

involve

a journal of mathematics

Spectra of Kohn Laplacians on spheres

John Ahn, Mohit Bansil, Garrett Brown,
Emilee Cardin and Yunus E. Zeytuncu



Spectra of Kohn Laplacians on spheres

John Ahn, Mohit Bansil, Garrett Brown,
Emilee Cardin and Yunus E. Zeytuncu

We study the spectrum of the Kohn Laplacian on the unit spheres in $\mathbb{C}^n$ and revisit Folland's classical eigenvalue computation. We also look at the growth rate of the eigenvalue counting function in this context. Finally, we consider the growth rate of the eigenvalues of the perturbed Kohn Laplacian on the Rossi sphere in $\mathbb{C}^2$.

1. Introduction

Background. The unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ is a CR manifold (of hypersurface type) with the CR structure induced from the ambient space. By following the standard setting we define the tangential Cauchy–Riemann complex with the operators $\bar{\partial}_b$ and $\bar{\partial}_b^*$ on the spaces of square integrable $(0, q)$ -forms $L^2_{(0,q)}(\mathbb{S}^{2n-1})$. (For simplicity we restrict our attention to $(0, q)$ forms instead of (p, q) forms.) The Kohn Laplacian (or $\bar{\partial}_b$ -Laplacian)

$$\square_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$$

is a linear, closed, densely defined self-adjoint operator from $L^2_{(0,q)}(\mathbb{S}^{2n-1})$ to itself. The analytic properties of this second-order differential operator are closely related to the geometry of the underlying manifold (although we work here on $\mathbb{S}^{2n-1}$, the same setup works on other CR manifolds). We refer the reader to [Chen and Shaw 2001, Chapter 7] for the details of this setup.

Spherical harmonics. We now list definitions and theorems that are needed in the rest of the paper. For a detailed study of spherical harmonics we refer the reader to [Axler et al. 1992].

We say a complex polynomial $p(z)$ is homogeneous of degree k if $p(\lambda z) = \lambda^k p(z)$ for all $z \neq 0$. Similarly, $p(z, \bar{z})$ is called homogeneous of bidegree (p, q) if $f(\lambda_1 z, \lambda_2 \bar{z}) = \lambda_1^p \lambda_2^q p(z, \bar{z})$ for all $z \neq 0$. We say a twice-differentiable function f

MSC2010: primary 32V05; secondary 32V30.

Keywords: Kohn Laplacian, spherical harmonics, Gershgorin's circle theorem.

This work is supported by NSF (DMS-1659203). The work of Cardin is also partially supported by a grant from the Simons Foundation (#353525).

is harmonic if $\Delta f = 0$, where the Laplacian is defined by

$$\Delta f = 4 \sum_{i=1}^n \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i}.$$

A spherical harmonic is the restriction to $\mathbb{S}^{2n-1}$ of a complex polynomial that is harmonic on $\mathbb{C}^n$. We use $\mathcal{H}_k(\mathbb{C}^n)$ to denote the space of all harmonic, homogeneous polynomials of degree k on $\mathbb{C}^n$ and $\mathcal{H}_{p,q}(\mathbb{C}^n)$ for the space of all harmonic, homogeneous polynomials of bidegree (p, q) . Similarly we use $\mathcal{H}_k(\mathbb{S}^{2n-1})$ and $\mathcal{H}_{p,q}(\mathbb{S}^{2n-1})$ to denote the restrictions of these spaces on $\mathbb{S}^{2n-1}$. The following decomposition theorem is fundamental in our study of $\square_b$ on $L^2(\mathbb{S}^{2n-1})$.

Theorem 1.1 [Klima 2004, Theorem 3.7]. *The spaces $\mathcal{H}_{p,q}(\mathbb{S}^{2n-1})$ are pairwise orthogonal, and*

$$L^2(\mathbb{S}^{2n-1}) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}_{p,q}(\mathbb{S}^{2n-1}).$$

By using a standard counting argument one obtains the following formula for the dimensions of the spaces of spherical harmonics.

Lemma 1.2 [Klima 2004, Corollary 3.10]. *For $p, q \geq 1$,*

$$\begin{aligned} \dim(\mathcal{H}_{p,q}(\mathbb{S}^{2n-1})) &= \binom{n+p-1}{p} \binom{n+q-1}{q} - \binom{n+p-2}{p-1} \binom{n+q-2}{q-1} \\ &= \frac{(n-1)(n+p+q-1)}{pq} \binom{n+p-2}{p-1} \binom{n+q-2}{q-1}. \end{aligned}$$

Notation. In the rest of the note we use the standard Ω and O notation to denote asymptotic lower and upper bounds, respectively. That is, given two functions f and g , we say $f = \Omega(g)$ if there exists a constant $c > 0$ such that $f(x) \geq cg(x)$ as $x \rightarrow \infty$. Similarly, $f = O(g)$ if there exists $c > 0$ such that $f(x) \leq cg(x)$ as $x \rightarrow \infty$. Finally, we say $f = \Theta(g)$ if $f = \Omega(g)$ and $f = O(g)$.

Results. Folland [1972] computed the eigenvalues and eigenforms of $\square_b$ on $L^2_{(0,q)}(\mathbb{S}^{2n-1})$ by using unitary representations.

Theorem 1.3. $\mathcal{H}_{p,q}(\mathbb{S}^{2n-1})$ is an eigenspace for $\bar{\partial}_b^* \bar{\partial}_b$ with the associated eigenvalue $2q(p+n-1)$.

In Section 2 of this note we go over these computations on the space of square integrable functions (i.e., $L^2(\mathbb{S}^{2n-1})$) by using spherical harmonics and present eigenvalue computations in an accessible way. This more elementary approach enables us to write code¹ in SymPy that computes the eigenvalues of $\square_b$ and other similar second-order differential operators defined on $L^2(\mathbb{S}^{2n-1})$. Furthermore, by

¹The code can be downloaded at <https://goo.gl/kBsUzA>.

using the explicit forms of the eigenvalues and formulas for the dimensions of spherical harmonic subspaces of $L^2(\mathbb{S}^{2n-1})$, we study the growth rate for the counting function of the eigenvalues. For $m \in \mathbb{Z}$, let $N(m)$ be the number of eigenvalues of $\square_b$ on $L^2(\mathbb{S}^{2n-1})$ that are less than or equal to m , counting multiplicity.

Theorem 1.4. *There exists a real $c > 0$ so that $\frac{1}{c}m^n \leq N(m) \leq cm^n$; that is, $N(m) = \Theta(m^n)$.*

In other words, here we prove that

$$\limsup_{m \rightarrow \infty} \frac{N(m)}{m^n} \in (0, \infty).$$

It would be interesting to compute the exact limit and check if it is related to the surface area of $\mathbb{S}^{2n-1}$. Indeed, in the case of the Laplace–Beltrami operator, Weyl’s law states that this ratio is the surface area of $\mathbb{S}^{2n-1}$.

In addition to the induced CR structure from the ambient manifold, one can define different intrinsic CR structures on a given manifold; see [Bogges 1991, Chapter 8]. The most famous example of these abstract CR manifolds is the Rossi sphere. It is known that the Rossi sphere is not globally CR embeddable into any $\mathbb{C}^n$ [Burns 1979]. This can be seen by explicitly studying the perturbed Kohn Laplacian (defined by the abstract CR structure) and looking at its essential spectrum. In [Abbas et al. 2019], the authors studied the bottom of the spectrum of the perturbed Kohn Laplacian by using spherical harmonics. In the last section of this note we continue this study and provide the growth rate of the largest eigenvalues from each subspace of spherical harmonics.

2. Eigenvalues of $\square_b$ on $L^2(\mathbb{S}^{2n-1})$

Explicit eigenvalue computation. Since $\bar{\partial}_b^*$ is identically zero on $L^2(\mathbb{S}^{2n-1})$, $\square_b$ simplifies on $L^2(\mathbb{S}^{2n-1})$ as

$$\square_b = \bar{\partial}_b^* \bar{\partial}_b.$$

Before we compute the eigenvalues we present the operators $\bar{\partial}_b$ and $\bar{\partial}_b^*$ in coordinate forms. A smooth differential 1-form ω on $\mathbb{S}^{2n-1}$ can be expressed as

$$\omega = \sum_{k=1}^n (A_k dz_k + B_k d\bar{z}_k) = A_1 dz_1 + B_1 d\bar{z}_1 + \cdots + A_n dz_n + B_n d\bar{z}_n,$$

where $A_k, B_k \in C^\infty(\mathbb{C}^n)$. As computed in [Folland 1972], for a smooth function f on $\mathbb{S}^{2n-1}$ we have

$$\bar{\partial}_b f = \sum_{i=1}^n \left(\frac{\partial f}{\partial \bar{z}_i} - z_i \sum_{a=1}^n \bar{z}_a \frac{\partial f}{\partial \bar{z}_a} \right) d\bar{z}_i.$$

Furthermore, following the normalization of inner products as in [Folland 1972] we have

$$\langle d\bar{z}_i, d\bar{z}_j \rangle = 2\delta_{ij} \quad \text{and} \quad \langle dz_i, d\bar{z}_j \rangle = 0.$$

Using integration by parts, we obtain the following expression for the adjoint operator.

Lemma 2.1. *For a smooth 1-form $\omega = \sum_{k=1}^n (A_k dz_k + B_k d\bar{z}_k)$,*

$$\bar{\partial}_b^* \omega = -2 \sum_{i=1}^n \left(\frac{\partial}{\partial z_i} B_i - \sum_{a=1}^n \frac{\partial}{\partial z_a} z_a \bar{z}_i B_i \right).$$

Proof. Let g be a smooth function on $\mathbb{S}^{2n-1}$. Since we are working on a compact manifold, we don't get any boundary terms when we integrate by parts:

$$\begin{aligned} & \left\langle \bar{\partial}_b^* \left(\sum_{k=1}^n (A_k dz_k + B_k d\bar{z}_k) \right), g \right\rangle \\ &= \left\langle \sum_{k=1}^n (A_k dz_k + B_k d\bar{z}_k), \bar{\partial}_b g \right\rangle \\ &= \left\langle \sum_{k=1}^n A_k dz_k + \sum_{k=1}^n B_k d\bar{z}_k, \sum_{i=1}^n \left(\frac{\partial g}{\partial \bar{z}_i} - z_i \sum_{a=1}^n \bar{z}_a \frac{\partial g}{\partial \bar{z}_a} \right) d\bar{z}_i \right\rangle \\ &= 2 \sum_{i=1}^n \left\langle B_i, \frac{\partial g}{\partial \bar{z}_i} - z_i \sum_{a=1}^n \bar{z}_a \frac{\partial g}{\partial \bar{z}_a} \right\rangle \\ &= 2 \sum_{i=1}^n \left(\left\langle B_i, \frac{\partial g}{\partial \bar{z}_i} \right\rangle - \sum_{a=1}^n \left\langle B_i, z_i \bar{z}_a \frac{\partial g}{\partial \bar{z}_a} \right\rangle \right) \\ &= 2 \sum_{i=1}^n \left(- \left\langle \frac{\partial}{\partial z_i} B_i, g \right\rangle + \sum_{a=1}^n \left\langle \frac{\partial}{\partial z_a} z_a \bar{z}_i B_i, g \right\rangle \right) \\ &= -2 \sum_{i=1}^n \left(\left\langle \frac{\partial}{\partial z_i} B_i, g \right\rangle - \sum_{a=1}^n \left\langle \frac{\partial}{\partial z_a} z_a \bar{z}_i B_i, g \right\rangle \right) \\ &= -2 \sum_{i=1}^n \left\langle \frac{\partial}{\partial z_i} B_i - \sum_{a=1}^n \frac{\partial}{\partial z_a} z_a \bar{z}_i B_i, g \right\rangle \\ &= \left\langle -2 \sum_{i=1}^n \left(\frac{\partial}{\partial z_i} B_i - \sum_{a=1}^n \frac{\partial}{\partial z_a} z_a \bar{z}_i B_i \right), g \right\rangle. \end{aligned}$$

By comparing the beginning and ending of the identity we prove the lemma. $\square$

Before we look at the action of $\square_b$ on a square integrable function we look at the action of two other operations on the spherical harmonics.

Lemma 2.2. *If $f \in \mathcal{H}_{p,q}(\mathbb{S}^{2n-1})$, then*

$$\sum_{k=1}^n z_k \frac{\partial f}{\partial z_k} = pf \quad \text{and} \quad \sum_{k=1}^n \bar{z}_k \frac{\partial f}{\partial \bar{z}_k} = qf.$$

Proof. Consider a polynomial $f \in \mathcal{H}_{p,q}$. So f is harmonic homogeneous of bidegree p, q . Then for each monomial term $g = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n}$ of f , we have

$$\begin{aligned} \sum_{k=1}^n z_k \frac{\partial g}{\partial z_k} &= \sum_{k=1}^n (\alpha_k) g = \left(\sum_{k=1}^n \alpha_k \right) g = pg, \\ \sum_{k=1}^n \bar{z}_k \frac{\partial g}{\partial \bar{z}_k} &= \sum_{k=1}^n (\beta_k) g = \left(\sum_{k=1}^n \beta_k \right) g = qg. \end{aligned}$$

So each monomial term g is scaled by p or q . By the linearity of differential operators, f is scaled by p or q as well. $\square$

By combining the lemmas above we obtain the eigenvalues of $\square_b$.

Theorem 1.3. $\mathcal{H}_{p,q}(\mathbb{S}^{2n-1})$ is an eigenspace for $\bar{\partial}_b^* \bar{\partial}_b$ with the associated eigenvalue $2q(p+n-1)$.

Proof. For $f \in \mathcal{H}_{p,q}(\mathbb{S}^{2n-1})$,

$$\begin{aligned} \bar{\partial}_b^* \bar{\partial}_b f &= \bar{\partial}_b^* \left[\sum_{i=1}^n \left(\frac{\partial f}{\partial \bar{z}_i} - z_i \sum_{a=1}^n \bar{z}_a \frac{\partial f}{\partial \bar{z}_a} \right) d\bar{z}_i \right] \\ &= \bar{\partial}_b^* \left[\sum_{i=1}^n \left(\frac{\partial f}{\partial \bar{z}_i} - z_i qf \right) d\bar{z}_i \right] \\ &= -2 \sum_{i=1}^n \left[\frac{\partial}{\partial z_i} \left(\frac{\partial f}{\partial \bar{z}_i} - z_i qf \right) - \sum_{a=1}^n \frac{\partial}{\partial z_a} z_a \bar{z}_i \left(\frac{\partial f}{\partial \bar{z}_i} - z_i qf \right) \right] \\ &= -2 \sum_{i=1}^n \left[\left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_i} - \frac{\partial}{\partial z_i} z_i qf \right) - \sum_{a=1}^n \left(\frac{\partial}{\partial z_a} z_a \bar{z}_i \frac{\partial f}{\partial \bar{z}_i} - \frac{\partial}{\partial z_a} z_a \bar{z}_i z_i qf \right) \right] \\ &= -2 \sum_{i=1}^n \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i} + 2 \sum_{i=1}^n \frac{\partial}{\partial z_i} z_i qf + 2 \sum_{i=1}^n \sum_{a=1}^n \frac{\partial}{\partial z_a} z_a \bar{z}_i \frac{\partial f}{\partial \bar{z}_i}. \end{aligned}$$

We start with the first term. Because f is harmonic, we know

$$0 = \Delta(f) = 4 \sum_{i=1}^n \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i}.$$

Thus, we have

$$0 = \sum_{i=1}^n \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i} = -2 \sum_{i=1}^n \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i}.$$

For the second and third terms, we apply the product rule:

$$\begin{aligned} 2 \sum_{i=1}^n \frac{\partial}{\partial z_i} z_i q f &= 2q \sum_{i=1}^n \frac{\partial}{\partial z_i} z_i f \\ &= 2q \sum_{i=1}^n \left(z_i \frac{\partial f}{\partial z_i} + f \right) \\ &= 2q \left[\sum_{i=1}^n z_i \frac{\partial f}{\partial z_i} + \sum_{i=1}^n f \right] = 2q(p+n)f, \\ 2 \sum_{i=1}^n \sum_{a=1}^n \frac{\partial}{\partial z_a} z_a \bar{z}_i \frac{\partial f}{\partial \bar{z}_i} &= 2 \sum_{a=1}^n \frac{\partial}{\partial z_a} z_a \sum_{i=1}^n \bar{z}_i \frac{\partial f}{\partial \bar{z}_i} \\ &= 2 \sum_{a=1}^n \frac{\partial}{\partial z_a} z_a q f \\ &= 2q \sum_{a=1}^n \left(z_a \frac{\partial f}{\partial z_a} + f \right) = 2q(p+n)f. \end{aligned}$$

Now recall that on $\mathbb{S}^{2n-1}$ we have $z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n = 1$. Thus,

$$\sum_{a=1}^n \sum_{i=1}^n z_i \bar{z}_i f = \sum_{a=1}^n f = nf.$$

We also go over the following explicit computation (again by using linearity we can assume f is a monomial and $f = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n}$):

$$\begin{aligned} \sum_{a=1}^n z_a \frac{\partial}{\partial z_a} \sum_{i=1}^n z_i \bar{z}_i f &= \sum_{a=1}^n z_a \frac{\partial}{\partial z_a} (z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n) f \\ &= \sum_{a=1}^n z_a \left(\frac{\partial}{\partial z_a} z_1 \bar{z}_1 f + \cdots + \frac{\partial}{\partial z_a} z_a \bar{z}_a f + \cdots + \frac{\partial}{\partial z_a} z_n \bar{z}_n f \right) \\ &= \sum_{a=1}^n z_a \left(\frac{\alpha_a}{z_a} z_1 \bar{z}_1 f + \cdots + \frac{\alpha_a + 1}{z_a} z_a \bar{z}_a f + \cdots + \frac{\alpha_a}{z_a} z_n \bar{z}_n f \right) \\ &= \sum_{a=1}^n ((\alpha_a) z_1 \bar{z}_1 f + \cdots + (\alpha_a + 1) z_a \bar{z}_a f + \cdots + (\alpha_a) z_n \bar{z}_n f) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n (\alpha_1 + \cdots + (\alpha_i + 1) + \cdots + \alpha_n) z_i \bar{z}_i f \\
&= \sum_{i=1}^n (p+1) z_i \bar{z}_i f = (p+1) \sum_{i=1}^n z_i \bar{z}_i f = (p+1) f.
\end{aligned}$$

We are now ready to compute the fourth term of the $\bar{\partial}_b^* \bar{\partial}_b f$ expansion:

$$\begin{aligned}
-2 \sum_{i=1}^n \sum_{a=1}^n \frac{\partial}{\partial z_a} z_a \bar{z}_i z_i q f &= -2q \left(\sum_{a=1}^n \frac{\partial}{\partial z_a} z_a \sum_{i=1}^n z_i \bar{z}_i f \right) \\
&= -2q \left(\sum_{a=1}^n \left(z_a \frac{\partial}{\partial z_a} + I \right) \sum_{i=1}^n z_i \bar{z}_i f \right) \\
&= -2q \left(\sum_{a=1}^n z_a \frac{\partial}{\partial z_a} \sum_{i=1}^n z_i \bar{z}_i f + \sum_{a=1}^n \sum_{i=1}^n z_i \bar{z}_i f \right) \\
&= -2q(p+1+n)f.
\end{aligned}$$

Returning to our original computation of $\bar{\partial}_b^* \bar{\partial}_b f$, we now have

$$\begin{aligned}
&\bar{\partial}_b^* \bar{\partial}_b f \\
&= -2 \sum_{i=1}^n \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i} + 2 \sum_{i=1}^n \frac{\partial}{\partial z_i} z_i q + 2 \sum_{i=1}^n \sum_{a=1}^n \frac{\partial}{\partial z_a} z_a \bar{z}_i \frac{\partial f}{\partial \bar{z}_i} - 2 \sum_{i=1}^n \sum_{a=1}^n \frac{\partial}{\partial z_a} z_a \bar{z}_i z_i q \\
&= 0 + 2q(p+n)f + 2q(p+n)f - 2q(p+1+n)f \\
&= 2q(p+n-1)f.
\end{aligned}$$

□

Asymptotics of counting function. We now look at the counting function $N(m)$.

Definition 2.3. For $m \in \mathbb{Z}$, let $N(m)$ be the number of eigenvalues of $\square_b$ on $L^2(\mathbb{S}^{2n-1})$ that are less than or equal to m , counting multiplicity.

Similar functions and relations between their asymptotics and geometry of the underlying manifold were studied in [Métivier 1981; Fu 2005; 2008]. In particular in some cases the growth rate of $N(m)$ carries information about the type of the manifold [Fu 2005; 2008]. Furthermore, in the case of the Laplace–Beltrami operator, Weyl’s law states that the limit of the ratio $N(m)/m^n$ gives the surface area of $\mathbb{S}^{2n-1}$. Before we state our result, we recall Lemma 1.2.

Lemma 1.2. For $p, q \geq 1$,

$$\dim(\mathcal{H}_{p,q}(\mathbb{S}^{2n-1})) = \frac{(n-1)(n+p+q-1)}{pq} \binom{n+p-2}{p-1} \binom{n+q-2}{q-1}.$$

Note that ignoring multiplicity would induce a function with linear growth. Indeed for any even $\hat{m}$ with $m \geq \hat{m} > 2(n-1)$, we can solve $\hat{m} = 2q(p+n-1)$ after fixing $q = 1$. Additionally, by convention, we set $N(m) = 0$ when $m < 0$.

We note that when $n = 1$, the eigenvalue of $\bar{\partial}_b^* \bar{\partial}_b$ is equal to 0. Indeed, when $n = 1$ and when p and q are both nonzero, Lemma 1.2 gives us that the dimension of $\mathcal{H}_{p,q}$ is 0. This is because the only harmonic homogeneous polynomials on $\mathbb{C}$ are of the form z^p or $\bar{z}^q$, which belong to $\mathcal{H}_{p,0}$ or $\mathcal{H}_{0,q}$, respectively. Thus, $\mathcal{H}_{p,q}$ is nontrivial only when either p or q is zero. However, on such spaces, the eigenvalue of $\bar{\partial}_b^* \bar{\partial}_b$ on $\mathcal{H}_{p,q}$ is 0.

Lemma 2.4. *There exists a real constant $c > 0$ so that $cm^n \leq N(m)$; that is, $N(m) \in \Omega(m)$.*

Proof. Fix even m ; then $N(m) - N(m-2)$ is the multiplicity of the eigenvalue m , since all the eigenvalues are even by Theorem 1.3. This requires computing the sum of the dimensions of all $\mathcal{H}_{p,q}(\mathbb{S}^{2n-1})$ such that the pair (p, q) satisfies the equation $E(p, q) = m$, where $E(p, q) = 2q(p+n-1)$. Now assuming $m > 2(n-1)$, there exists a positive integer solution $p = \hat{p}$ to $E(p, q) = m$ when $q = 1$. Define the solution set $A = \{(p, q) \mid E(p, q) = m\}$. Then we have

$$N(m) - N(m-2) = \sum_{(p,q) \in A} \dim \mathcal{H}_{p,q} \geq \dim \mathcal{H}_{\hat{p},1}.$$

Note that $\dim \mathcal{H}_{\hat{p},1} = \Omega(m^{n-1})$, which follows from Lemma 1.2. Namely, since asymptotically $\hat{p} = m/2$, we have

$$\begin{aligned} \dim \mathcal{H}_{\hat{p},1(\mathbb{S}^{2n-1})} &= \frac{(n-1)(n+\hat{p})}{\hat{p}} \binom{n+\hat{p}-2}{n-1} \binom{n-1}{n-1} \\ &\geq \binom{n+\hat{p}-2}{n-1} \geq \frac{1}{(n-1)!} \hat{p}^{n-1} \\ &= \Omega\left(\frac{m}{2}\right)^{n-1} = \Omega(m^{n-1}). \end{aligned}$$

Putting it all together, we have

$$\begin{aligned} 2N(m) &\geq N(m) + N(m-1) = \sum_{j=0}^m (N(j) - N(j-2)) \\ &\geq \sum_{j=0}^m \Omega(j^{n-1}) \geq \Omega(m^n). \end{aligned} \quad \square$$

Lemma 2.5. *There exists a real constant $c > 0$ so that $N(m) \leq cm^n$; that is, $N(m) = O(m^n)$.*

Proof. Again, fix an even m and inspect $N(m) - N(m-2)$. Note that asymptotically, we can let our eigenvalue equation be $E(p, q) = 2qp$. Thus, asymptotically we

have

$$N(m) - N(m-2) = \sum_{(p,q) \in A} \dim \mathcal{H}_{p,q} \lesssim \sum_{(p,q) \in A} (p+q)(pq)^{n-2} = \sigma(m) O(m^{n-2}),$$

where $\sigma(m)$ is the sum of all divisors of m . Thus, we have

$$N(m) \lesssim \sum_{x \leq m} 2x^{n-2} \sigma(x) \lesssim 2m^{n-2} \sum_{x \leq m} \sigma(x) = O(m^n).$$

The last equality follows since $\sum_{x \leq m} \sigma(x) = O(m^2)$. A proof of this fact can be found in Chapter 3.6 of [Apostol 1976]. $\square$

By combining the last two lemmas we obtain the following statement.

Theorem 1.4. *There exists a real $c > 0$ so that $\frac{1}{c}m^n \leq N(m) \leq cm^n$; that is, $N(m) = \Theta(m^n)$.*

We note that the constants in Lemma 2.4, Lemma 2.5, and Theorem 1.4 do depend on the dimension n . This dependence also agrees with the explicit constants calculated by Weyl for the Laplace–Beltrami operator.

3. Spectra of other second-order differential operators on $L^2(\mathbb{S}^{2n-1})$

Another interesting class of second-order differential operators are sum of squares operators $\mathcal{M}_b$, introduced in the fourth chapter of [Klima 2004]. These operators capture *half* of the action of $\square_b$ on $\mathbb{S}^3$; in higher dimensions they lead to the study of various possible perturbations of $\square_b$.

We define the sum of squares operator $\mathcal{M}_b$ on $L^2(\mathbb{S}^{2n-1})$ as

$$\mathcal{M}_b = -(M_{12}\bar{M}_{12} + M_{13}\bar{M}_{13} + \cdots + M_{1n}\bar{M}_{1n}),$$

where $M_{1k} = \bar{z}_1(\partial/\partial z_k) - \bar{z}_k(\partial/\partial z_1)$ and $\bar{M}_{1k} = z_1(\partial/\partial \bar{z}_k) - z_k(\partial/\partial \bar{z}_1)$. Note that one can easily consider M_{ik} for $i \neq 1$; for simplicity we focus on the case $i = 1$.

For any $f \in \mathcal{H}_{p,q}(\mathbb{S}^{2n-1})$, the specific degrees of the $z_k, \bar{z}_k$ may vary. For example, both $z_1^2 z_2 \bar{z}_1^3 \bar{z}_2^2$ and $z_1 z_2^2 \bar{z}_1^3 \bar{z}_2^3$ are in $\mathcal{H}_{3,5}(\mathbb{S}^3)$. In previous arguments, such specificity was unnecessary, but we find that for $\mathcal{M}_b$, the eigenvalues can directly depend on the exact degrees of the $z_k, \bar{z}_k$. To that end, for nonnegative integer tuples $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$, we use $\mathcal{H}_{p,q}^*(\mathbb{C}^n)$ to denote the space of all harmonic, homogeneous polynomials where p_k is the degree of z_k , and q_k is the degree of $\bar{z}_k$. Then we use $\mathcal{H}_{p,q}^*(\mathbb{S}^{2n-1})$ to denote the restriction of this space on $\mathbb{S}^{2n-1}$. For example, now $z_1^2 z_2 \bar{z}_1^3 \bar{z}_2^2 \in \mathcal{H}_{(2,1),(3,2)}^*(\mathbb{S}^3)$ but $z_1 z_2^2 \bar{z}_1^3 \bar{z}_2^3 \in \mathcal{H}_{(1,2),(2,3)}^*(\mathbb{S}^3)$. Note that $\mathcal{H}_{p,q}^*(\mathbb{S}^{2n-1})$ is a subspace of $\mathcal{H}_{\bar{p},\bar{q}}(\mathbb{S}^{2n-1})$, where $\bar{p} = \sum_{i=1}^n p_i$ and $\bar{q} = \sum_{i=1}^n q_i$. Now for certain $\mathcal{H}_{p,q}^*(\mathbb{S}^{2n-1})$, we have the following result.

Lemma 3.1. Consider two nonnegative integer tuples $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$. Suppose that for each $1 \leq k \leq n$, at least one of p_k or q_k is 0. Then the eigenvalue of $\mathcal{M}_b$ on $\mathcal{H}_{p,q}^*(\mathbb{S}^{2n-1})$ is

$$p_1 \sum_{k=2}^n q_k + q_1 \sum_{k=2}^n p_k + (n-1)q_1 + \sum_{k=2}^n q_k.$$

Proof. Take $f \in \mathcal{H}_{p,q}^*(\mathbb{S}^{2n-1})$, where $p_k = 0$ or $q_k = 0$ for each k . By linearity, we can inspect the action of each $-M_{1k}\bar{M}_{1k}$ piece of $\mathcal{M}_b$ on f . We have

$$\begin{aligned} -M_{1k}\bar{M}_{1k}f &= -\left(\bar{z}_1 \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial z_1}\right) \left(z_1 \frac{\partial}{\partial \bar{z}_k} - z_k \frac{\partial}{\partial \bar{z}_1}\right) f \\ &= -\bar{z}_1 \frac{\partial}{\partial z_k} z_1 \frac{\partial}{\partial \bar{z}_k} f + \bar{z}_1 \frac{\partial}{\partial z_k} z_k \frac{\partial}{\partial \bar{z}_1} f + \bar{z}_k \frac{\partial}{\partial z_1} z_1 \frac{\partial}{\partial \bar{z}_k} f - \bar{z}_k \frac{\partial}{\partial z_1} z_k \frac{\partial}{\partial \bar{z}_1} f \\ &= -z_1 \bar{z}_1 \frac{\partial}{\partial z_k} \frac{\partial}{\partial \bar{z}_k} f + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \frac{\partial}{\partial z_k} z_k f + \bar{z}_k \frac{\partial}{\partial \bar{z}_k} \frac{\partial}{\partial z_1} z_1 f - z_k \bar{z}_k \frac{\partial}{\partial z_1} \frac{\partial}{\partial \bar{z}_1} f \\ &= 0 + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \frac{\partial}{\partial z_k} z_k f + \bar{z}_k \frac{\partial}{\partial \bar{z}_k} \frac{\partial}{\partial z_1} z_1 f - 0 \\ &= \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \left(z_k \frac{\partial}{\partial z_k} + I\right) f + \bar{z}_k \frac{\partial}{\partial \bar{z}_k} \left(z_1 \frac{\partial}{\partial z_1} + I\right) f \\ &= \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} z_k \frac{\partial}{\partial z_k} f + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} f + \bar{z}_k \frac{\partial}{\partial \bar{z}_k} z_1 \frac{\partial}{\partial z_1} f + \bar{z}_k \frac{\partial}{\partial \bar{z}_k} f \\ &= q_1 p_k f + q_1 f + q_k p_1 f + q_k f. \end{aligned}$$

Thus, we have

$$\begin{aligned} \mathcal{M}_b(f) &= \sum_{k=2}^n -M_{1k}\bar{M}_{1k}f = \sum_{k=2}^n (q_1 p_k + q_1 + q_k p_1 + q_k) f \\ &= \left(\sum_{k=2}^n q_1 p_k + \sum_{k=2}^n q_1 + \sum_{k=2}^n q_k p_1 + \sum_{k=2}^n q_k \right) f \\ &= \left(q_1 \sum_{k=2}^n p_k + (n-1)q_1 + p_1 \sum_{k=2}^n q_k + \sum_{k=2}^n q_k \right) f. \quad \square \end{aligned}$$

The above lemma tells us that $z_1^2 z_2 \bar{z}_1^3 \bar{z}_2^2 \in \mathcal{H}_{(2,1),(3,2)}^*(\mathbb{S}^3)$ has eigenvalue $2(2) + 3(1) + (2-1)(3) + (2) = 12$. On the other hand, $z_1 z_2^2 \bar{z}_1^2 \bar{z}_2^3 \in \mathcal{H}_{(1,2),(3,2)}^*(\mathbb{S}^3)$ has eigenvalue $1(2) + 3(2) + (2-1)(3) + (2) = 13$. More generally, the lemma tells us that $\mathcal{H}_{p,0}(\mathbb{S}^{2n-1})$ is in the null space of $\mathcal{M}_b$ for all $p \in \mathbb{N}$. Furthermore, the eigenvalue of $\mathcal{M}_b$ on $\mathcal{H}_{0,q}^*(\mathbb{S}^{2n-1})$ is $(n-1)q_1 + q_2 + \dots + q_n$. On other $\mathcal{H}_{p,q}(\mathbb{S}^{2n-1})$ spaces, computational results suggest that we have integer eigenvalues, and matrix representations follow a pattern as well. We will leave the investigation

of other eigenvalues to a future study. We invite the interested reader to see other computational results by downloading our code.²

4. Eigenvalues of $\square_b^t$ on the Rossi sphere

Previously in [Abbas et al. 2019], the authors studied the spectrum of the perturbed Kohn Laplacian $\square_b^t$ on the Rossi sphere. They obtained an upper bound for the lowest eigenvalue for $\square_b^t$ on each $\mathcal{H}_k(\mathbb{S}^3)$. In our project, we look at the asymptotics of the spectrum of the (perturbed) Kohn Laplacian on the Rossi sphere, in particular the asymptotics of $\lambda_k^{\max}$, the maximum eigenvalue of $\square_b^t$ on $\mathcal{H}_k(\mathbb{S}^3)$.

In [Abbas et al. 2019] the authors prove tridiagonal representation results for spaces of homogeneous polynomials of odd degree, $\mathcal{H}_{2k-1}(\mathbb{S}^3)$. However, their proof actually works for arbitrary degrees, $\mathcal{H}_k(\mathbb{S}^3)$. We restate the steps to construct the tridiagonal matrix representations here, and one can refer to [Abbas et al. 2019] for details. We first recall the definition of differential operators $\mathcal{L}$, $\bar{\mathcal{L}}$, and $\square_b^t$ on $L^2(\mathbb{S}^3)$.

Definition 4.1. We define $\mathcal{L}$ and $\bar{\mathcal{L}}$ as

$$\begin{aligned}\mathcal{L} &= \bar{z}_1 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial z_1}, \\ \bar{\mathcal{L}} &= z_1 \frac{\partial}{\partial \bar{z}_2} - z_2 \frac{\partial}{\partial \bar{z}_1}, \\ \square_b^t &= -\mathcal{L}_t \frac{1 + |t|^2}{(1 - |t|^2)^2} \bar{\mathcal{L}}_t.\end{aligned}$$

The motivation for these operators arises from the CR-manifold $(\mathbb{S}^3, \mathcal{L}_t)$, which is not CR-embeddable [Rossi 1965]. Note that $\mathcal{L}_t = \mathcal{L} + t\bar{\mathcal{L}}$ and $|t| < 1$.

Theorem 4.2 [Abbas et al. 2019]. *Let $\{f_0, \dots, f_k\}$ be an orthogonal basis for $\mathcal{H}_{0,k}(\mathbb{S}^3)$. Then $\{\bar{\mathcal{L}}^\sigma f_0, \dots, \bar{\mathcal{L}}^\sigma f_k\}$ is an orthogonal basis for $\mathcal{H}_{\sigma,k-\sigma}(\mathbb{S}^3)$.*

The proof of Theorem 4.2 follows from induction on inner products. The main two steps are the fact that $-\mathcal{L}$ is the adjoint of $\bar{\mathcal{L}}$, and that $\mathcal{L}\bar{\mathcal{L}}$ scales elements of $\mathcal{H}_{p,q}(\mathbb{S}^3)$ by a constant factor based on their bidegree.

Now one can consider an orthogonal basis $\{f_0, \dots, f_k\}$ for $\mathcal{H}_{0,k}(\mathbb{S}^3)$ and define the following two subspaces for even k :

$$\begin{aligned}V_i &= \text{span}\{f_i, \bar{\mathcal{L}}^2 f_i, \bar{\mathcal{L}}^4 f_i, \dots, \bar{\mathcal{L}}^{k-2} f_i, \bar{\mathcal{L}}^k f_i\}, \\ W_i &= \text{span}\{\bar{\mathcal{L}} f_i, \bar{\mathcal{L}}^3 f_i, \bar{\mathcal{L}}^5 f_i, \dots, \bar{\mathcal{L}}^{k-3} f_i, \bar{\mathcal{L}}^{k-1} f_i\},\end{aligned}$$

²The code can be downloaded at <https://goo.gl/kBsUzA>.

and similarly for odd k :

$$\begin{aligned} V_i &= \text{span}\{f_i, \bar{\mathcal{L}}^2 f_i, \bar{\mathcal{L}}^4 f_i, \dots, \bar{\mathcal{L}}^{k-3} f_i, \bar{\mathcal{L}}^{k-1} f_i\}, \\ W_i &= \text{span}\{\bar{\mathcal{L}} f_i, \bar{\mathcal{L}}^3 f_i, \bar{\mathcal{L}}^5 f_i, \dots, \bar{\mathcal{L}}^{k-2} f_i, \bar{\mathcal{L}}^k f_i\}. \end{aligned}$$

The motivation to define such spaces follows by inspecting the expanded form of $\square_b^t$, which is equal to $\mathcal{L}\bar{\mathcal{L}} + \bar{\mathcal{L}}\mathcal{L} + \mathcal{L}^2 + \bar{\mathcal{L}}^2$ up to constants. Previous work has shown that $\mathcal{L}\bar{\mathcal{L}}$ and $\bar{\mathcal{L}}\mathcal{L}$ scale elements of $\mathcal{H}_{p,q}(\mathbb{S}^3)$ by a constant factor, and the actions of $\mathcal{L}^2$ and $\bar{\mathcal{L}}^2$ suggest that invariant subspaces will involve basis elements that differ by $2j$ applications of $\bar{\mathcal{L}}$. Indeed, it was shown in [Abbas et al. 2019] that $\square_b^t$ is invariant on V_i and W_i . On these finite-dimensional invariant subspaces one can obtain a matrix representation for the second-order operator $\square_b^t$.

Theorem 4.3 [Abbas et al. 2019]. *The matrix representation of $\square_b^t$, $m(\square_b^t)$, on $V_i, W_i \subset \mathcal{H}_k(\mathbb{S}^3)$ is*

$$h \begin{pmatrix} d_1 & u_1 & & & \\ -\bar{t} & d_2 & u_2 & & \\ & -\bar{t} & d_3 & \ddots & \\ & & \ddots & \ddots & u_{k-1} \\ & & & -\bar{t} & d_k \end{pmatrix},$$

where h is a constant and on V_i ,

$$\begin{aligned} u_j &= -4t \cdot (j)(2j-1)(k-j)(2k-1-2j), \\ d_j &= (2j-1)(2k+1-2j) + 4|t|^2(j-1)(k+1-j); \end{aligned}$$

on W_i ,

$$\begin{aligned} u_j &= -4t \cdot (j)(2j+1)(k-j)(2k-1-2j), \\ d_j &= 4(j)(k-j) + |t|^2(2j-1)(2k+1-2j). \end{aligned}$$

Moreover, the matrix above is similar to

$$B = \begin{pmatrix} d_1 & c_1 & & & \\ c_1 & d_2 & c_2 & & \\ & c_2 & d_3 & \ddots & \\ & & \ddots & \ddots & c_{k-1} \\ & & & c_{k-1} & d_k \end{pmatrix},$$

where $c_j = (-\bar{t} \cdot u_j)^{1/2} = |t|\sqrt{-u_j/t}$.

After recalling these results, we also introduce the numerical range of a matrix.

Definition 4.4. Given an $n \times n$ square matrix A , we define its numerical range $W(A) = \{\langle Ax, x \rangle \mid x \in \mathbb{C}^n, \|x\| = 1\}$.

Also recall that $\lambda_k^{\max}$ denotes the maximum eigenvalue of $m(\square_b^t)$ on $\mathcal{H}_k(\mathbb{S}^3)$. We first prove the following lower bound.

Lemma 4.5. *There exists a real constant $c > 0$ so that $\frac{1}{c}k^2 \leq \lambda_k^{\max}$; that is, $\lambda_k^{\max} = \Omega(k^2)$.*

Proof. For a square matrix A , $\sup W(A)$ is an upper bound for the eigenvalues of A . Furthermore, if A is Hermitian then the maximum eigenvalue equals $\sup W(A)$.

Let $A = m(\square_b^t)$ on W_i . By the above discussion, since A is similar to a Hermitian matrix B , it suffices to show that $\sup W(B) = \Omega(k^2)$.

Fix $x = e_{k/2}$ for k even, and $x = e_{(k+1)/2}$ for k odd. Since $\langle Be_i, e_j \rangle = a'_{ij}$, by the above matrix representation we have that for k even

$$\begin{aligned} \langle Be_{k/2}, e_{k/2} \rangle &= B_{k/2, k/2} = d_{k/2} \\ &= 4\left(\frac{k}{2}\right)\left(k - \frac{k}{2}\right) + |t|^2\left(2\frac{k}{2} - 1\right)\left(2k + 1 - 2\frac{k}{2}\right) \\ &= k^2 + |t|^2(k-1)(k+1) \\ &= \Omega(k^2). \end{aligned}$$

A similar result follows for k odd. Now since $\langle Be_{k/2}, e_{k/2} \rangle \in W(B)$, we have $\sup W(B) = \Omega(k^2)$. $\square$

For the lower bound we invoke Gershgorin's circle theorem.

Theorem 4.6 [Gershgorin 1931]. *Suppose A is a complex square matrix, and R_i is the sum of the absolute values of the off-diagonal entries in the i -th row. Then every eigenvalue of A must lie within one of the closed discs $D(a_{ii}, R_i) \subset \mathbb{C}$.*

Recall that $m(\square_b^t)$ on V_i , W_i is similar to the real symmetric matrix B . Since B is Hermitian, Theorem 4.6 will give us interval bounds on the real line. Furthermore, the tridiagonal structure of B makes these bounds tight.

Lemma 4.7. *There exists a real constant $c > 0$ so that $\lambda_k^{\max} \leq ck^2$; that is, $\lambda_k^{\max} = O(k^2)$.*

Proof. Applying Theorem 4.6 on B , we have

$$D(b_{ii}, R_i) = (d_i - (c_{i-1} + c_i), d_i + (c_{i-1} + c_i)),$$

since the i -th row of B has only two off-diagonal entries, c_{i-1} and c_i , both of which are nonnegative by Theorem 4.3. Note that for the extremal cases of the first and last rows, the radii of these discs will involve only one off-diagonal entry. Now it suffices to show that an upper bound for $M_i = d_i + c_{i-1} + c_i$ is $O(k^2)$. By inspection, c_{i-1} , and c_i are $O(k^2)$ because u_{i-1} , u_i are $O(k^4)$. Since d_i is $O(k^2)$ as well, we have our result. $\square$

By combining the last two lemmas we obtain the following statement.

Theorem 4.8. *There exists a real $c > 0$ so that $\frac{1}{c}k^2 \leq \lambda_k^{\max} \leq ck^2$; that is, $\lambda_k^{\max} = \Theta(k^2)$.*

In addition to the asymptotics $\lambda_k^{\max}$, we computed $\lambda_k^{\max}$ explicitly by using SymPy. Similar codes also work to compute the largest eigenvalues of other operators, such as $\mathcal{M}_b$, on finite-dimensional invariant spaces.

Finally we note that, in this section we studied perturbed Kohn Laplacians on $\mathbb{S}^3$. One can define similar perturbations on higher-dimensional spheres and investigate the corresponding spectra. Although in higher dimensions the Boutet de Monvel theorem [1975] guarantees embeddability of strongly pseudoconvex abstract CR manifolds, it would be still worthwhile to compute the distribution of eigenvalues.

Acknowledgements

We thank the anonymous referee for constructive comments. This research was conducted at the NSF REU Site (DMS-1659203) in Mathematical Analysis and Applications at the University of Michigan-Dearborn. We would like to thank the National Science Foundation, National Security Agency, and University of Michigan-Dearborn for their support.

References

- [Abbas et al. 2019] T. Abbas, M. M. Brown, A. Ramasami, and Y. E. Zeytuncu, “Spectrum of the Kohn Laplacian on the Rossi sphere”, *Involve* **12**:1 (2019), 125–140. MR Zbl
- [Apostol 1976] T. M. Apostol, *Introduction to analytic number theory*, Springer, 1976. MR Zbl
- [Axler et al. 1992] S. Axler, P. Bourdon, and W. Ramey, *Harmonic function theory*, Graduate Texts in Math. **137**, Springer, 1992. MR Zbl
- [Boggess 1991] A. Boggess, *CR manifolds and the tangential Cauchy–Riemann complex*, CRC Press, Boca Raton, FL, 1991. MR Zbl
- [Boutet de Monvel 1975] L. Boutet de Monvel, “Intégration des équations de Cauchy–Riemann induites formelles”, exposé 9 in *Séminaire Goulaouic–Lions–Schwartz 1974–1975*, Centre Math., École Polytech., Paris, 1975. MR Zbl
- [Burns 1979] D. M. Burns, Jr., “Global behavior of some tangential Cauchy–Riemann equations”, pp. 51–56 in *Partial differential equations and geometry* (Park City, UT, 1977), edited by C. I. Byrnes, Lecture Notes in Pure and Appl. Math. **48**, Dekker, New York, 1979. MR Zbl
- [Chen and Shaw 2001] S.-C. Chen and M.-C. Shaw, *Partial differential equations in several complex variables*, AMS/IP Studies in Adv. Math. **19**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl
- [Folland 1972] G. B. Folland, “The tangential Cauchy–Riemann complex on spheres”, *Trans. Amer. Math. Soc.* **171** (1972), 83–133. MR Zbl
- [Fu 2005] S. Fu, “Hearing pseudoconvexity with the Kohn Laplacian”, *Math. Ann.* **331**:2 (2005), 475–485. MR Zbl
- [Fu 2008] S. Fu, “Hearing the type of a domain in $\mathbb{C}^2$ with the $\bar{\partial}$ -Neumann Laplacian”, *Adv. Math.* **219**:2 (2008), 568–603. MR Zbl
- [Gershgorin 1931] S. Gershgorin, “Über die Abgrenzung der Eigenwerte einer Matrix”, *Izv. Akad. Nauk SSSR* **1931**:6 (1931), 749–754. In Russian. Zbl

- [Klima 2004] O. Klima, *Analysis of a subelliptic operator on the sphere in complex n -space*, master's thesis, University of New South Wales, 2004, available at <https://tinyurl.com/klimamast>.
- [Métivier 1981] G. Métivier, "Spectral asymptotics for the $\bar{\partial}$ -Neumann problem", *Duke Math. J.* **48**:4 (1981), 779–806. MR Zbl
- [Rossi 1965] H. Rossi, "Attaching analytic spaces to an analytic space along a pseudoconcave boundary", pp. 242–256 in *Proc. Conf. Complex Analysis* (Minneapolis, 1964), edited by A. Aepli et al., Springer, 1965. MR Zbl

Received: 2018-09-05 Accepted: 2018-12-26

jtahn@bowdoin.edu	<i>Bowdoin College, Brunswick, ME, United States</i>
bansilmo@msu.edu	<i>Michigan State University, East Lansing, MI, United States</i>
garrettbrown@college.harvard.edu	<i>Harvard University, Cambridge, MA, United States</i>
elcardin@email.wm.edu	<i>College of William and Mary, Williamsburg, VA, United States</i>
zeytuncu@umich.edu	<i>Department of Mathematics and Statistics, University of Michigan-Dearborn, Dearborn, MI, United States</i>

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA	Chi-Kwong Li	College of William and Mary, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Robert B. Lund	Clemson University, USA
Martin Bohner	Missouri U of Science and Technology, USA	Gaven J. Martin	Massey University, New Zealand
Nigel Boston	University of Wisconsin, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of N Carolina, Chapel Hill, USA	Frank Morgan	Williams College, USA
Pietro Cerone	La Trobe University, Australia	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Scott Chapman	Sam Houston State University, USA	Zuhair Nashed	University of Central Florida, USA
Joshua N. Cooper	University of South Carolina, USA	Ken Ono	Emory University, USA
Jem N. Corcoran	University of Colorado, USA	Yuval Peres	Microsoft Research, USA
Toka Diagana	Howard University, USA	Y.-F. S. Pétermann	Université de Genève, Switzerland
Michael Dorff	Brigham Young University, USA	Jonathon Peterson	Purdue University, USA
Sever S. Dragomir	Victoria University, Australia	Robert J. Plemmons	Wake Forest University, USA
Joel Foisy	SUNY Potsdam, USA	Carl B. Pomerance	Dartmouth College, USA
Errin W. Fulp	Wake Forest University, USA	Vadim Ponomarenko	San Diego State University, USA
Joseph Gallian	University of Minnesota Duluth, USA	Bjorn Poonen	UC Berkeley, USA
Stephan R. Garcia	Pomona College, USA	József H. Przytycki	George Washington University, USA
Anant Godbole	East Tennessee State University, USA	Richard Rebarber	University of Nebraska, USA
Ron Gould	Emory University, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Javier Rojo	Oregon State University, USA
Jim Haglund	University of Pennsylvania, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Johnny Henderson	Baylor University, USA	Hari Mohan Srivastava	University of Victoria, Canada
Glenn H. Hurlbert	Arizona State University, USA	Andrew J. Sterge	Honorary Editor
Charles R. Johnson	College of William and Mary, USA	Ann Trenk	Wellesley College, USA
K. B. Kulasekera	Clemson University, USA	Ravi Vakil	Stanford University, USA
Gerry Ladas	University of Rhode Island, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
David Larson	Texas A&M University, USA	John C. Wierman	Johns Hopkins University, USA
Suzanne Lenhart	University of Tennessee, USA	Michael E. Zieve	University of Michigan, USA

PRODUCTION

Silvio Levy, Scientific Editor

Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2019 is US \$195/year for the electronic version, and \$260/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1444-4184 electronic, 1444-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

involve

2019

vol. 12

no. 5

Orbigraphs: a graph-theoretic analog to Riemannian orbifolds	721
KATHLEEN DALY, COLIN GAVIN, GABRIEL MONTES DE OCA, DIANA OCHOA, ELIZABETH STANHOPE AND SAM STEWART	
Sparse neural codes and convexity	737
R. AMZI JEFFS, MOHAMED OMAR, NATCHANON SUAYSOM, ALEINA WACHTEL AND NORA YOUNGS	
The number of rational points of hyperelliptic curves over subsets of finite fields	755
KRISTINA NELSON, JÓZSEF SOLYMOSI, FOSTER TOM AND CHING WONG	
Space-efficient knot mosaics for prime knots with mosaic number 6	767
AARON HEAP AND DOUGLAS KNOWLES	
Shabat polynomials and monodromy groups of trees uniquely determined by ramification type	791
NAIOMI CAMERON, MARY KEMP, SUSAN MASLAK, GABRIELLE MELAMED, RICHARD A. MOY, JONATHAN PHAM AND AUSTIN WEI	
On some edge Folkman numbers, small and large	813
JENNY M. KAUFMANN, HENRY J. WICKUS AND STANISŁAW P. RADZISZOWSKI	
Weighted persistent homology	823
GREGORY BELL, AUSTIN LAWSON, JOSHUA MARTIN, JAMES RUDZINSKI AND CLIFFORD SMYTH	
Leibniz algebras with low-dimensional maximal Lie quotients	839
WILLIAM J. COOK, JOHN HALL, VICKY W. KLIMA AND CARTER MURRAY	
Spectra of Kohn Laplacians on spheres	855
JOHN AHN, MOHIT BANSIL, GARRETT BROWN, EMILEE CARDIN AND YUNUS E. ZEYTUNCU	
Pairwise compatibility graphs: complete characterization for wheels	871
MATTHEW BEAUDOUIN-LAFON, SERENA CHEN, NATHANIEL KARST, DENISE SAKAI TROXELL AND XUDONG ZHENG	
The financial value of knowing the distribution of stock prices in discrete market models	883
AYELET AMIRAN, FABRICE BAUDOIN, SKYLYN BROCK, BEREND COSTER, RYAN CRAVER, UGONNA EZEAKA, PHANUEL MARIANO AND MARY WISHART	



1944-4176(2019)12:5;1-0