_0,\pi_1\rangle \geq A_n$ with $n \geq 5$.*

(1) *If $|\pi_0| \neq |\pi_1|$, then $\Gamma=\langle(\pi_0,\pi_1),(\pi_1,\pi_0)\rangle$ must contain $A_n \times A_n$.*

(2) *$\Gamma=\langle(\pi_0,\pi_1,\mathrm{id}),(\mathrm{id},\pi_0,\pi_1),(\pi_1,\mathrm{id},\pi_0)\rangle$ must contain $A_n\times A_n\times A_n$.*

Proof. Suppose that $\mathrm{id}\neq\rho\in A_n$. Observe that $\langle\tau^{-1}\rho\tau:\tau\in A_n\rangle$ is a normal subgroup of A_n. If $n\geq 5$, then A_n is simple and therefore, $A_n=\langle\tau^{-1}\rho\tau:\tau\in A_n\rangle$.

First, we consider statement (1). Suppose that $(\rho,\mathrm{id})\in\Gamma$. We want to show that $A_n\times\langle\mathrm{id}\rangle$ is a subgroup of Γ. There is a homomorphism $\mathrm{proj}: S_n\times S_n\to S_n$, which is a projection from the first component. Since $A_n\leq\langle\pi_0,\pi_1\rangle$, we have $\mathrm{proj}(\Gamma)\geq A_n$. Therefore, for all $\tau\in A_n$ there exists $\tau'\in S_n$ such that $(\tau,\tau')\in\Gamma$. Conjugating (ρ,id) by all (τ,τ') shows that $A_n\times\langle\mathrm{id}\rangle\leq\Gamma$. Note that the same argument can be used to show $\langle\mathrm{id}\rangle\times A_n\leq\Gamma$ via projection in the other component. Statement (1) then follows as long as $\rho\neq\mathrm{id}$ exists. Furthermore, the argument to establish statement (2) would proceed in an identical fashion, presuming $\rho\neq\mathrm{id}$ exists.

To establish existence of ρ in the case of statement (1), we claim that there exists an element of the form $(\rho, \mathrm{id}) \in \Gamma$, where $\rho \neq \mathrm{id}$. Without loss of generality, assume $|\pi_0| > |\pi_1|$, and then consider $(\pi_0, \pi_1)^{|\pi_1|}, (\pi_1, \pi_0)^{|\pi_1|}$, in which case we may let $\rho = \pi_0^{|\pi_1|}$.

Now we prove such an element exists in the case of statement (2) for $n > 2$. If $|\pi_0| \neq |\pi_1|$, then the proof is analogous to the argument for statement (1). Otherwise $|\pi_0| = |\pi_1| = r$ and we want to find some element $\pi \in A_n$ such that $|\pi| \nmid r$. One can show that such a π exists by proving that, for $n > 2$, there must be some prime q *not* dividing $|\pi_0| = r$. One can show q exists by using the fact that

$$n < \sum_{\substack{p \leq n \\ p \text{ prime}}} p \quad \text{for } n > 2.$$

Using all three generators of Γ, one can produce the element $(\pi_0^{k_1}, \pi, \pi_1^{k_2}) \in A_n^3$, where $k_1, k_2 \in \mathbb{Z}$. By raising this element to the r-th power, we produce the element $(\mathrm{id}, \pi^r, \mathrm{id}) \in \Gamma$ and let $\rho = \pi^r$. □

Corollary A.6. *Let H be a simple group. Suppose $\pi_0, \pi_1 \in S_n$ with $\langle \pi_0, \pi_1 \rangle \geq H$.*

(1) *If $|\pi_0| \neq |\pi_1|$, then $\Gamma = \langle (\pi_0, \pi_1), (\pi_1, \pi_0) \rangle$ must contain $H \times H$.*

(2) *$\Gamma = \langle (\pi_0, \pi_1, \mathrm{id}), (\mathrm{id}, \pi_0, \pi_1), (\pi_1, \mathrm{id}, \pi_0) \rangle$ must contain $H \times H \times H$.*

Remark A.7. In [Adrianov et al. 1997], Adrianov, Kochetkov, and Suvorov classify all the possible primitive, and thus simple, monodromy groups of plane trees.

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